Computing endomorphism rings of supersingular elliptic curves

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AGCCT 2023

joint work with Fuselier, Iezzi, Kozek, Namoijam

Computing supersingular endomorphism rings

Problem

Given a prime p and a supersingular elliptic curve $E/\mathbb{F}_{p^2}$, compute $\mathrm{End}(E)$, i.e. give a basis and multiplication table for $\mathrm{End}(E) \otimes \mathbb{Q}$, along with four elements of $\mathrm{End}(E) \otimes \mathbb{Q}$ generating $\mathrm{End}(E)$ as a $\mathbb{Z}$-module.

Computing End(E): ordinary vs supersingular

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  - $\mathrm{End}(E)$ is contained in the unique maximal order in the imaginary quadratic field $\mathrm{End}^0(E) \otimes \mathbb{Q}$, so computing $[\mathrm{End}(E) : \mathbb{Z}[\pi_E]]$ determines $\mathrm{End}(E)$.

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- If E is supersingular:
  - Computing one endomorphism $\alpha \in \mathrm{End}(E) \setminus \mathbb{Z}$ is hard...
  - ... and given an order $\mathcal{O} \subseteq \mathrm{End}(E)$ of finite index, there can be exponentially many (in the size of $\mathcal{O}$!) pairwise nonisomorphic maximal orders containing $\mathcal{O}$, any one of which could be $\mathrm{End}(E)$.

Prior algorithms for computing the endomorphism ring

- If $E/\mathbb{F}_q$ is ordinary: Bisson-Sutherland (2011), Robert (2022) give a subexponential, in $\log q$, algorithm for computing $\mathrm{End}(E)$.
- If E is supersingular: Kohel gives a $O(p^{1+\epsilon})$ algorithm to compute a suborder of $\mathrm{End}(E)$.
- Eisenträger-Hallgren-Leonardi-M.-Park: $\tilde{O}(p^{1/2})$ algorithm, assuming GRH and heuristics, which computes a Bass suborder of $\mathrm{End}(E)$.
- Today: (joint with Fuselier-Iezzi-Kozek-Namoijam): compute $\mathrm{End}(E)$ using *inseparable endomorphisms* in time $\tilde{O}(p^{1/2})$ assuming only GRH.

Computing End(E) with inseparable endomorphisms

- Every inseparable endomorphism $\alpha \in \mathrm{End}(E)$ factors as

$$\alpha = \pi \circ \phi,$$

where $\phi\colon E \to E^{(p)}$ is an isogeny and $\pi\colon E \to E^{(p)}$ is the p-power Frobenius isogeny.

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- The order

$$\mathbb{Z} + P \subseteq \mathrm{End}(E)$$

has index p in $\mathrm{End}(E)$, and $\mathrm{End}(E)$ is the unique maximal order containing $\mathbb{Z} + P$.

Supersingular isogeny graphs

Let $p \neq \ell$ be primes. Let $G(p, \ell)$ be the directed graph whose

- vertices are a complete set of representatives of the isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}_p}$
- (directed) edges are ℓ-isogenies $E_i \to E_j$

$\sigma_p = (x \mapsto x^p) \in Gal(\mathbb{F}_{p^2}/\mathbb{F}_p)$ acts on $G(p, \ell)$ as an automorphism:

- map E_i to $E_i^{(p)}$,
- and map $\phi\colon E_i \to E_j$ to $\phi^{(p)}\colon E_i^{(p)} \to E_j^{(p)}$

Curves close to their Galois conjugate

Cyclic, inseparable endomorphisms with trace zero, i.e. embeddings

$$\mathbb{Z}\left[\sqrt{-dp}\right] \hookrightarrow \mathrm{End}(E).$$

are used in Charles-Goren-Lauter (2011), EHLMP (2020), Chenu-Smith 2022.

