# Dimension of paramodular forms with Atkin-Lehner signs

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# Classical results on supersingular elliptic curves

Let $D$ be a definite quaternion algebra over $\mathbb{Q}$ of discriminant $p$. Let $H$ be the class number of $D$ and $T$ be the number of isomorphism classes of maximal orders of $D$.

## Theorem (Deuring, Eichler)

*(1) $H$ is the number of isomorphism classes over $\overline{\mathbb{F}_p}$ of supersingular elliptic curves, and given by*

$$H = \frac{p-1}{12} + \frac{1}{4}\left(1 - \left(\frac{-1}{p}\right)\right) + \frac{1}{3}\left(1 - \left(\frac{-3}{p}\right)\right).$$

*(2) $2T - H$ is the number of isomorphism classes over $\overline{\mathbb{F}_p}$ of supersingular elliptic curves defined over $\mathbb{F}_p$, and given by $a_p h(\sqrt{-p})$, where $a_p = 1/2, 1, 2$ for $p \equiv 1, 7$(or $2$), $3 \bmod 8$, respectively.*

# Algebrain modular forms vs modular forms: Eichler

Fix a maximal order $O$ of $D$ and put $U = D_\infty^\times \prod_p O_p^\times$. Put

$$\mathfrak{M}_0(p) = \{f : D_A^\times \to \mathbb{C} : f(uxa) = f(x)\}$$

where $u \in U$, $x \in D_A^\times$, $a \in D^\times$. Spanned by char. func, of $U \backslash D_A^\times / D^\times$. Let $A_2(\Gamma_0(p))$ be the space of elliptic modular forms.

**Theorem**

*There exists a Hecke equivariant isomorphism $A_2(\Gamma_0(p)) \cong \mathfrak{M}_0(p)$.*

$$H = \dim A_2(\Gamma_0(p)) = 1 + \dim S_2(\Gamma_0(p)),$$
$$2T - H = 1 + \dim S_2^-(\Gamma_0(p)) - \dim S_2^+(\Gamma_0(p)).$$

The weight can be generalized to any even $k \geq 2$:
$S_k(\Gamma_0(p)) \cong \mathfrak{M}_{k-2}(p) = \{f : D_A^\times \to \mathbb{C}^{k-2}; f(uxa) = \rho_{k-2}(u)f(x)\}$.
Here $\rho_{k-2} : U \to D_\infty^\times \to GL_{k-2}(\mathbb{C})$ is the sym. tensor rep of $\deg = k - 2$.

# Paramodular groups and paramodular forms

We define a paramodular form of level $N$ by

$$K(N) = Sp(2,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

$$Sp(2,\mathbb{Q}) = \{g \in SL_4(\mathbb{Q}); {}^t gJg = J\}, \quad J = \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}.$$

Let $\rho_{k,j} = \det^k Sym(j)$ be the irreducible representation where det is the determinant and $Sym(j)$ is the symmetric tensor representation of degree $j$. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{Q})$ and a holomorphic function $F(Z)$ on $H_2 = \{Z = X + iY, X = {}^t X, Y = {}^t Y > 0\}$, we define

$$(F|_{k,j}[g])(Z) = \rho_{k,j}(CZ+D)^{-1}F(gZ).$$

# Paramodular cusp forms and involution

The space of paramodular cusp forms of level $N$ is defined by

$$S_{k,j}(K(p)) = \Big\{ F : H_2 \to \mathbb{C}^{j+1} : F|_{k,j}[\gamma] = F \quad \text{for all } \gamma \in K(N), \\ \text{vanishing on the boundary of } \overline{K(N)\backslash H_2}^{Satake} \Big\}$$

The Atkin-Lehner type involution is given by

$$\rho_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & N & 0 & 0 \\ N & 0 & 0 & 0 \end{pmatrix}.$$

We have eigenspaces of $\rho_N$ such that $F|_{k,j}[\rho] = \pm F$. We denote by $S^{\pm}_{k,j}(K(N))$ the plus and the minus eigenspaces.

# Why paramodular forms are important?

(1) An ab. var. $A$ is called supersingular if $A \sim E^g$ (isogeny).
Isomorphism classes $\rightarrow$ the number of irreducible components of supersingular locus in p. p. abelian surfaces over $\overline{\mathbb{F}_p}$.
$E/\mathbb{F}_p$(i.e, $j \in \mathbb{F}_p$) $\rightarrow$ Irreducible components defined over $\mathbb{F}_p$.
Class of ideals $\rightarrow$ Class of quaternion hermitian lattices in $D^2$.
Elliptic modular forms $\rightarrow$ Paramodular forms.
(2) A simplest model of Langlands correspondence for $rank > 1$.
$Sp(2, \mathbb{Q})$ vs $G = \{g \in M_2(D); gg^* = 1_2\}$.
Paramodular forms of level $p$ vs algebraic modular forms on $G$ w.r.t. $U$ defined by the non-principal genus of maximal lattices.
(3) Shimura-Taniyama like conjecture: the zeta functions of "generic" abelian surfaces over $\mathbb{Q}$ are the zeta functions of paramodular forms of weight 2. (Brumer-Kramer, Poor-Yuen).
(4) Roberts and Schmidt exploited a good new form theory of paramodular forms with various levels.

# Results for $N = p$: prime

**Theorem (Ibu. 1985 ,2007)**

*We have an explicit formula for* $\dim S_{k,j}(K(p))$ *for any prime* $p$ *and* $k \geq 3$, *consisting of* 22 *complicated terms (essentially polynomials in* $k$ *and* $p$ *depending on* $k \bmod 60$, $p \bmod 60$.*)*

This was obtained by using the Selberg trace formula.

