Practical Encryption from Isogenies between Elliptic Products

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SIDH Attacks in a Nutshell

Given $(\varphi(P), \varphi(Q))$ under a secret isogeny $\varphi\colon E \to E'$ of degree d, where $\langle P, Q\rangle = E[n]$, it is possible to recover φ.

FESTA = Party in Italian and Basque (and many other languages)

Trapdoor Function

Triple of algorithms $(\mathsf{KeyGen}, f, f^{-1})$

- $\mathsf{KeyGen}(\lambda) \xrightarrow{\$} (\mathsf{sk}, \mathsf{pk})$
- $f(\mathsf{pk}, x) \to y$
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One-way
Given pk and y, finding x st $f(\mathsf{pk}, x) = y$ is hard.

FESTA Trapdoor

$$\binom{P_b}{Q_b}$$

$$E_0$$

FESTA Trapdoor

$\mathsf{sk} = (\mathbf{A}, \varphi_A)$

FESTA Trapdoor

$$\begin{pmatrix} P_b \\ Q_b \end{pmatrix} \qquad \begin{pmatrix} R_A \\ S_A \end{pmatrix} = \mathbf{A} \begin{pmatrix} \varphi_A(P_b) \\ \varphi_A(Q_b) \end{pmatrix}$$

$$E_0 \xrightarrow{\varphi_A} E_A$$

$$\boxed{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} [\alpha]P_1 + [\beta]P_2 \\ [\gamma]P_1 + [\delta]P_2 \end{pmatrix}}$$

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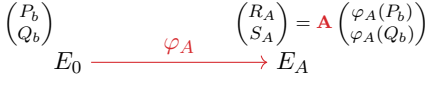

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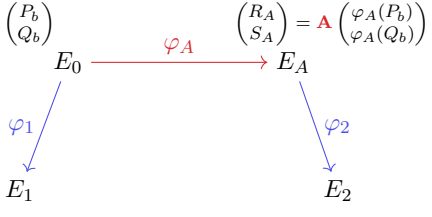

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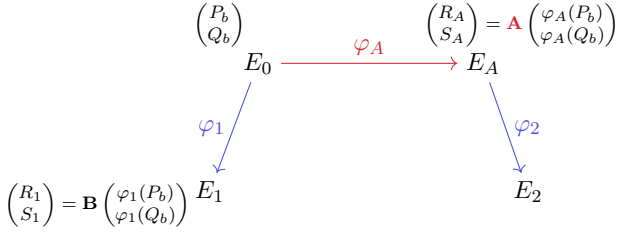

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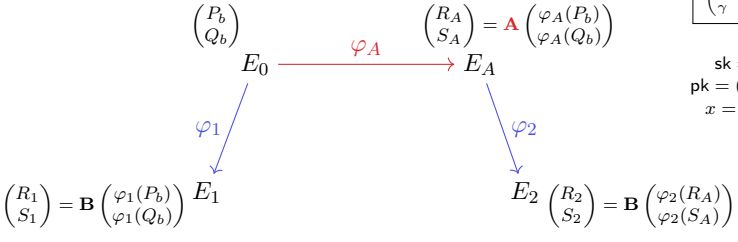

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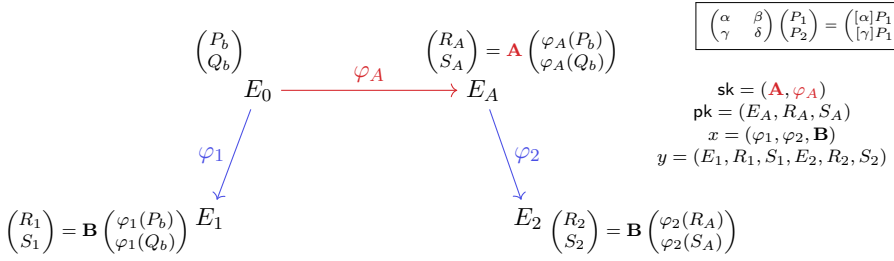

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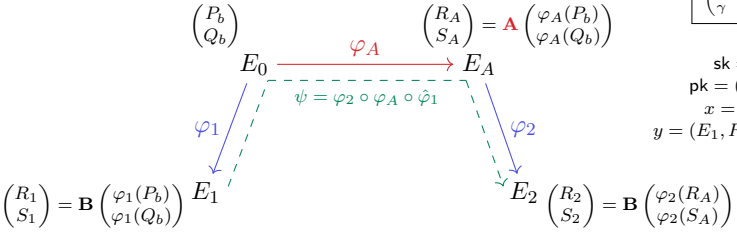

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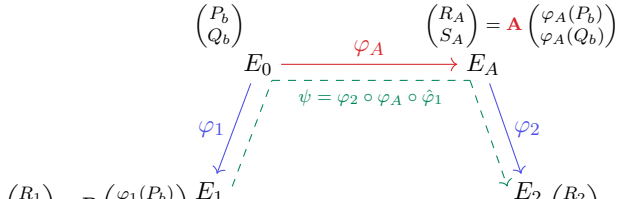

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$$\mathbf{B} \cdot \mathbf{A} \cdot \mathbf{B}^{-1} \begin{pmatrix} \psi(R_1) \\ \psi(S_1) \end{pmatrix} = \deg(\varphi_1) \begin{pmatrix} R_2 \\ S_2 \end{pmatrix}$$