Definition (Chenu-Smith)

Let $\epsilon = \pm 1$ and let d be a squarefree positive integer. A *supersingular* (d, ϵ)*-structure* is a pair (E, ψ) where $E/\mathbb{F}_{p^2}$ is a supersingular elliptic curve and $\psi\colon E \to E^{(p)}$ satisfies $\psi^{(p)} = \epsilon\widehat{\psi}$.

Chenu and Smith show $\mathbb{Z}[\pi\psi] \simeq \mathbb{Z}[\sqrt{-dp}]$ if d is squarefree and (E, ψ) is a (d, ϵ)-structure.

Inseparable reflections

Definition

Let p be a prime, and let d_1, d be integers with d square-free and with d_1, d, p all pairwise coprime. An **inseparable reflection of degree** dpd_1^2 of a supersingular elliptic curve $E/\mathbb{F}_{p^2}$ is an endomorphism

$$\alpha := \pi \circ \widehat{\phi^{(p)}} \circ \psi \circ \phi$$

such that

1. $\phi\colon E \to E'$ is a cyclic d_1-isogeny,
2. (E', ψ) is a (d, ϵ)-structure,
3. and ϕ does not factor nontrivially through an isogeny $\phi'\colon E \to E''$ such that E'' has a (d, ϵ)-structure.

In our algorithms, we use $d = 1$ or 2 and $d_1 = \ell^t$ for small prime ℓ.

Orders generated by inseparable reflections

Theorem (Fuselier-Iezzi-Kozek-M.-Namoijam)

Let $E/\mathbb{F}_{p^2}$ be supersingular, $p > 3$. For $i = 1, 2$ let d, p, d_i be pairwise coprime, and let

$$\alpha_i = \pi\widehat{\phi_i^{(p)}}\psi\phi_i$$

be inseparable reflections of degree dpd_i^2. Then:

1. $$\mathbb{Z}[\alpha_i] \simeq \mathbb{Z}[\deg(\phi_i)\sqrt{-p}];$$
2. *If $-dp \not\equiv 1 \pmod 4$, then*

$$\Lambda := \mathbb{Z} + \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_1\alpha_2$$

is a Bass suborder of $\mathrm{End}(E)$ with index $\mathrm{disc}(\frac{\alpha_1\alpha_2}{p})$ in $\mathbb{Z} + P$.

Computing End(E) from a Bass suborder

Given $E/\mathbb{F}_{p^2}$ supersingular:

1. Take random walks of length $O(\log p)$ beginning at E in the 3-isogeny graph until finding a curve E_1 with a $(2, \epsilon)$-structure (E_1, ψ) yielding one inseparable reflection

$$\alpha_1 = \pi \widehat{\phi_1^{(p)}} \psi_1 \phi_1$$

   of degree $2p3^{2e_3}$
2. Repeat the above in the 5-isogeny graph to obtain inseparable reflection α_2 of degree $2p5^{2e_5}$
3. Let $\Lambda = \langle 1, \alpha_1, \alpha_2, \alpha_1\alpha_2 \rangle$.
4. Enumerate maximal orders $\mathcal{O} \subseteq \Lambda$ until $\mathcal{O} = \mathrm{End}(E)$

First two steps take expected $\tilde{O}(\sqrt{p})$ time, assuming GRH. We prove that the final step takes $O(p^\epsilon)$ time for every $\epsilon > 0$, using that Λ is Bass.

Heuristics (re)moved

- With two calls to an algorithm for computing an inseparable reflection, we provably compute a generating set for a Bass suborder of $\mathrm{End}(E)$.
- What is the expected number of calls to such an algorithm until finding a generating set of $\mathbb{Z} + P$?

