Algorithms for hyperelliptic Mumford curves

Steffen Müller (University of Groningen)

joint with Enis Kaya, Marc Masdeu and Marius van der Put

Arithmétique, Géométrie, Cryptographie et Théorie des Codes
June 6 2023

Tate curves

Let p be prime and $K/\mathbb{Q}_p$ finite with

- residue field k
- uniformizer π
- a fixed embedding $K \hookrightarrow \mathbb{C}_p$.

Tate curves

Let p be prime and $K/\mathbb{Q}_p$ finite with

- residue field k
- uniformizer π
- a fixed embedding $K \hookrightarrow \mathbb{C}_p$.

Question. Let C/K be a nice (i.e. smooth, projective and geometrically irreducible) curve.

Does C have a p-adic uniformization?

Tate curves

Let p be prime and $K/\mathbb{Q}_p$ finite with

- residue field k
- uniformizer π
- a fixed embedding $K \hookrightarrow \mathbb{C}_p$.

Question. Let C/K be a nice (i.e. smooth, projective and geometrically irreducible) curve.

Does C have a p-adic uniformization?

Theorem (Tate '59). If E/K is an elliptic curve with split multiplicative reduction, then there is a p-adic uniformization

$$E(K) \simeq K^\times / q^{\mathbb{Z}}; \quad |q| < 1.$$

Examples

A nice curve C/K of genus $g \geq 2$ has split degenerate reduction, if it has a stable model $\mathcal{C}/\mathcal{O}_K$ such that

Examples

A nice curve C/K of genus $g \geq 2$ has split degenerate reduction, if it has a stable model $\mathcal{C}/\mathcal{O}_K$ such that the special fiber $\mathcal{C}_k$ satisfies

- all irreducible components have genus 0 (degenerate),

Examples

A nice curve C/K of genus $g \geq 2$ has split degenerate reduction, if it has a stable model $\mathcal{C}/\mathcal{O}_K$ such that the special fiber $\mathcal{C}_k$ satisfies

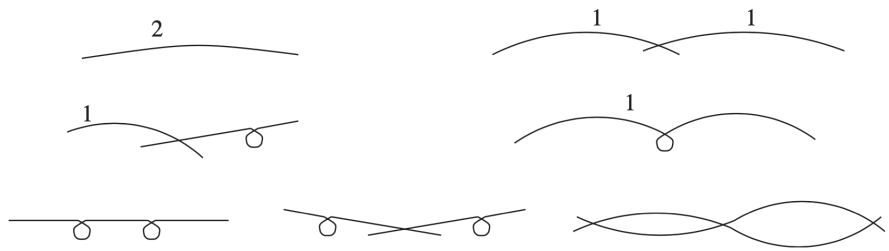
- all irreducible components have genus 0 (degenerate),
- all double points are k-rational with k-rational branches (split).

Examples

A nice curve C/K of genus $g \geq 2$ has split degenerate reduction, if it has a stable model $\mathcal{C}/\mathcal{O}_K$ such that the special fiber $\mathcal{C}_k$ satisfies

- all irreducible components have genus 0 (degenerate),
- all double points are k-rational with k-rational branches (split).

Example: There are 7 stable reduction types in genus 2:



The bottom three pictures are the degenerate ones.

Schottky groups

Definition. A Schottky group over K is a discrete, finitely generated and free subgroup $\Gamma \leq \mathrm{PGL}_2(K)$.

Example. If $q \in K^\times$ with $|q| < 1$, then $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ generates a Schottky group.

Schottky groups

Definition. A Schottky group over K is a discrete, finitely generated and free subgroup $\Gamma \leq \mathrm{PGL}_2(K)$.

Example. If $q \in K^\times$ with $|q| < 1$, then $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ generates a Schottky group.

A Schottky $\Gamma \leq \mathrm{PGL}_2(K)$ acts on $\mathbb{P}^1(\mathbb{C}_p)$ via fractional linear transformations.