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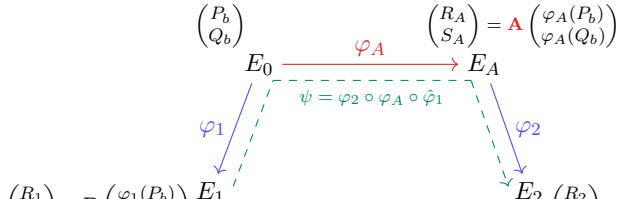

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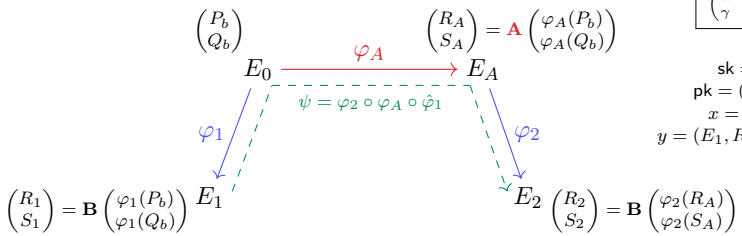

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$$\mathbf{A} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{or} \quad \mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

Kani's Criterion

- $\varphi_{N_1} \colon E_0 \to E_1$, $\varphi_{N_2} \colon E_0 \to E_2$ s.t. $\gcd(N_1, N_2) = 1$
- $K := \langle([N_2]\varphi_{N_1}(P), [N_1]\varphi_{N_2}(P)), ([N_2]\varphi_{N_1}(Q), [N_1]\varphi_{N_2}(Q))\rangle$, where $\langle P, Q\rangle = E_0[N_1 + N_2]$

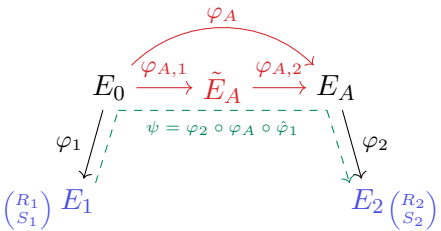
Theorem

The $(N_1 + N_2, N_1 + N_2)$-polarised isogeny Φ with kernel K has matrix form

$$\begin{pmatrix} \widehat{\varphi}_{N_1} & -\widehat{\varphi}_{N_2} \\ g_{N_2} & \widehat{g}_{N_1} \end{pmatrix},$$

where g_{N_i} are N_i-isogenies such that $\varphi_{N_2} \circ \widehat{\varphi}_{N_1} = g_{N_1} \circ g_{N_2}$.

Concrete Inversion



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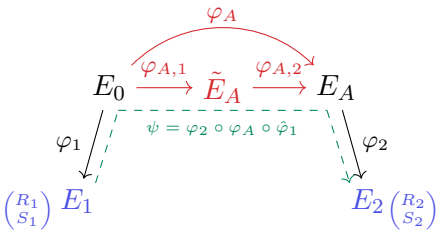

- $d_{A,i} = \deg(\varphi_{A,i})$, $d_i = \deg(\varphi_i)$
- $m_1^2 d_{A,1} d_1 + m_2^2 d_{A,2} d_2 = 2^b$
- $\varphi_{N_1} = [m_1] \circ \varphi_1 \circ \widehat{\varphi}_{A,1}$,
  $\varphi_{N_2} = [m_2] \circ \varphi_2 \circ \varphi_{A,2}$
- $d_i > 2^{2\lambda}$, $d_{A,1} \cdot d_{A,2} > 2^{2\lambda}$

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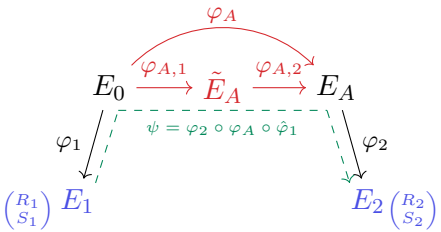

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$$K := \left\langle \begin{matrix} ([m_2 d_{A,2} d_2] R_1, [d_1 m_1] R_2'), \\ ([m_2 d_{A,2} d_2] S_1, [d_1 m_1] S_2') \end{matrix} \right\rangle, \quad \begin{pmatrix} R_2' \\ S_2' \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} R_2 \\ S_2 \end{pmatrix}$$

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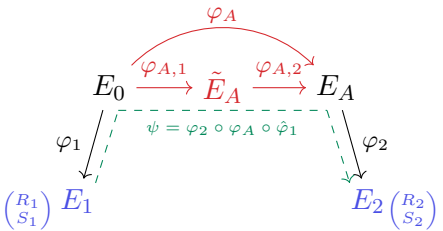

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$$\Phi = \begin{pmatrix} [m_1] \circ \varphi_{A,1} \circ \widehat{\varphi}_1 & -[m_2] \circ \widehat{\varphi}_{A,2} \circ \widehat{\varphi}_2 \\ [m_2] \circ g_{d_2 d_{A,2}} & [m_1] \circ \widehat{g}_{d_{A,1} d_1} \end{pmatrix}$$

Proof of Concept

https://github.com/FESTA-PKE/FESTA-SageMath

```
================================================================================
                              Running FESTA_128
================================================================================
================================================================================
                            Keygen took: 4.853 seconds
================================================================================
--------------------------------------------------------------------------------
                         Compressed public key: 561 bytes
--------------------------------------------------------------------------------
================================================================================
                            Encrypt took: 3.513 seconds
================================================================================
--------------------------------------------------------------------------------
                        Compressed ciphertext: 1122 bytes
--------------------------------------------------------------------------------
================================================================================
                            Decrypt took: 10.102 seconds
================================================================================
```

Thanks for your attention!

Questions?