Intersections of symmetric determinantal varieties, theta characteristics, and an application to arithmetic.

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June 5, 2023

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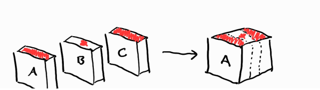
Throughout:

- k is a field of characteristic not 2 or 3.
- $\mathbf{x} := (x_0, \ldots, x_n)$ will be variables for $\mathbb{P}^n$
- $\mathbf{y} := (y_0, \ldots, y_m)$ will be variables for $\mathbb{P}^m$
- $\widehat{\mathbb{P}}^n$ is the dual projective space of $\mathbb{P}^n$

We'll implicitly identify points in $\mathbb{P}^n_k$ and chosen representatives in k^{n+1}, for the sake of brevity.

Let A, B, C be 4×4 symmetric matrices. Let $\mathcal{A}$ be the tensor given by stacking $A, B, C \in \mathrm{Sym}_2 k^4$.



Then

$$X := Z(\det(x_0 A + x_1 B + x_2 C)) = \det Z(\mathcal{A}(\mathbf{x}, \cdot, \cdot))$$

is (generically) a canonical genus 3 curve in $\mathbb{P}^2$.

On the other hand,

$$\mathfrak{C} := Z(\mathbf{y}^T A\mathbf{y}, \mathbf{y}^T B\mathbf{y}, \mathbf{y}^T C\mathbf{y}) = Z(\mathcal{A}(\cdot, \mathbf{y}, \mathbf{y}))$$

is an intersection of 3 quadrics in $\mathbb{P}^3$. In this case, $\mathfrak{C}$ is called the Cayley octad.

Theorem (Hesse, 1855)

The 28 bitangents are in bijection with the 28 pairs of points of the Cayley octad.

If $(p_0 : p_1 : p_2) \in X$, then $\ker(p_0 A + p_1 B + p_2 C)$ is nontrivial.

$$\operatorname{rk}\ker(p_0 A + p_1 B + p_2 C) = 1 \text{ at any smooth point of } p \in X.$$

Thus, there is a map

$$\varphi\colon p \mapsto \ker(p_0 A + p_1 B + p_2 C) = \ker \mathcal{A}(p, \cdot, \cdot) \in \mathbb{P}^3$$

This is called the kernel map.

Write $X = Z(F)$. The map

$$\theta_X\colon p \mapsto \nabla_p F \in \widehat{\mathbb{P}}^2$$

is called the Gauss map.

Finally, we have the map

$$\psi\colon \mathbf{y} \mapsto (\mathbf{y}^T A\mathbf{y}, \mathbf{y}^T B\mathbf{y}, \mathbf{y}^T C\mathbf{y}) = \mathcal{A}(\cdot, \mathbf{y}, \mathbf{y})$$

Important observation: $\nabla_p F = \psi(\varphi(p))$

We can also look at the singularities of $X = Z(\det \mathcal{A}(\mathbf{x}, \cdot, \cdot)) = Z(F)$.

If p is a singular point, then $\nabla_p F = 0$. Either,

- $\operatorname{rk} \ker \mathcal{A}(p, \cdot, \cdot) \geq 2$ (essential singularity)
- $\operatorname{rk} \ker \mathcal{A}(p, \cdot, \cdot) = 1$ (accidental singularity)

$$\mathcal{A}(\cdot, \varphi(p), \varphi(p)) = 0, \text{or equivalently } \varphi(p) \in \mathfrak{C}.$$

We will see later this implies $\mathfrak{C}$ is singular at $\varphi(p)$.

A symmetroid is a scheme of the form

$$X := Z(\det \mathcal{A}(\mathbf{x}, \cdot, \cdot))$$

for some $\mathcal{A} \in k^{n+1} \otimes \mathrm{Sym}_2 k^{m+1}$.

If $\mathcal{A}^{(1)} \in k^{n+1} \otimes \mathrm{Sym}_2 k^{d_1}, \ \ldots \ , \mathcal{A}^{(r)} \in k^{n+1} \otimes \mathrm{Sym}_2 k^{d_r}$

$$X := Z(\det \mathcal{A}^{(1)}(\mathbf{x}, \cdot, \cdot), \ldots, \det \mathcal{A}^{(r)}(\mathbf{x}, \cdot, \cdot))$$

is an intersection of symmetroids, to which we associate the block-diagonal tensor

$$\mathcal{A} := \begin{bmatrix} \mathcal{A}^{(1)} & & \\ & \ddots & \\ & & \mathcal{A}^{(r)} \end{bmatrix}$$

If $\mathcal{A} \in k^{n+1} \otimes \bigoplus_{j=1}^{r} \mathrm{Sym}_2 k^{d_j}$ is a block-diagonal tensor, we represent

$$q = (q_1, q_2, \ldots, q_r) \in \mathbb{P}^m, \quad m + 1 = d_1 + \ldots + d_r.$$

Theorem (K.-Vemulapalli)

(a) *If $\mathfrak{C}$ is a complete intersection of $n+1$ quadrics, then: There exists a $p \in k^{n+1}$ such that $\mathcal{A}(p, q, \cdot) = 0$ if and only if $q \in \operatorname{sing} \mathfrak{C}$.*

(b) *If $\mathcal{A}(p, q, \cdot) = 0$ and each q_l is non-zero, then $p \in \operatorname{sing} X$. If X is a complete intersection and p is an accidental singularity of X, then there exists a $q \in \mathfrak{C}$ such that $\mathcal{A}(p, q, \cdot) = 0$.*

- If H is tangent to X at a smooth point p, then $X \cap H$ is an intersection of symmetroids with *an accidental singularity at p*.