Schottky groups

Definition. A Schottky group over K is a discrete, finitely generated and free subgroup $\Gamma \leq \mathrm{PGL}_2(K)$.

Example. If $q \in K^\times$ with $|q| < 1$, then $\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ generates a Schottky group.

A Schottky $\Gamma \leq \mathrm{PGL}_2(K)$ acts on $\mathbb{P}^1(\mathbb{C}_p)$ via fractional linear transformations. Let $\Omega_\Gamma \neq \varnothing$ be the largest subset of $\mathbb{P}^1(\mathbb{C}_p)$ on which Γ acts discontinuously. Then Ω_Γ/Γ is a rigid analytic space.

(Here $\mathbb{P}^1(\mathbb{C}_p) \setminus \Omega_\Gamma$ is the set of limit points of Γ: the set of all $x \in \mathbb{P}^1(\mathbb{C}_p)$ such that for some $y \in \mathbb{P}^1(\mathbb{C}_p)$, there is a sequence $\{\gamma_n\}_n$ of distinct $\gamma_n \in \Gamma$ such that $\gamma_n(y) \to x$.)

Mumford Curves

Theorem (Mumford, '72):

(a) For a Schottky $\Gamma \leq \mathrm{PGL}_2(K)$ there is an analytic isomorphism

$$\Omega_\Gamma/\Gamma \simeq C^{\mathrm{an}},$$

where C/K is a nice curve with split degenerate reduction.

Mumford Curves

Theorem (Mumford, '72):

(a) For a Schottky $\Gamma \leq \mathrm{PGL}_2(K)$ there is an analytic isomorphism

$$\Omega_\Gamma/\Gamma \simeq C^{\mathrm{an}},$$

where C/K is a nice curve with split degenerate reduction.

(b) The map $\Gamma \mapsto \Omega_\Gamma/\Gamma$ induces a bijection

$$\left\{\begin{array}{c}\text{conjugacy classes}\\ \text{of Schottky}\\ \Gamma \leq \mathrm{PGL}_2(K)\\ \text{of rank } g \geq 2\end{array}\right\} \to \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{nice curves } C/K \text{ of}\\ \text{genus } g \geq 2 \text{ with split}\\ \text{degenerate reduction}\end{array}\right\}.$$

We call curves with split degenerate reduction Mumford curves.

Theta functions

Let Γ be a Schottky group over K of rank g. Set

$$\Theta(a, b; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}, \quad a, b, z \in \Omega := \Omega_\Gamma .$$

Theta functions

Let Γ be a Schottky group over K of rank g. Set

$$\Theta(a, b; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}, \quad a, b, z \in \Omega := \Omega_\Gamma .$$

For fixed $a, b \in \Omega$, the function $z \mapsto \Theta(a, b; z)$ is meromorphic

Theta functions

Let Γ be a Schottky group over K of rank g. Set

$$\Theta(a, b; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}, \quad a, b, z \in \Omega := \Omega_\Gamma .$$

For fixed $a, b \in \Omega$, the function $z \mapsto \Theta(a, b; z)$ is meromorphic and automorphic: for all $\gamma \in \Gamma$ there is $c(a, b, \gamma) \in K^\times$ such that

$$\Theta(a, b; z) = c(a, b, \gamma) \cdot \Theta(a, b; \gamma(z)) \text{ for all } z \in \Omega .$$

Theta functions

Let Γ be a Schottky group over K of rank g. Set

$$\Theta(a, b; z) := \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}, \quad a, b, z \in \Omega := \Omega_\Gamma .$$

For fixed $a, b \in \Omega$, the function $z \mapsto \Theta(a, b; z)$ is meromorphic and automorphic: for all $\gamma \in \Gamma$ there is $c(a, b, \gamma) \in K^\times$ such that

$$\Theta(a, b; z) = c(a, b, \gamma) \cdot \Theta(a, b; \gamma(z)) \text{ for all } z \in \Omega .$$

