$\mathbb{Q}$-Rational torsion of generalised modular Jacobians

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The modular curve $X_0(N)$

Given a positive integer N, let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

and $X_0(N)/\mathbb{Q}$ the projective non-singular **modular curve** associated to $\Gamma_0(N)$.

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$$\begin{array}{ccccc} X_0(N)/\mathbb{C} & \simeq & \mathcal{H}/\,\Gamma_0(N) & \cup & \mathbb{P}^1(\mathbb{Q})/\Gamma_0(N) \\ & & \downarrow & & \downarrow \\ & & Y_0(N) & & \mathrm{Cusps}(X_0(N)) \end{array}$$

$X_0(N)$ is the moduli space of $\{(E, C) : E/\mathbb{C}$ elliptic curve $, \mathbb{Z}/N\mathbb{Z} \simeq C \subset E\}/\simeq$.

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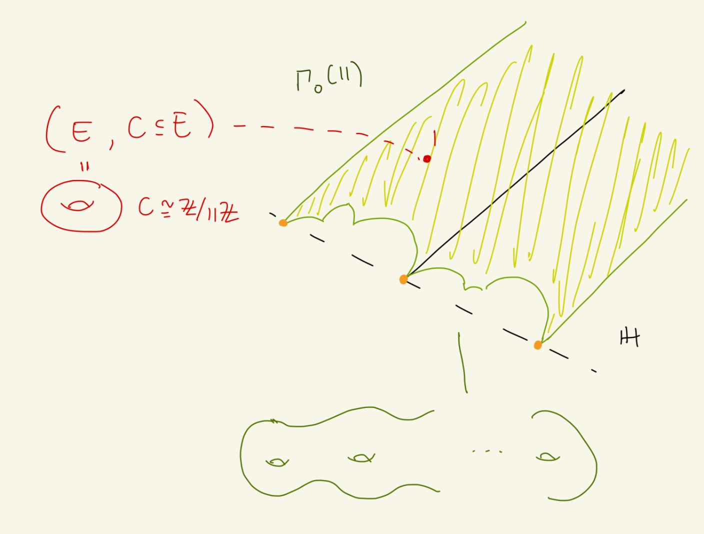

The Jacobian $J_0(N)$

Let $J_0(N) := \mathrm{Jac}(X_0(N)) = \mathrm{Pic}^0(X_0(N)) = \mathrm{Div}^0(X_0(N))/\sim$.
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By Mordell-Weil, we have

$$J_0(N)(\mathbb{Q}) \simeq \mathbb{Z}^r \oplus J_0(N)(\mathbb{Q})_{\mathrm{tor}}.$$

Ogg's Conjecture

Any $[D] \in J_0(N)$ with $D \in \mathrm{Div}^0_{\mathrm{cusp}}(X_0(N))$ is torsion. (Manin-Drinfeld, 1973)

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We have $\mathrm{Cusps}(X_0(N)) = \{0, \infty\} \subseteq X_0(N)(\mathbb{Q})$.

Ogg's Conjecture (1975)

The group

$$J_0(N)(\mathbb{Q})_{\mathrm{tor}} \simeq \langle [0 - \infty] \rangle \simeq \mathbb{Z}/\mathrm{num}\left(\frac{N-1}{12}\right)\mathbb{Z}$$

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Proofs of Ogg's conjecture: Mazur (1977).

Cuspidal subgroup

Take $C_N := \mathbf{im}(\mathbf{Div}^0_{\mathbf{cusp}}(X_0(N))\ \mathbf{in}\ J_0(N)))$ the **cuspidal subgroup**. However, in general $\mathrm{Cusps}(X_0(N)) \not\subseteq X_0(N)(\mathbb{Q})$.

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Large evidence in favour of the conjecture! (Ling, Lorenzini, Otha, Luopian, Lui, Ren, Conrad-Edixhoven-Stein, Ribet-Wake ...)

Cuspidal subgroup

Theorem (Yoo, 2019, 2021)

For any positive integer N and any odd prime ℓ such that ℓ^2 does not divide $3N$ we have

$$J_0(N)(\mathbb{Q})_{\text{tor}}[\ell^\infty] \simeq \left(\bigoplus_{d \in D_1(N)} \langle [Z_\ell(d)] \rangle\right)[l^\infty]$$

for certain divisors $\{[Z_\ell(d)] \in C_N(\mathbb{Q}) : d \in D_1(N)\}$, where $D_1(N) := \{\text{divisors of N}\} \setminus \{1\}$.

The divisors $Z_\ell(d)$ and their orders can be explicitly computed.