**Theorem (Ibu. 2022)**

*We have an explicit formula for* $\dim S^{\pm}_{k,j}(K(p))$ *for any prime* $p$ *and* $k \geq 3$, *consisting of class numbers of* $\mathbb{Q}(\sqrt{-dp})$ *(*$d = 1, 2, 3$*), the second generalized Bernoulli number* $B_{2,\chi}$, *and "polynomials".*

This was obtained by comparison with algebraic modular forms of compact $Sp(2)$ and $SO(5)$ (dummigan et al.+T.Asai's class #.)
Dimensions including non-cusp forms can be obtained similarly.

# Comparison I (omit the case $j > 0$ for simplicity)

## Theorem (Ib. 1985: Assume $k \geq 3$)

$$\dim S_k(K(p)) - \left(2\dim S_k(Sp(2,\mathbb{Z})) - \delta_{k,even}\dim S_{2k-2}(SL_2(\mathbb{Z}))\right)$$
$$= \dim \mathfrak{M}_{k-3,k-3}(U(p)) - \dim S_2(\Gamma_0(p)) \times \dim S_{2k-2}(\Gamma_0(p))$$
$$- \delta_{k,odd}\dim S_{2k-2}(SL_2(\mathbb{Z})) - \delta_{k3}.$$

## Conjecture (Ib. 1985: Assume $k \geq 3$)

*There exists a Hecke equivariant isomorphism $S_k^{new}(K(p))$ and $\mathfrak{M}_{k-3,k-3}(p)$ up to lifting part.*

## Theorem (van Hoften, Rösner and Weissauer(2019,2021))

*The above conjecture is true.*

# Comparison II (with involution)

In 2021, Dummigan, Pacetti, Rama and Tornaría generalized the isomorphims to the one between Atkin-Lehner plus and minus part. This essentially gives an abstract proof of the following relation.

**Theorem (Assume $k \geq 3$)**

$$\begin{aligned}\dim S_k^+(K(p)) = &\dim \mathfrak{M}^-_{k-3,k-3}(p) + \dim S_k(Sp(2,\mathbb{Z})) \\ &- \dim S_2^+(\Gamma_0(p)) \times S_{2k-2}(SL_2(\mathbb{Z})).\end{aligned}$$

**Theorem (Ib. 2022; Assume $k \geq 3$)**

*We have an explicit formula for $\mathfrak{M}^{\pm}_{k-3,k-3}(p)$, so also have an explicit formula for* $\dim S_k^{\mp}(K(p))$.

# . Bias of plus and minus. Vanishing $p$ for $k = 3$

### Theorem

*The number of irred. comp. of locus of ppss av surfaces is given by* $1 + \dim S_3(K(p))$.
$\#$(*those defined over* $\mathbb{F}_p$) $= 1 + \dim S_3^-(K(p)) - \dim S_3^+(K(p))$.
*For any* $k \geq 3$, *we always have*

$$(-1)^k(\dim S_k^+(K(p)) - \dim S_k^-(K(p))) \geq 0.$$

*This is* 0 *if and only if* $\dim S_k(K(p)) = 0$, *i.e. if and only if*

$$(p,k) = (2,3),(2,4),(2,5),(2,6),(2,7),(2,9),(2,13)\\ (3,3),(3,4),(3,5),(3,7),(5,3),(5,4),(7,3),(11,3).$$

### Theorem

$\dim S_3(K(p)) = 0$ *if and only if* $p = 2,\ 3,\ 5,\ 7,\ 11$.
$\dim S_3^+(K(p)) = 0$ *if and only if* $p$ *is a prime* $\leq 163$ *or* $p = 179,\ 181,\ 191,\ 193,\ 199,\ 211,\ 229,\ 241$.

# Example of the dimension

We denote by $A_k^+(K(p))$ paramodular forms of weight $k$ including non-cusp forms. Then, for $p = 79$, we have

$$\sum_{k=0}^{\infty} \dim A_k^+(K(79))t^k = \frac{P_+^{(79)}(t)}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})}$$

with

$$\begin{aligned} P_+^{(79)}(t) =& 1 + t^2 + 22t^4 + 76t^6 + 17t^7 + 162t^8 + 45t^9 + 265t^{10} + \\ & 74t^{11} + 352t^{12} + 92t^{13} + 406t^{14} + 92t^{15} + 406t^{16} + \\ & 91t^{17} + 352t^{18} + 75t^{19} + 267t^{20} + 47t^{21} + 164t^{22} + \\ & 18t^{23} + 76t^{24} - t^{25} + 21t^{26} - t^{27} + t^{30}. \end{aligned}$$

# Example of the dimensions (2)

We denote by $A_k(K(p))$ paramodular forms of weight $k$ including non-cusp forms. Then we have

$$\sum_{k=0}^{\infty} \dim A_k(K(79)) t^k = \frac{P^{(79)}(t)}{(1-t^4)(1-t^5)(1-t^6)(1-t^{12})},$$

where

$$\begin{aligned} P^{(79)}(t) =& 1 + t^2 + 8t^3 + 22t^4 + 43t^5 + 82t^6 + 128t^7 + 191t^8 + \\ & 254t^9 + 324t^{10} + 386t^{11} + 439t^{12} + 476t^{13} + 503t^{14} + \\ & 508t^{15} + 498t^{16} + 480t^{17} + 439t^{18} + 392t^{19} + 331t^{20} + \\ & 266t^{21} + 198t^{22} + 134t^{23} + 82t^{24} + 42t^{25} + 16t^{26} + \\ & 2t^{27} - 5t^{29} + t^{30}. \end{aligned}$$

# Reference

(1) T. Ibukiyama, Dimensions of paramodular forms and compact twist modular forms with involutions. arXiv:2208.13578

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