Fix generators $\gamma_1, \ldots, \gamma_g$ of Γ and let $\Lambda \subset (K^\times)^g$ be spanned by

$$(c(a, \gamma_i(a), \gamma_j))_{j \in \{1,\ldots,g\}} \in (K^\times)^g ; \quad i \in \{1, \ldots, g\}, \ a \in \Gamma.$$

Theorem (Manin–Drinfeld '72). Let C/K be the Mumford curve corresponding to Γ. Then there is an analytic isomorphism

$$\mathrm{Jac}_C(K) \simeq (K^\times)^g/\Lambda.$$

Computing theta functions

Idea (Morrison-Ren '15.) Given generators of a Schottky group Γ, can approximate

$$\Theta(a, b; z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)}$$

Computing theta functions

Idea (Morrison-Ren '15.) Given generators of a Schottky group Γ, can approximate

$$\Theta(a, b; z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)} = \prod_{m=0}^{\infty} \Theta_m(a, b; z)\,,$$

where

- $\Theta_m(a, b; z) = \prod_{\gamma \in \Gamma_m} \frac{z-\gamma(a)}{z-\gamma(b)}$,
- $\Gamma_m = \{\text{reduced words of length} \leq m \text{ in the generators}\}$.

Computing theta functions

Idea (Morrison-Ren '15.) Given generators of a Schottky group Γ, can approximate

$$\Theta(a, b; z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma(a)}{z - \gamma(b)} = \prod_{m=0}^{\infty} \Theta_m(a, b; z)\,,$$

where

- $\Theta_m(a, b; z) = \prod_{\gamma \in \Gamma_m} \frac{z-\gamma(a)}{z-\gamma(b)}$,
- $\Gamma_m = \{\text{reduced words of length} \leq m \text{ in the generators}\}$.

Kaya–Masdeu–M.–van der Put: Sage-implementation

- Get an exponential time algorithm. Convergence rate depends on position of a, b, z and the fixed points of the generators.
- To certify correctness, need to tweak error bounds due to Morrison–Ren.

Computing theta functions

Idea (Morrison-Ren '15.) Given generators of a Schottky group Γ, can approximate

$$\Theta(a,b;z) = \prod_{\gamma\in\Gamma} \frac{z-\gamma(a)}{z-\gamma(b)} = \prod_{m=0}^{\infty} \Theta_m(a,b;z)\,,$$

where

- $\Theta_m(a,b;z) = \prod_{\gamma\in\Gamma_m} \frac{z-\gamma(a)}{z-\gamma(b)}$,
- $\Gamma_m = \{\text{reduced words of length} \leq m \text{ in the generators}\}$.

Kaya–Masdeu–M.–van der Put: Sage-implementation

- Get an exponential time algorithm. Convergence rate depends on position of a, b, z and the fixed points of the generators.
- To certify correctness, need to tweak error bounds due to Morrison–Ren.

Masdeu–Xarles ('23). Polynomial time algorithm using overconvergent functions.

Algorithmic goals

Goals. Given a Mumford curve C/K, we want algorithms to

(1) compute Γ such that $C^{\mathrm{an}} \simeq \Omega_\Gamma/\Gamma$;

(2) lift a given $P \in C(K)$ to Ω_Γ.

Algorithmic goals

Goals. Given a Mumford curve C/K, we want algorithms to

(1) compute Γ such that $C^{\mathrm{an}} \simeq \Omega_\Gamma/\Gamma$;

(2) lift a given $P \in C(K)$ to Ω_Γ.

Kaya–Masdeu–M.–van der Put: Algorithms and Sage-implementations for these when C is hyperelliptic.

Some applications.