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Example

Take $N = pq$ and ℓ s.t. $\mathrm{val}_\ell(p-1) \leq \mathrm{val}_\ell(q-1)$. We have $D_1(N) = \{p, q, pq\}$ and

- $Z_\ell(p) = (q-p)0 - (q-1)P_p + (p-1)P_q$ has order $\mathrm{num}\left(\frac{(p-1)(q^2-1)}{24}\right)$.
- $Z_\ell(q) = 0 - P_q$ has order $\mathrm{num}\left(\frac{(p^2-1)(q-1)}{24}\right)$.
- $Z_\ell(pq) = q0 - qP_p + P_q - \infty$ has order $\mathrm{num}\left(\frac{(p-1)(q-1)\gcd(p+1,q+1)}{24\gcd(p-1,q-1)}\right)$.

Generalised Jacobians

Jacobians help us explore smooth curves.

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Let $\mathbf{m} \in \mathrm{Div}(X_0(N))$ an effective rational divisor (call it a **modulus**).
Construct $X_0(N)_{\mathbf{m}}$, the singular curve obtained from $X_0(N)$ by "glueing" the points in $\mathbf{m}$ into a single point Q.

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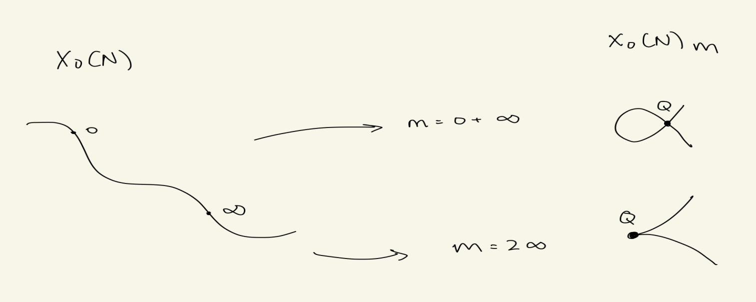

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Definition

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Properties generalised Jacobian

Given M a set of points in $X_0(N)$, there exists a modulus $\mathbf{m}$ with $M = \mathrm{Supp}(\mathbf{m})$, and a map

$$\iota_{\mathbf{m}} : X_0(N) \setminus M \to J_0(N)_{\mathbf{m}}$$

with universal property.

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Properties generalised Jacobian

Let $\mathrm{Div}^0_{\mathbf{m}}(X_0(N))$ be the set of divisors coprime to $\mathbf{m}$. Then the generalised Jacobian $J_0(N)_{\mathbf{m}}$ is

$$J(X)_{\mathbf{m}} = \mathrm{Div}^0_{\mathbf{m}}(X)/\sim_{\mathbf{m}}.$$

where $\sim_{\mathbf{m}}$ denotes the linear equivalence

$$D_1 \sim_{\mathbf{m}} D_2 \text{ iff there is } f \in k(X) \text{ with } \mathrm{div}(f) = D_1 - D_2 \text{ and } f \equiv 1 \pmod{\mathbf{m}}.$$

Cuspidal modulus

This gives

$$0 \to L_{\mathbf{m}} \to J_0(N)_{\mathbf{m}} \to J_0(N) \to 0 \qquad (*)$$

The linear group $L_{\mathbf{m}}$ is isomorphic to a product of copies of $\mathbb{G}_m$'s and $\mathbb{G}_a$'s.

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Motivation

If $\mathbf{m} \subset \text{Div}_{\text{cusp}}(X_0(N))$, $(*)$ becomes explicit and $J_0(N)_{\mathbf{m}}$ seems to be related to (weakly) modular forms! (Gross, Bruinier-Li)

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What can we say about $J_0(N)_{\mathbf{m}}(\mathbb{Q})_{\mathrm{tor}}$?

Main result

Theorem (C.I., 2022)

Let N be an odd positive integer. For any odd prime ℓ with ℓ^2 not dividing $3N$ we can construct divisors $E_\ell(d) \in \mathrm{Div}^0_{\mathrm{cusp}}(X_0(N))$ such that

$$J_0(N)_{\mathbf{m}}(\mathbb{Q})_{\mathrm{tor}}[\ell^\infty] \simeq \left(\bigoplus_{d \in D_2(N)} \langle [E_\ell(d)] \rangle \right) [\ell^\infty]$$

where $D_2(N) = \{d | N, \text{ divisible by at least 2 primes}\}$.

Extend results of Yamazaki-Yang (2016) and Wei-Yamazaki (2019).

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Example

Let $N = p^2 q$. Then $D_2(N) = \{pq, p^2 q\}$ and

- $E_\ell(pq) = (q+1)Z_\ell(p) - Z_\ell(pq)$ and its order is num $\left(\frac{(p-1)(q-1)}{24}\right)$;
- $E_\ell(p^2 q) = (q+1)Z_\ell(p^2) - Z_\ell(p^2 q)$ and its order is num $\left(\frac{(p^2-1)(q-1)}{24}\right)$.

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$$\delta([D]) = \left(c_{d'}(f) \cdot c_N(f)^{-1}\right)_{d'|N, d' \neq N} \otimes \frac{1}{m}.$$

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5. Compute order$([E_\ell(d)])$ for all $d \in D_2(N)$.

Thank you!