Heuristic algorithm

Suppose α_{ij} are inseparable reflections for $1 \leq i, j \leq 2$

$$\Lambda_1 = \langle \alpha_{11}, \alpha_{12} \rangle, \quad \Lambda_2 = \langle \alpha_{21}, \alpha_{22} \rangle, \quad \Lambda = \langle \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \rangle$$

$$\rho_i = \frac{1}{p} \alpha_{i1} \alpha_{i2}, \quad D_i = -\operatorname{disc}(\rho_i)$$

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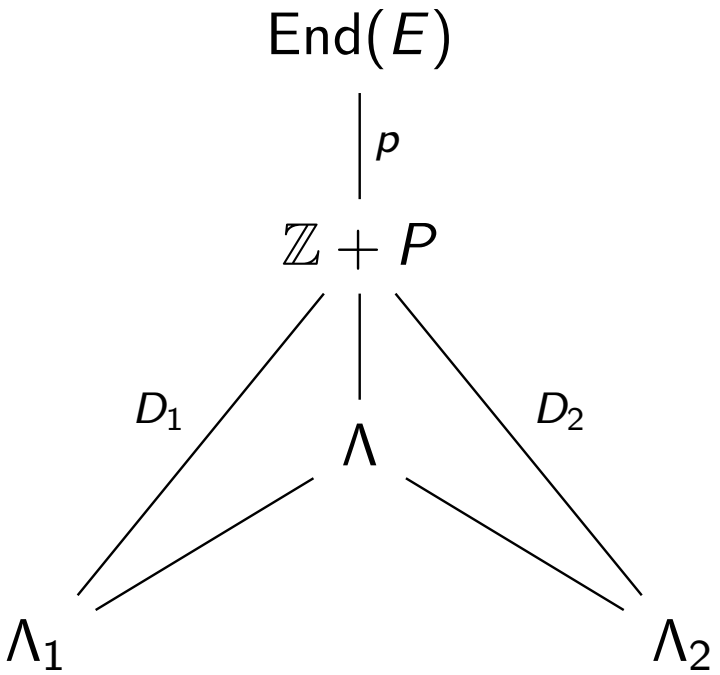