- p-adic uniformization of $\mathrm{Jac}_C(K)$
- computing isogenies (Kadziela '07)
- local p-adic height pairing on C, useful for
  - quadratic Chabauty
  - gathering numerical evidence for (and formulating!) a p-adic version of the Birch and Swinnerton–Dyer conjecture for split degenerate abelian varieties of GL_2-type

Hyperelliptic Mumford curves

Theorem (van der Put, '78). Given a genus g Mumford curve

$$C : y^2 = f(x) = \prod_i (x - r_i), \quad \mathcal{R} := \{r_i\} \subset K \,,$$

Hyperelliptic Mumford curves

Theorem (van der Put, '78). Given a genus g Mumford curve

$$C\colon y^2 = f(x) = \prod_i (x - r_i), \quad \mathcal{R} := \{r_i\} \subset K\,,$$

there are $s_0, \ldots, s_g \in \mathrm{PGL}_2(K)$ of order 2 such that

- the free product $\Gamma' := \langle s_0 \rangle * \cdots * \langle s_g \rangle \leq \mathrm{PGL}_2(K)$ is discrete;

Hyperelliptic Mumford curves

Theorem (van der Put, '78). Given a genus g Mumford curve

$$C\colon y^2 = f(x) = \prod_i (x - r_i), \quad \mathcal{R} := \{r_i\} \subset K\,,$$

there are $s_0, \ldots, s_g \in \mathrm{PGL}_2(K)$ of order 2 such that

- the free product $\Gamma' := \langle s_0 \rangle * \cdots * \langle s_g \rangle \leq \mathrm{PGL}_2(K)$ is discrete;
- $\Gamma := \langle s_0 s_1, \ldots, s_0 s_g \rangle \leq \Gamma'$ is Schottky and $[\Gamma' : \Gamma] = 2$;
- we have $C^{\mathrm{an}} \simeq \Omega_\Gamma/\Gamma$;

Hyperelliptic Mumford curves

Theorem (van der Put, '78). Given a genus g Mumford curve

$$C\colon y^2 = f(x) = \prod_i (x - r_i), \quad \mathcal{R} := \{r_i\} \subset K\,,$$

there are $s_0, \ldots, s_g \in \mathrm{PGL}_2(K)$ of order 2 such that

- the free product $\Gamma' := \langle s_0 \rangle * \cdots * \langle s_g \rangle \leq \mathrm{PGL}_2(K)$ is discrete;
- $\Gamma := \langle s_0 s_1, \ldots, s_0 s_g \rangle \leq \Gamma'$ is Schottky and $[\Gamma' : \Gamma] = 2$;
- we have $C^{\mathrm{an}} \simeq \Omega_\Gamma/\Gamma$;
- for suitable fixed $a, b \in \Omega := \Omega_\Gamma$,

$$F(z) := \prod_{\gamma \in \Gamma'} \frac{z - \gamma(a)}{z - \gamma(b)}$$

  defines a meromorphic Γ'-invariant function on Ω, and induces an analytic isomorphism $\Omega/\Gamma' \simeq (\mathbb{P}^1)^{\mathrm{an}}$;
- $F^{-1}(\mathcal{R}) = \mathcal{FP}$, where $\mathcal{FP}$ is the set of fixed points of the s_i.

(1) Compute Γ' (and hence Γ)

The generators s_i are determined by $\mathcal{FP}$. We want to find $\mathcal{FP}$ using

$$F^{-1}(\mathcal{R}) = \mathcal{FP}, \quad F(z) = \prod_{\gamma \in \Gamma'} \frac{z - \gamma(a)}{z - \gamma(b)} = \prod_{m \geq 1} F_m(z).$$

(1) Compute Γ' (and hence Γ)

The generators s_i are determined by $\mathcal{FP}$. We want to find $\mathcal{FP}$ using

$$F^{-1}(\mathcal{R}) = \mathcal{FP}, \quad F(z) = \prod_{\gamma \in \Gamma'} \frac{z - \gamma(a)}{z - \gamma(b)} = \prod_{m \geq 1} F_m(z).$$

But F is defined in terms of Γ', which we don't know!

