The refined Humbert invariant with automorphism group of a genus 2 curve

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AGCCT
5 – 9 June, 2023
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- In this talk, I will give a list for the refined Humbert invariants q_C according to $\mathrm{Aut}(C)$.

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$$q_{(A,\theta)}(D) = (D.\theta)^2 - 2(D.D), \text{ for } D \in \mathrm{NS}(A),$$

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- The refined Humbert invariant $q_{(A,\theta)}$ can be seen as an integral quadratic form.

Automorphism groups of genus 2 curves

- For a positive ternary form q, let $r_4(q) = |\{(x, y, z) \in \mathbb{Z}^3 : q(x, y, z) = 4\}|$, and let

$$a(q) = \max(1, r_4(q), 3r_4(q) - 12).$$

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- By Kani (2022), when q_C is ternary, we have the following possibilities:

$\mathrm{Aut}(C)$	$a(q_C)$	$r_4(q_C)$
C_2	1	0
$C_2 \times C_2$	2	2
D_4	4	4
D_6	6	6
$C_3 \rtimes D_4$	12	8
$\mathrm{GL}_2(3)$	24	12

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- **Aim:** Find all ternary quadratic forms q such that $q \sim q_C$, for some genus 2 curve C and calculate the corresponding $r_4(q)$.

The classification

Theorem (K. 2022)

Let f be an **imprimitive** positive ternary form. Then f is equivalent to a form $q_{(A,\theta)}$, for some (A,θ) if and only if
(i) $\frac{1}{2}f$ is an improperly primitive form and
(ii) $f(x_0, y_0, z_0) = (2n)^2$ for some $x_0, y_0, z_0, n \in \mathbb{Z}$ with $\gcd(n, \mathrm{disc}(f)) = 1$.

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- Kani (2021): **primitive case**.

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$$q(x,y,z) = ax^2 + by^2 + cz^2 + ryz + sxz + txy \;=:\; [a,b,c,r,s,t],$$
with $a,b,c,r,s,t \in \mathbb{Z}$.

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Theorem (K. 2023 +)

Let $q = [a, b, c, 2r, 2s, 2t]$ be an Eisenstein reduced positive ternary quadratic form. Let us put $u = |\operatorname{Aut}^{+}(q)|$. If $3 \mid u$ and $u > 6$, then
i) $u = 12 \Leftrightarrow q = [a, a, c, 0, 0, -a]$ or $q = [a, b, b, -b, 0, 0]$ with $a \neq b$.
ii) $u = 24 \Leftrightarrow q = [a, a, a, 0, 0, 0]$ or $q = [a, a, a, \kappa a, \kappa a, \kappa a]$ with $\kappa = 1$ or $-2/3$.

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- Discuss it for all other u's. ($u \in \{2,4,6,8\}$).

Imprimitive Case

Table: List of imprimitive (reduced) ternary forms q_C when $\mathrm{Aut}(C)$ is not trivial.

$\mathrm{Aut}(C)$	q_C	$\lvert \mathrm{Aut}^+(q_C) \rvert$
$C_2 \times C_2$	$[4, b, c, 4, 4, 4]$ with $b \neq c$	2
	$[4, b, c, 2r, 0, -4]$ with $r < 0$	2
	$[4, b, c, 2r, -4, 0]$ with $r < 0$	2
	$[4, b, c, 0, 0, -4]$	4
	$[4, b, b, 2r, 4, 4]$ with $r > 0$	4
	$[4, b, c, 0, -4, 0]$ with $b \neq c$	4
	$[4, b, c, -b, -4, 0]$ with $b \neq c$	4
	$[4, c, c, -4, 0, 0]$	4
D_4	$[4, 4, c, 0, -4, 0]$	4
	$[4, 4, c, -4, -4, 0]$	8
D_6	$[4, 4, c, 4, 4, 4]$	6
	$[4, 4, c, 0, 0 - 4]$	12
$C_3 \rtimes D_4$	$[4, 4, 4, 0, 0 - 4]$	12
$\mathrm{GL}_2(3)$	$[4, 4, 4, 4, 4, 4]$	24

Notation: Suppose that $|2r| \leq b$, $4 \mid b, c, 2r$ and $4 < b \leq c$ and that if $r < 0$, then $b \neq -2r$.

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	$[4, b, b, 2r, 4, 4]$ with $r > 0$	4
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	$[4, 4, c, 0, 0, 0]$	8
	$[4, 4, c', -4, -4, 0]$	8
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Notation: Suppose that $4 < c, c'$, $c \equiv 1 \pmod 4$ and $c' \equiv 1 \pmod 8$.

Applications

- Kani introduced the concept of a *generalized Humbert scheme* $H(q)$ which is associated to a given quadratic form q. Given any integral positive definite quadratic form q in r variables, let

$$H(q) = \{\langle A, \theta\rangle \in \mathcal{A}_2(K) \mid q_{(A,\theta)} \text{ primitively represents } q\},$$

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- If $q(x) = Nx^2$, i.e., $r = 1$, then $H(q) = H_N$ is the **classical Humbert surface of invariant N.**

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Let $q = [4,4,4]$, and let $c > 1$ with $c \equiv 0,1 \ (\mathit{mod}\ 4)$ such that q does not primitively represent c. Then

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- The similar result in D_4 case.

Elliptic subcovers

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- Consider $C \in H(q_{1,5})$.
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- Consider $C \in H(q_{1,5})$.
- $J_C \simeq E_1 \times E_2$. The degree map $\mathrm{disc}(q_{E_1,E_2}) = \frac{\mathrm{disc}(q_{1,5})}{16} = -11$.
- The j-invariants $j(E_1) = j(E_2) = -2^{15}$ and that the E_i/K are given by the equation $y^2 + y = x^3 - x^2 - 7x + 10$.

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- $|H^*([4,4,4]) \cap H_9| = |\mathcal{L}(D_6, 3)|$.
- $H^*([4,4,4]) \cap H_9 \;=\; H(q_{1,5}) \cup H(q_{1,9}) \cup H(q_{2,5}) \cup H(q_{2,9})$.
- $|H^*([4,4,4]) \cap H_9| \;=\; |H(q_{1,5})| \cup |H(q_{1,9})| \cup |H(q_{2,5})| \cup |H(q_{2,9})| = 1+3+1+1 = 6$.
- Consider $C \in H(q_{1,5})$.
- $J_C \simeq E_1 \times E_2$. The degree map $\mathrm{disc}(q_{E_1,E_2}) = \frac{\mathrm{disc}(q_{1,5})}{16} = -11$.
- The j-invariants $j(E_1) = j(E_2) = -2^{15}$ and that the E_i/K are given by the equation $y^2 + y = x^3 - x^2 - 7x + 10$.
- By Gélin, Howe and Ritzenthaler (2019),

$$C : y^2 = 2x^6 + 11x^3 - 22.$$

Some References

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THANK YOU

Thank you!