A genus 4 curve X can be written as

$$X := X_2 \cap X_3 \subset \mathbb{P}^3$$

where X_2 is a symmetroid quadric and X_3 is a symmetroid cubic if and only if X has:

- a vanishing even theta characteristic θ_0 with an effective representative defined over $\mathbb{Q}$.
- a [2]-torsion point ϵ defined over k such that, viewing $\theta_0 \colon J_X[2] \to \mathbb{F}_2$ as a quadratic form,

$$q_{\theta_0}(\epsilon) = 0.$$

If $\mathcal{A} \in k^4 \otimes (\mathrm{Sym}_2 k^2 \oplus \mathrm{Sym}_2 k^3)$, the Cayley variety $\mathfrak{C} \subset \mathbb{P}^4$ is

- dimension 0
- degree 16
- has the natural automorphism $(y_0 : y_1 : y_2 : y_3 : y_4) \mapsto (-y_0 : -y_1 : y_2 : y_3 : y_4)$.

Corollary

The $\binom{16}{2} = 120$ *secants of* $\mathfrak{C}$ *are contracted by* ψ *to* $56 + 8$ *points of* $\widehat{\mathbb{P}}^3$.

- 8 *define representatives of* θ_0.
- 56 *define representatives of odd theta characteristics of* X.

When X is an intersection of symmetroids:

$$\mathcal{A} := \begin{bmatrix} \mathcal{A}^{(1)} & \\ & \mathcal{A}^{(2)} \end{bmatrix}$$

there's a natural projection $\pi\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$. There's a 4-to-1 cover

$$\left\{ \begin{array}{c} \text{The 112 secant lines } \ell \text{ of } \mathfrak{C} \text{ defining} \\ \text{odd theta characteristics of } X \end{array} \right\} \to \left\{ \text{ 28 lines } \pi(\ell) \ \right\}$$

and a double cover

$$\left\{ \begin{array}{c} \text{The 112 secant lines } \ell \text{ of } \mathfrak{C} \text{ defining} \\ \text{odd theta characteristics of } X \end{array} \right\} \to \left\{ \text{ 56 tritangents } \psi(\ell) \ \right\}.$$

This correspondence is a special case of a correspondence from **Bruin-Sertöz**, inspired by a 1926 paper of Milne.

If S is a del Pezzo surface of degree one, then there is a double cover

$$|(\kappa_S^\vee)^{\otimes 2}|\colon S \to X_2,$$

where X_2 is a quadric cone in $\mathbb{P}^3$. It is branched over a genus 4 curve X (and the node).

$(\operatorname{Pic} S)^\perp := \{D \in \operatorname{Pic} S : D.\kappa_S = 0\}$, and restriction gives a natural isomorphism

$$\frac{(\operatorname{Pic} S)^\perp}{2(\operatorname{Pic} S)^\perp} \to J_X[2]$$

There are 240 exceptional curves on S. If e is an exceptional curve, then $e \otimes \kappa_S^\vee \in (\operatorname{Pic} S)^\perp$.

Useful trick: If Q parametrizes a family of quadrics of (generically) even rank, there is a natural (possibly branched) double cover $\widetilde{Q} \to Q$ from labelling the maximal isotropic subspaces.

Observation: The cone X_2 parametrizes quadrics of generic rank 4. Its associated double cover is S.

If ℓ is a secant of $\mathfrak{C}$, then we define a subscheme of S by

$$\tau(\ell) := \left\{ (x, [\mathcal{L}]) \in S : \begin{array}{l} \mathcal{L} := \ell + \ker \mathcal{A}(x, \cdot, \cdot) \text{ is a} \\ \text{maximal isotropic sub-} \\ \text{space of } \mathcal{A}(x, \mathbf{y}, \mathbf{y}) \end{array} \right\}.$$

Proposition

The map $\tau\colon \operatorname{sec}\mathfrak{C} \to \operatorname{Pic} S$ *sends* 112 *secants to* 112 *exceptional curves in S. The set*

$$\{\tau(\ell) \otimes \kappa_S^\vee : \ell \in \operatorname{sec}\mathfrak{C}\}$$

generates a lattice of type D_8 *in* $(\operatorname{Pic} S)^\perp$.

Theorem (K.)

Let $\rho\colon \operatorname{Gal}(\mathbb{Q}^{\mathrm{sep}}/\mathbb{Q}) \to W_{D_8}$ *be a continuous homomorphism such that* $\operatorname{Im}(\rho)$ *is the Galois group of degree* 16 *univariate polynomial.*

Then there exists a nonsingular del Pezzo surface of degree one S such that each $\sigma \in \operatorname{Gal}(\mathbb{Q}^{\mathrm{sep}}/\mathbb{Q})$ *permutes the* 240 *exceptional curves of S as described by* $\rho(\sigma) \subset S_{240}$.

In summary:

- Symmetric determinantal representations link intersections of symmetroids and their Cayley varieties.
- Theta characteristics of special complete intersection canonical curves are described in terms of tensors
- Such connections can be used to study arithmetic questions

Thanks!