(1) Compute Γ' (and hence Γ)

The generators s_i are determined by $\mathcal{FP}$. We want to find $\mathcal{FP}$ using

$$F^{-1}(\mathcal{R}) = \mathcal{FP}, \quad F(z) = \prod_{\gamma \in \Gamma'} \frac{z - \gamma(a)}{z - \gamma(b)} = \prod_{m \geq 1} F_m(z).$$

But F is defined in terms of Γ', which we don't know!

Algorithm (Kadziela '07 + Kaya–Masdeu–M.–van der Put). For $\mathcal{R}$ and $\mathcal{FP}$ in suitable position, can simultaneously approximate both $\mathcal{FP}$ and F.

(1) Compute Γ' (and hence Γ)

The generators s_i are determined by $\mathcal{FP}$. We want to find $\mathcal{FP}$ using

$$F^{-1}(\mathcal{R}) = \mathcal{FP}, \quad F(z) = \prod_{\gamma \in \Gamma'} \frac{z - \gamma(a)}{z - \gamma(b)} = \prod_{m \geq 1} F_m(z).$$

But F is defined in terms of Γ', which we don't know!

Algorithm (Kadziela '07 + Kaya–Masdeu–M.–van der Put). For $\mathcal{R}$ and $\mathcal{FP}$ in suitable position, can simultaneously approximate both $\mathcal{FP}$ and F.

Idea. Find $\mathcal{FP}$ mod π^2, then continue digit-by-digit using

$$F(z) \bmod \pi^t = \prod_{m=0}^{t-2} F_m(z \bmod \pi^t) \text{ for } t \geq 3 \text{ and } z \in \mathcal{FP}\,.$$

(2) Lift points from C to Ω

Goal. Given $P = (x, y) \in C(K)$, compute a lift $z_P \in \Omega$ of P.

(2) Lift points from C to Ω

Goal. Given $P = (x, y) \in C(K)$, compute a lift $z_P \in \Omega$ of P.

Using Newton iteration, we can find $z_0 \in \Omega$ such that $F(z_0) = x$.
So for $\pi : \Omega \to \Omega_\Gamma/\Omega \to C$, we have $\pi(z_0) \in \{(x, y), (x, -y)\}$.

(2) Lift points from C to Ω

Goal. Given $P = (x, y) \in C(K)$, compute a lift $z_P \in \Omega$ of P.

Using Newton iteration, we can find $z_0 \in \Omega$ such that $F(z_0) = x$.
So for $\pi : \Omega \to \Omega_\Gamma/\Omega \to C$, we have $\pi(z_0) \in \{(x, y), (x, -y)\}$.

Theorem (Kaya–Masdeu–M.–van der Put). Let a_i, b_i be the fixed points of s_i. There is a constant c such that

$$H(z) := c \cdot \Theta(a, \gamma(a); z) \cdot \prod_{i=0}^{g} \Theta(a_i, b; z) \cdot \Theta(b_i, s_0(b); z)$$

satisfies:

(2) Lift points from C to Ω

Goal. Given $P = (x, y) \in C(K)$, compute a lift $z_P \in \Omega$ of P.

Using Newton iteration, we can find $z_0 \in \Omega$ such that $F(z_0) = x$.
So for $\pi : \Omega \to \Omega_\Gamma/\Omega \to C$, we have $\pi(z_0) \in \{(x, y), (x, -y)\}$.

Theorem (Kaya–Masdeu–M.–van der Put). Let a_i, b_i be the fixed points of s_i. There is a constant c such that

$$H(z) := c \cdot \Theta(a, \gamma(a); z) \cdot \prod_{i=0}^{g} \Theta(a_i, b; z) \cdot \Theta(b_i, s_0(b); z)$$

satisfies:

- H is Γ-invariant, but not Γ'-invariant

(2) Lift points from C to Ω

Goal. Given $P = (x, y) \in C(K)$, compute a lift $z_P \in \Omega$ of P.

