Cohen-Macaulay type of endomorphism rings of abelian varieties over finite fields

Stefano Marseglia

Utrecht University

AGC^{2}T 2023 - 6 June 2023.

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...or...

when an abelian variety met Bruns-Herzog's book.

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$$\pi_A \curvearrowright T_\ell A = \varprojlim A[l^n] \simeq_{\mathbb{Z}_\ell} \mathbb{Z}_\ell^{2g}, \quad \text{for a prime } \ell \neq p.$$

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- Ex. $C/\mathbb{F}_3 : Y^2 = X^6 + X + 1 \rightsquigarrow h_{\mathrm{Jac}(C)}(x) = x^4 + 3x^3 + 6x^2 + 9x + 9$.

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  - $\mathbb{Z}[\pi, q/\pi] \subseteq \mathrm{End}_{\mathbb{F}_q}(A) \subseteq \mathcal{O}_K$ are orders in K (an **order** R is a subring $R \subset K$ such that $R \simeq_{\mathbb{Z}} \mathbb{Z}^{\dim_{\mathbb{Q}} K}$).

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- If I is invertible, then $(I : I) = R$.

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- Ex: pick a prime $\ell \in \mathbb{Z}$. Then $\mathrm{type}_{\ell\mathcal{O}_K}(\mathbb{Z} + \ell\mathcal{O}_K) = \dim_{\mathbb{Q}} K - 1$.

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- By the Lemma:
  1. $\exists x \in I$ such that $m((x + \mathfrak{p}I) \otimes V) = \frac{xI^t + \mathfrak{p}R^t}{\mathfrak{p}R^t}$ equals W.
     By Nakayama's lemma: $I^t_{\mathfrak{p}} \simeq R^t_{\mathfrak{p}} \iff R_{\mathfrak{p}} \simeq I_{\mathfrak{p}}, \ldots$

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  2. ...or, $\exists y \in I^t$ such that $U \otimes m(U \otimes (y + \mathfrak{p})I^t) = W$ implying $I^t_\mathfrak{p} \simeq R_\mathfrak{p} \iff I_\mathfrak{p} \simeq R^t_\mathfrak{p}$.

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Fix a squarefree characteristic poly $h(x)$ of Frobenius π over $\mathbb{F}_q$.
Put $K = \mathbb{Q}[x]/h = \mathbb{Q}[\pi]$.
Let $\mathscr{I}_h$ be the corresponding isogeny class.

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References: Deligne, Howe, Centeleghe-Stix, Bergström-Karemaker-M.

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Let R be an order in K and $\mathfrak{p}$ a maximal ideal of R (possibly but not necessarily above p). Assume:

$$R = \overline{R}, \quad \mathfrak{p} = \overline{\mathfrak{p}}, \quad \textit{and} \quad \mathrm{type}_{\mathfrak{p}}(R) = 2.$$

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- *Abelian varieties over finite fields and their groups of rational points*
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Thank you!