Torsion on abelian surfaces with many endomorphisms

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Introduction

If A is an abelian variety over a number field F then

$$A(F) \simeq \mathbb{Z}^r \oplus A(F)_{\text{tors}}$$

Question

What can $A(F)_{\text{tors}}$ be?

Theorem (Mazur '77)

If $E/\mathbb{Q}$ is an elliptic curve then

$$E(\mathbb{Q})_{\text{tors}} \in \begin{cases} (\mathbb{Z}/N\mathbb{Z}) & N \in \{1 \ldots 10\} \cup \{12\}; \\ (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z}) & N \in \{2, 4, 6, 8\}. \end{cases}$$

Introduction

Let A be a geometrically simple abelian surface defined over $\mathbb{Q}$.

$$O := \mathrm{End}(A_{\overline{\mathbb{Q}}}), \qquad B := \mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

B is one of the following:

1. $\mathbb{Q}$ - *typical*;
2. $\mathbb{Q}(\sqrt{m}),\ m > 0$ - *GL_2-type* or *real multiplication (RM)*;
3. An indefinite quaternion algebra over $\mathbb{Q}$ - *quaternionic multiplication (QM)*;
4. A *CM* field - *complex multiplication (CM)*.

Introduction

Definition

A quaternion algebra B over $\mathbb{Q}$ is an algebra of the form

$$B \simeq \mathbb{Q} \cdot 1 + \mathbb{Q} \cdot i + \mathbb{Q} \cdot j + \mathbb{Q} \cdot ij$$

where

$$i^2 = a, \; j^2 = b, \; \textit{and } ij = -ji$$

for some $a, b \in \mathbb{Q}^\times$. B is denoted $\left(\frac{a,b}{F}\right)$.

Example:

$$M_2(\mathbb{Q}) \simeq \left(\frac{1,1}{\mathbb{Q}}\right), \quad i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- B is *split* if $B \simeq M_2(\mathbb{Q})$
- B is *indefinite* if $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})$
- B is characterized by the finitely many primes p such $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \not\simeq M_2(\mathbb{Q}_p)$. The product of these primes is the *discriminant*.

Introduction

Suppose $O := \mathrm{End}(A_{\overline{\mathbb{Q}}})$ is a maximal order in a nonsplit rational quaternion algebra over $\mathbb{Q}$. We say A has *potential quaternionic multiplication* (PQM), or is a *PQM surface*.

Fact

A PQM surface over $\mathbb{Q}$ has everywhere potentially good reduction.

$$A(\mathbb{Q})^{\text{odd}}_{\text{tors}} \hookrightarrow A(\mathbb{F}_2) \quad \text{and} \quad A(\mathbb{Q})[2^\infty] \hookrightarrow A(\mathbb{F}_3).$$

Theorem (Clark-Xarles)

If $A/\mathbb{Q}$ is a PQM surface then

$$\#A(\mathbb{Q})_{\text{tors}} \in [1,16] \cup [18,20] \cup \{22,24,25,28,30,36,48,60,72\}.$$

Introduction

Theorem (Laga, S., Shnidman, Voight)

Let $A/\mathbb{Q}$ be a PQM surface. Then

- *if $A[\ell](\mathbb{Q}) \neq 0$ for a prime ℓ, $\ell \in \{2,3\}$;*
- *each of the six groups*

$$\{1\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, (\mathbb{Z}/3\mathbb{Z})^2$$

  occurs as $A(\mathbb{Q})_{\mathrm{tors}}$ for infinitely many $\overline{\mathbb{Q}}$-isomorphism classes of PQM surfaces $A/\mathbb{Q}$;
- *if A is GL-type then $A(\mathbb{Q})_{\mathrm{tors}}$ is one of these groups minus $\mathbb{Z}/6\mathbb{Z}$;*
- *all of the remaining possible groups have been ruled out except*

$$\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^3, (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3,$$

$$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/4\mathbb{Z})^2, (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2.$$

Galois action

Let $I \subset O$ be a two sided ideal and consider the I-torsion

$$A[I](\overline{\mathbb{Q}}) = \{\ P \in A(\overline{\mathbb{Q}}) \mid x \cdot P = 0 \text{ for all } x \in I\ \}$$

- $A[I](\overline{\mathbb{Q}})$ is a left O-module and a right $\mathrm{Gal}_{\mathbb{Q}}$-module.
- O is a right $\mathrm{Gal}_{\mathbb{Q}}$-module via the action on the equations defining elements of $O = \mathrm{End}(A_{\overline{\mathbb{Q}}})$.