Using Newton iteration, we can find $z_0 \in \Omega$ such that $F(z_0) = x$.
So for $\pi\colon \Omega \to \Omega_\Gamma/\Omega \to C$, we have $\pi(z_0) \in \{(x, y), (x, -y)\}$.

Theorem (Kaya–Masdeu–M.–van der Put). Let a_i, b_i be the fixed points of s_i. There is a constant c such that

$$H(z) := c \cdot \Theta(a, \gamma(a); z) \cdot \prod_{i=0}^{g} \Theta(a_i, b; z) \cdot \Theta(b_i, s_0(b); z)$$

satisfies:

- H is Γ-invariant, but not Γ'-invariant
- $H(z)^2 = \prod_{w \in \mathcal{FP}} (F(z) - F(w)) = f(F(z))$
- $\pi := (F, H)$ uniformizes C

Corollary: If $H(z_0) = y$, then $z_P = z_0$. Otherwise $z_P = s_0(z_0)$.

Schneider–Werner pairing

Assumption. The matrix $L := (\log_p c(a, \gamma_i(a), \gamma_j))_{i,j} \in K^{g \times g}$ is invertible.

Schneider–Werner pairing

Assumption. The matrix $L := (\log_p c(a, \gamma_i(a), \gamma_j))_{i,j} \in K^{g \times g}$ is invertible.

Definition (Werner, '96). For $P, Q, R, S \in C(K)$ such that $\{P, Q\} \cap \{R, S\} = \varnothing$, choose preimages $z_P, z_Q, z_R, z_S \in \Omega$ Then the following defines a local height pairing:

$$h_p^{\mathrm{SW}}(P - Q, R - S) := \log_p \frac{\Theta(z_P, z_Q; z_R)}{\Theta(z_P, z_Q; z_S)} - \sum_{i=1}^{g} \chi_i \log_p c(z_P, z_Q, \gamma_i)$$

Schneider–Werner pairing

Assumption. The matrix $L := (\log_p c(a, \gamma_i(a), \gamma_j))_{i,j} \in K^{g \times g}$ is invertible.

Definition (Werner, '96). For $P, Q, R, S \in C(K)$ such that $\{P, Q\} \cap \{R, S\} = \varnothing$, choose preimages $z_P, z_Q, z_R, z_S \in \Omega$ Then the following defines a local height pairing:

$$h_p^{\mathrm{SW}}(P - Q, R - S) := \log_p \frac{\Theta(z_P, z_Q; z_R)}{\Theta(z_P, z_Q; z_S)} - \sum_{i=1}^{g} \chi_i \log_p c(z_P, z_Q, \gamma_i)$$

Schneider–Werner pairing

Assumption. The matrix $L := (\log_p c(a, \gamma_i(a), \gamma_j))_{i,j} \in K^{g\times g}$ is invertible.

Definition (Werner, '96). For $P, Q, R, S \in C(K)$ such that $\{P, Q\} \cap \{R, S\} = \varnothing$, choose preimages $z_P, z_Q, z_R, z_S \in \Omega$ Then the following defines a local height pairing:

$$h_p^{\mathrm{SW}}(P - Q, R - S) := \log_p \frac{\Theta(z_P, z_Q; z_R)}{\Theta(z_P, z_Q; z_S)} - \sum_{i=1}^{g} \chi_i \log_p c(z_P, z_Q, \gamma_i)$$

where

$$(\chi_1, \ldots, \chi_g) := \left(\log_p c(z_R, z_S, \gamma_1), \ldots, \log_p c(z_R, z_S, \gamma_g)\right) \cdot L^{-1}.$$

Schneider–Werner pairing

Assumption. The matrix $L := (\log_p c(a, \gamma_i(a), \gamma_j))_{i,j} \in K^{g\times g}$ is invertible.