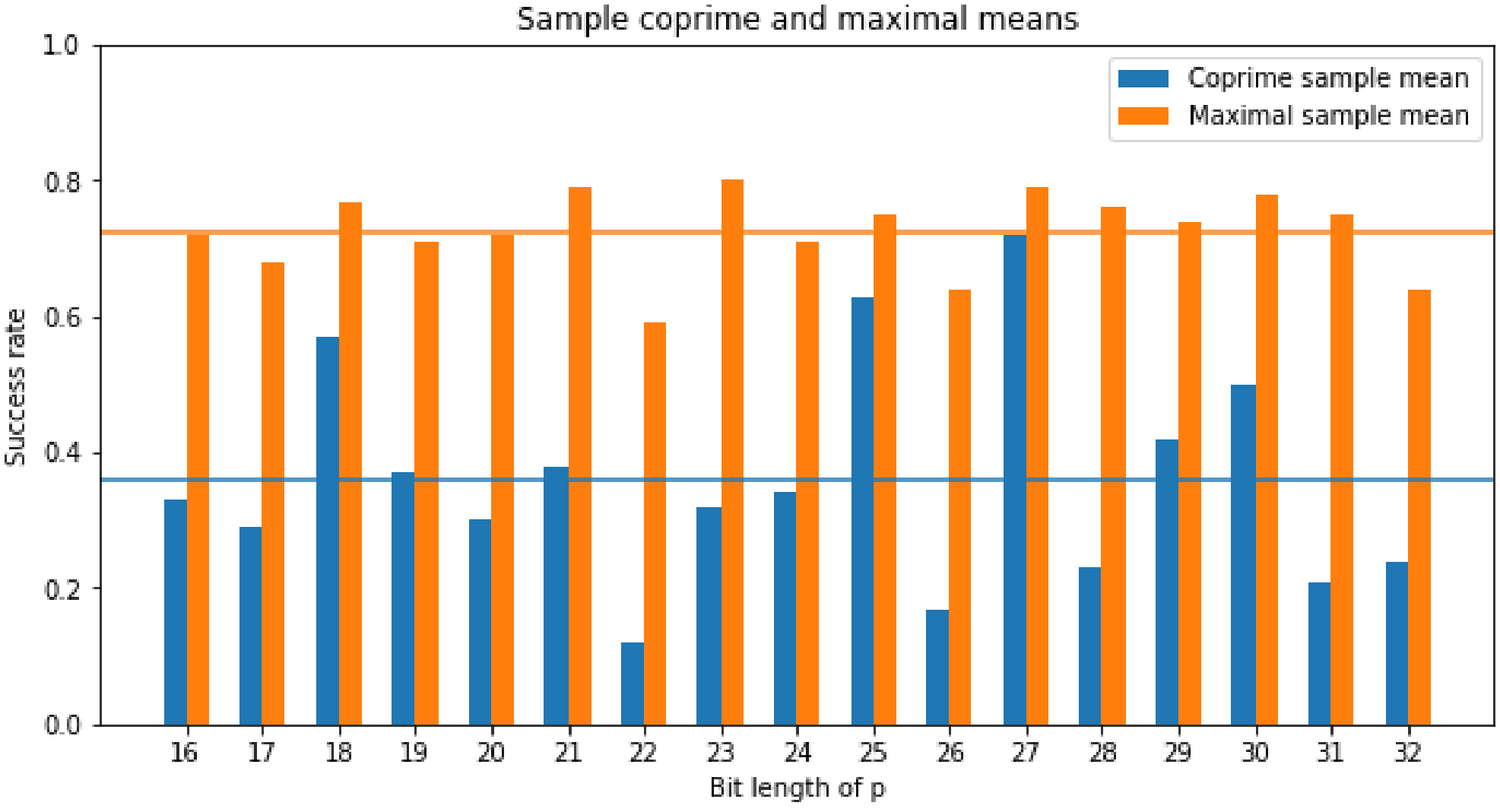
- $[\mathbb{Z} + P : \Lambda]$ divides $[\mathbb{Z} + P : \Lambda_i] = D_i$.
- If $\gcd(D_1, D_2) = 1$, then $\Lambda = \mathbb{Z} + P$.
- Heuristic: for some constant $c > 0$,

$$\Pr[\gcd(\operatorname{disc} \rho_1, \operatorname{disc} \rho_2) = 1] > c$$

Experiment 1

For $b \in [16, 32]$:

- Pick a prime of bit length b. Pick 100 pseudorandom supersingular $E/\mathbb{F}_{p^2}$.
- For each E: compute four inseparable reflections α_{i1}, α_{i2} and let $\rho_i = \alpha_{i1}\alpha_{i2}/p$ for $i = 1, 2$.
- Test if:
  1. $\gcd(\operatorname{disc} \rho_1, \operatorname{disc} \rho_2) = 1$, and
  2. $\langle \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \rangle = \mathbb{Z} + P$


Sample coprime and maximal means
Coprime sample mean
Maximal sample mean
Success rate
1.0
0.8
0.6
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0.2
0.0
16
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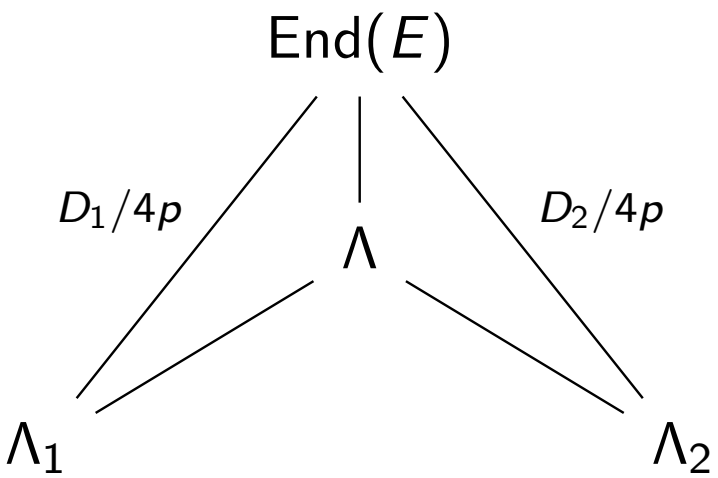
31
32
Bit length of p

What is the truth?