They are compatible in the sense that

$$(a \cdot P)^{\sigma} = a^{\sigma} \cdot P^{\sigma}.$$

The minimal field of definition L of $\mathrm{End}(A)$ satisfies $\mathrm{Gal}(L|\mathbb{Q}) \simeq C_2$ or $D_n, n \in 2, 3, 4, 6$ and either

- $\mathrm{Gal}(L|\mathbb{Q}) \simeq C_2$ and A is GL_2-type; or
- $\mathrm{End}(A) \simeq \mathbb{Z}$.

Proposition

For a prime $\ell \geqslant 5$, if $(\mathbb{Z}/\ell\mathbb{Z}) \leqslant A[\ell](\mathbb{Q})$ then A is of GL_2-type.

Bad reduction

Reduction properties also constrain the torsion.

Let $\mathcal{A}_{\mathbb{F}_p}$ denote the special fiber of the Néron model.

$$0 \longrightarrow \mathcal{A}^{\circ}_{\mathbb{F}_p} \longrightarrow \mathcal{A}_{\mathbb{F}_p} \longrightarrow \Phi \longrightarrow 0$$

where Φ is the group of connected components. The identity component $\mathcal{A}^{\circ}_{\mathbb{F}_p}$ fits into

$$0 \longrightarrow U \times T \longrightarrow \mathcal{A}^{\circ}_{\mathbb{F}_p} \longrightarrow B \longrightarrow 0;$$

U unipotent; T toric; B abelian variety.

- If A has good reduction over M then $|M : \mathbb{Q}_p|$ kills $\Phi^{(p)}$ (Lorenzini).
- If A has purely additive reduction $(\mathrm{Dim}(U) = 2)$ then on the prime-to-p torsion $A(\mathbb{Q})^{(p)}_{\mathrm{tors}} \leqslant \Phi^{(p)}$.

Proposition

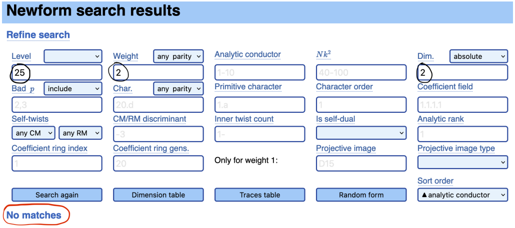
For a prime ℓ if $(\mathbb{Z}/\ell\mathbb{Z}) \leqslant A[\ell](\mathbb{Q}) \neq 0$ then $\ell \in \{2,3\}$.

Proof idea: suppose $\ell \geqslant 5, (\mathbb{Z}/\ell\mathbb{Z}) \leqslant A(\mathbb{Q})_{\text{tors}}$. If $p \neq \ell$ were a prime of bad reduction:

- A is GL_2-type, it follows that it has purely additive reduction at p.
- $(\mathbb{Z}/\ell\mathbb{Z}) \leqslant A(\mathbb{Q})_{\text{tors}}^{(p)} \leqslant \Phi^{(p)}$.
- $|M : \mathbb{Q}_p|$ is coprime to $\ell \notin \{2,3\}$ (Jordan-Morrison).

$|M : \mathbb{Q}_p|$ cannot kill $\Phi^{(p)}$, so A has good reduction at p.

$$A \rightsquigarrow f \in S_2(\Gamma_0(\ell^2)).$$



Torsion

This still leaves a lot of possibilities for $A(\mathbb{Q})_{\text{tors}}$:

$$\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^3, \mathbb{Z}/8\mathbb{Z},$$
$$(\mathbb{Z}/9\mathbb{Z}), (\mathbb{Z}/3\mathbb{Z})^2, (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3, \mathbb{Z}/12\mathbb{Z}, (\mathbb{Z}/4\mathbb{Z})^2, (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z},$$
$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^4 \ldots$$

Many have been ruled out by analysing the Galois modules, reduction properties mod p and ad hoc methods.