Definition (Werner, '96). For $P, Q, R, S \in C(K)$ such that $\{P, Q\} \cap \{R, S\} = \varnothing$, choose preimages $z_P, z_Q, z_R, z_S \in \Omega$ Then the following defines a local height pairing:

$$h_p^{\mathrm{SW}}(P - Q, R - S) := \log_p \frac{\Theta(z_P, z_Q; z_R)}{\Theta(z_P, z_Q; z_S)} - \sum_{i=1}^{g} \chi_i \log_p c(z_P, z_Q, \gamma_i)$$

where

$$(\chi_1, \ldots, \chi_g) := \left(\log_p c(z_R, z_S, \gamma_1), \ldots, \log_p c(z_R, z_S, \gamma_g)\right) \cdot L^{-1}.$$

This is a special case of a pairing defined by Schneider, which features in p-adic BSD for split multiplicative elliptic curves.

Algorithm (Kaya–Masdeu–M.–van der Put). Compute h_p^{SW} for hyperelliptic Mumford curves using the above.

p-adic BSD

Let A_f be the abelian variety associated to a newform $f \in S_2(\Gamma_0(N))$ with good ordinary reduction at p.

Conjecture (Balakrishnan–M.–Stein '16). The Mordell-Weil rank r of $A_f/\mathbb{Q}$ equals $\operatorname{ord}_{s=1}(L_p(A_f, s))$ and

$$\frac{L_p^*(A_f, 1)}{\epsilon_p(A_f)} = \frac{\operatorname{Reg}_p(A_f/\mathbb{Q}) \cdot |Ш(A_f/\mathbb{Q})| \cdot \prod_v c_v(A_f)}{|A_f(\mathbb{Q})_{\mathrm{tors}}| \cdot |A_f^\vee(\mathbb{Q})_{\mathrm{tors}}|}.$$

This conjecture

- is equivalent to BSD in rank 0,
- reduces to Mazur-Tate-Teitelbaum if $g = 1$,
- is consistent with the main conjecture of Iwasawa theory for abelian varieties, via work of Perrin-Riou and Schneider.
- has been verified for various examples by Balakrishnan–M.–Stein ('16) and Gajović–M..
- should also work for nonsplit semistable reduction, but there's no data supporting it yet.

Let $E = E_f$ be an elliptic curve associated to a newform $f \in S_2(\Gamma_0(N))$ with split multiplicative reduction at p.

Conjecture (Mazur–Tate–Teitelbaum '86). The Mordell-Weil rank r of $E_f/\mathbb{Q}$ equals $\mathrm{ord}_{s=1}(L_p(E_f, s)) - 1$ and

$$\frac{L_p^*(E_f, 1)}{\mathcal{L}_p} = \frac{\mathrm{Reg}_p^{\mathrm{SW}}(E_f/\mathbb{Q}) \cdot |Ш(E_f/\mathbb{Q})| \cdot \prod_v c_v(E_f)}{|E_f(\mathbb{Q})_{\mathrm{tors}}|^2},$$

where $\mathcal{L}_p = \log_p(q_E)/\mathrm{ord}_p(q_E)$ and q_E is the Tate period of E.

How does this conjecture extend to dimension > 1?

Good position

Let $\gamma_1, \ldots, \gamma_g$ be generators of a Schottky group. We call $F \subset \mathbb{P}^1(\mathbb{C}_p)$ a good fundamental domain with respect to $\gamma_1, \ldots, \gamma_g$ if

$$F = \mathbb{P}^1(\mathbb{C}_p) \setminus (B_1 \cup B_{-1} \cup B_2 \cup B_{-2} \cup \ldots \cup B_g \cup B_{-g}),$$

where the B_i are open balls such that the corresponding closed balls $\overline{B_i}$ are disjoint and for all $i \in \{1, \ldots, g\}$,

$$\gamma_i(\mathbb{P}^1 \setminus B_{-i}) = \overline{B_i}$$
$$\gamma_i^{-1}(\mathbb{P}^1 \setminus \overline{B_i}) = \overline{B_{-i}}$$

We say that the fixed points of the γ_i are in good position if they have a good fundamental domain.