Suppose α_{ij} are arbitrary in $\mathrm{End}(E)$ (of degree up to some bound X) for $1 \le i, j \le 2$

$$\Lambda_1 = \langle \alpha_{11}, \alpha_{12} \rangle, \quad \Lambda_2 = \langle \alpha_{21}, \alpha_{22} \rangle, \quad \Lambda = \langle \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \rangle$$

$$\rho_i = (\mathrm{Trd}\,\alpha_1)\alpha_2 + (\mathrm{Trd}\,\alpha_2)\alpha_1 - 2\alpha_1\alpha_2, \quad \mathrm{disc}(\Lambda_i) = \left(\frac{1}{4}\,\mathrm{disc}\,\rho_i\right)^2.$$



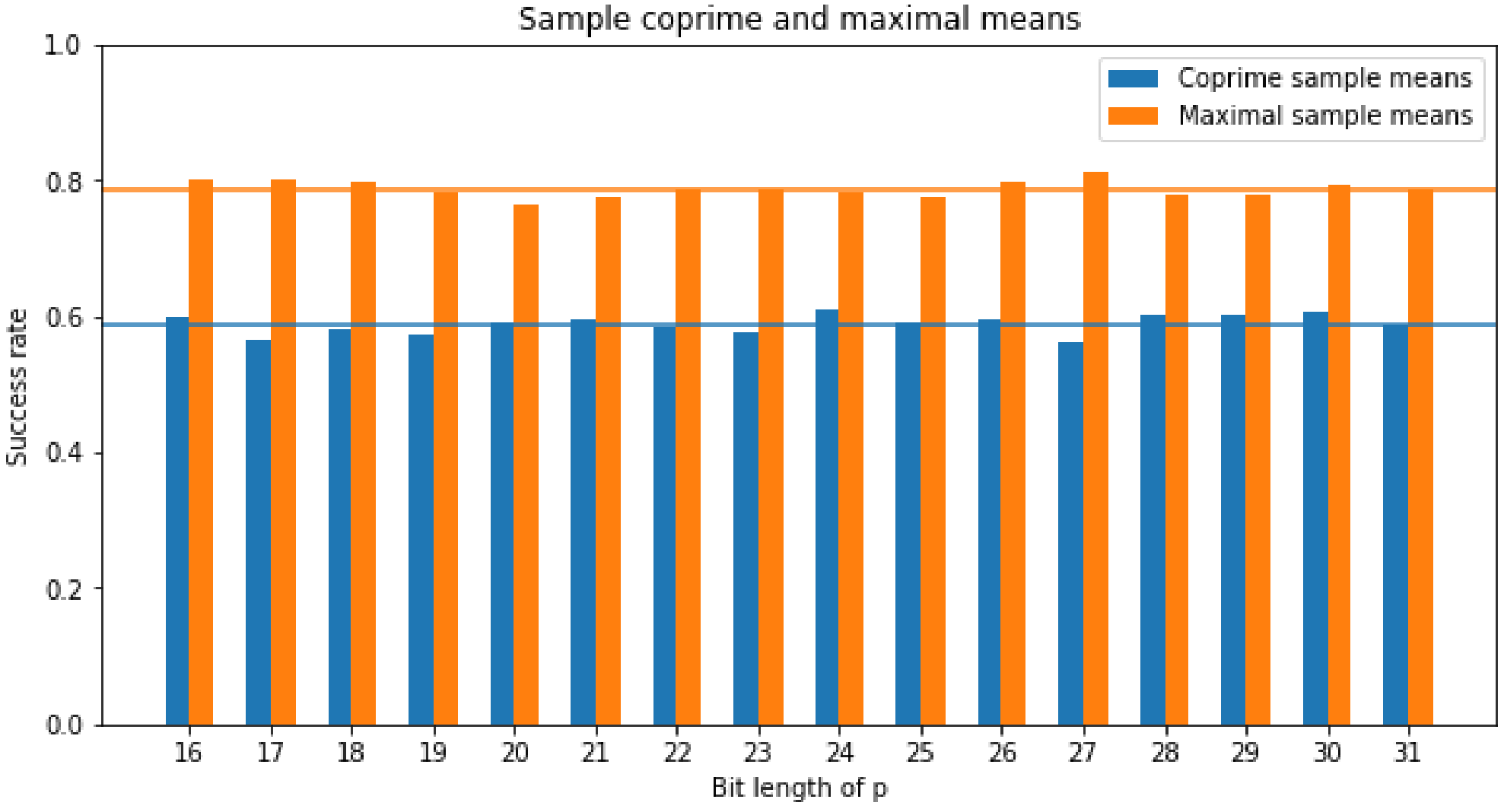
- $[\mathrm{End}(E) : \Lambda]$ divides $[\mathrm{End}(E) : \Lambda_i] = \frac{1}{4p} D_i$.
- If $\gcd(D_1, D_2) = 4p$, then $\Lambda = \mathrm{End}(E)$.
- Heuristic: for some constant $c > 0$,