This leaves

Examples Known : $\{1\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, (\mathbb{Z}/3\mathbb{Z})^2.$

Unkown : $\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^3, (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/3,$
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, (\mathbb{Z}/4\mathbb{Z})^2, (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^2.$

Shimura curves

$A(\mathbb{Q})_{\text{tors}}$	$y^2 = f(x)$
$(\mathbb{Z}/2\mathbb{Z})$	$y^2 = -1550x^6 + 20460x^5 - 73470x^4 + 123070x^3 - 930x^2 - 24180x - 1550$
$(\mathbb{Z}/2\mathbb{Z})^2$	$y^2 = -180x^6 - 159x^5 + 894x^4 + 1691x^3 + 246x^2 - 672x + 80$
$(\mathbb{Z}/3\mathbb{Z})$	$y^2 = -10x^6 - 300x^5 + 420x^4 + 800x^3 + 120x^2 - 1200x - 560$
$(\mathbb{Z}/3\mathbb{Z})^2$	$y^2 = -15x^6 - 270x^5 + 315x^4 - 270x^3 - 45x^2 + 270x + 105$
$(\mathbb{Z}/6\mathbb{Z})$	$y^2 = 5x^6 + 21x^5 - 63x^4 - 49x^3 + 294x^2 - 343$

Table: Examples of PQM surfaces with torsion

Baba-Granath computed Igusa-Invariants for a family of genus 2 curves with PQM Jacobians with $\mathrm{Disc}(O) = 6$:

$$(96j + 96 : 576j : 23040j^2 + 18432j : 4096j^3).$$

This corresponds to the *Shimura curve* $X^*(6,1) := N_{B^\times}(O)\backslash\mathbb{H}$.
The covers

$$\phi : X \longrightarrow X^*(6,1).$$

are computable in low degree via Belyi maps (Musty-Schiavone-Sijsling-Voight).

Shimura curves

There is a Galois representation

$$\overline{\rho}_{A,N} : \mathrm{Gal}_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(O) \ltimes (O/N)^{\times}.$$

We can consider subgroups $H \leqslant \mathrm{Aut}(O) \ltimes (O/N)^{\times}$ which leave a subset of $A[N](\overline{\mathbb{Q}})$ fixed.

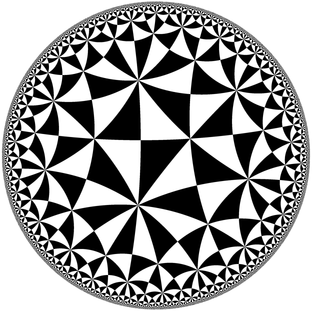


Figure: Shimura curve $X^*(6,1)$

There is a Fuchsian group $\Gamma_H \leqslant \mathrm{PSL}_2(\mathbb{R})$ associated to H and

$$X_H := \Gamma_H \backslash \mathbb{H}$$

will parameterise PQM surfaces with $\overline{\rho}_{A,N}(\mathrm{Gal}_{\mathbb{Q}}) \subseteq H$.

Shimura curves

Genus(X_H)	$A(\mathbb{Q})_{\text{tors}}$	Index	Atkin-Lehner
0	$(\mathbb{Z}/2\mathbb{Z})$	6	w_6
0	$(\mathbb{Z}/3\mathbb{Z})$	4	w_2, w_3, w_6
0	$(\mathbb{Z}/2\mathbb{Z})^2$	6	w_3
1	$(\mathbb{Z}/6\mathbb{Z})$	12	w_2, w_3, w_6
0	$(\mathbb{Z}/3\mathbb{Z})^2$	8	w_2
1	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$	24	w_2, w_3, w_6
3	$(\mathbb{Z}/2\mathbb{Z})^3$	48	w_3
3	$(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$	48	w_3
2	$(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/3\mathbb{Z})$	48	w_2, w_3, w_6

Table: Shimura curves X_H

Thank you for listening!