Tori and surfaces violating a local-to-global principle for rationality

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The talk is based on arXiv:2305.03481.

Motivating example

In a recent preprint "A threefold violating a local-to-global principle for rationality" (see `arXiv:2304.09306`), Sarah Frei and Lena Ji exhibited a smooth, projective, $\mathbb{Q}$-unirational threefold X, a smooth intersection of two quadrics in $\mathbb{P}^5$, such that $X_v := X \times_{\mathbb{Q}} \mathbb{Q}_v$ is $\mathbb{Q}_v$-rational for all places v of $\mathbb{Q}$ but X is not $\mathbb{Q}$-rational (conditionally on the Birch and Swinnerton-Dyer conjecture for the Jacobian of a certain genus 2 curve).

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First result

Theorem 1. *For any global field k there exists a smooth, projective toric k-variety X for which the Brauer obstruction is absent, $X_v := X \times_k k_v$ is k_v-rational for all places v of k but X is not k-rational.*

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Second result

Theorem 2. *For any global field k of characteristic $\neq 2$ there exists a smooth, projective, k-unirational, geometrically rational k-surface X for which the Brauer obstruction is absent, $X_v := X \times_k k_v$ is k_v-rational for all places v of k but X is not k-rational.*

Meaning

A simple-minded meaning of Theorems 1 and 2 can be formulated as follows: even within a class of varieties where the Brauer obstruction is the only obstruction to the local-to-global principle for the existence of rational points (Hasse principle), this obstruction, even in a stronger, base change invariant form, may be insufficient for explaining counter-examples to the local-to-global principle for rationality.

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Almost in the same time (1983), rationality properties of algebraic tori were studied in my PhD thesis. It turns out that some examples considered there can be used for proving Theorem 1.

Some recollections on tori

- A k-torus T is an algebraic k-group such that $T \times_k \bar{k} \cong \mathbb{G}^d_{\mathrm{m},\bar{k}}$.

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- Many rationality properties of T can be expressed in terms of M.

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We have irreversible implications
$(\mathrm{i}) \Rightarrow (\mathrm{ii}) \Rightarrow (\mathrm{iii}) \Rightarrow (\mathrm{iv}) \cap (\mathrm{v}) = (\mathrm{vi}) \Rightarrow (\mathrm{iv})(\text{ or}(\mathrm{v}))$.

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Note that $H^1(k, F) = \mathrm{Br}(X)/\mathrm{Br}(k) = \mathrm{Br}_{\mathrm{nr}}(T)$ (the unramified Brauer group of T).

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We then have the implications $(\mathrm{i}) \Rightarrow (\mathrm{ii}) \Rightarrow (\mathrm{ii}') \Rightarrow (\mathrm{iii})$, all irreversible except possibly for the leftmost one, whose reversibility is a notoriously difficult long-standing problem (important in torus-based cryptography, Rubin and Silberberg, 2003).

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In case (iii), we sometimes say that the Brauer obstruction is absent.

Relation to the properties of modules

We have the following relations: T is stably rational (resp. retract rational, resp. Br-trivial) if and only if the module F in a flasque resolution of $M = \hat{T}$ is stably permutation (resp. invertible, resp. H-trivial).

Idea of an example

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How to guarantee that such a T is everywhere k_v-rational?

Shafarevich's theorem

Theorem. Let k be a global field, and let G be a finite solvable group. Then there exists a Galois field extension K/k with group G such that all decomposition groups G_v are cyclic.

An explicit example

Let $\Pi = (\mathbb{Z}/2\mathbb{Z})^3 = \langle \alpha, \beta, \gamma \rangle$. Choose a subgroup of Π of order 4, say, $\Pi_0 = \langle \alpha, \beta \rangle$. Let $I := \ker[\mathbb{Z}[\Pi_0] \to \mathbb{Z}]$ denote the augmentation ideal of $\mathbb{Z}[\Pi_0]$.

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Given a field k and a Galois extension L/k with group Π, let $L_0 = k(\sqrt{a}, \sqrt{b}) = L^{\gamma}$, $L_1 = k(\sqrt{c}) = L^{\Pi_0}$. The extensions L_0 and L_1 are linearly disjoint and $L = L_0 L_1 = k(\sqrt{a}, \sqrt{b}, \sqrt{c})$ is their compositum.

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Further, denote by T_0 the k-torus with character module I split by L_0. Let $N\colon L_1 \to k$ denote the norm map. We denote by the same letter the norm map $N\colon R_{L_1/k}(T_0 \times_k L_1) \to T_0$ and define $T := \ker N$ (which is isomorphic to the quotient $R_{L_1/k}(T_0 \times_k L_1)/T_0$).

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Here is an explicit example for $k = \mathbb{Q}$, in the spirit of Tsfasman's example: take $L = \mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{89})$.

Surfaces

It turns out that one can do better and reduce the dimension of counter-examples to two: the celebrated example of a cubic k-surface which is stably k-rational but not k-rational (Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer, 1985) also works in our context.

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We start with the affine cubic surface $V \subset \mathbb{A}^3$ given over an arbitrary global field k with $\mathrm{char}(k) \neq 2$ by

$$y^2 - az^2 = f(x),$$

where a is not a square in k and $f \in k[x]$ is a separable irreducible polynomial of degree 3 with discriminant a.

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Theorem 2 is proven.

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But this is another story ...

THANKS FOR YOUR ATTENTION!