$$\Pr[\gcd(\mathrm{disc}\,\rho_1, \mathrm{disc}\,\rho_2) = 4p] > c$$

Experiment 2

For $b \in [16, 32]$:

- Pick a prime of bit length b. Pick 100 pseudorandom maximal orders $\mathcal{O}$ in the quaternion algebra ramified at p, ∞.
- For each $\mathcal{O}$: compute four random elements $\alpha_{i1}, \alpha_{i2} \in \mathcal{O}$ and $\rho_i = (\mathrm{Trd}\,\alpha_1)\alpha_2 + (\mathrm{Trd}\,\alpha_2)\alpha_1 - 2\alpha_1\alpha_2$ for $i = 1, 2$.
- Test if:
  1. $\gcd(\mathrm{disc}\,\rho_1, \mathrm{disc}\,\rho_2) = 4p$, and
  2. $\langle \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22} \rangle = \mathcal{O}$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Thank you!

From a lattice to a quaternion algebra: Schoof's algorithm

- Suppose we have a finitely generated subgroup

$$\Lambda := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3 + \mathbb{Z}\alpha_4 \subseteq \mathbb{Z} + P.$$

  How can we tell if $\Lambda = \mathbb{Z} + P$?

- Compute $G = (\mathit{trace}(\alpha_i\alpha_j))_{1\leq i,j\leq 4}$.

$$\det(G) = p^4 \iff \Lambda = \mathbb{Z} + P.$$

  Here, $\mathit{trace}(\alpha)$ is the integer t such that $[t] = \alpha + \widehat{\alpha}$.

- We can compute $\mathit{trace}(\alpha)$ efficiently with Schoof's algorithm (efficiently: polynomial in the length of a compact representation of the endomorphism α).

Computing $\mathrm{End}(E) \otimes \mathbb{Q}$ with inseparable endomorphisms

Compute $\mathrm{End}(E) \otimes \mathbb{Q}$...

1. ...as a **vector space**: compute three isogenies $\phi_i \colon E \to E^{(p)}$ to get three (linearly independent, over $\mathbb{Q}$) endomorphisms $\gamma_i := \pi \circ \phi_i$ generating a lattice Λ. Then $\gamma_0 = 1, \gamma_1, \gamma_2, \gamma_3$ span $\mathrm{End}(E) \otimes \mathbb{Q}$

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Next, compute more $\pi\gamma$ until we've generated $\mathbb{Z} + P$ and compute $\mathrm{End}(E)$ with Voight's algorithm.