Ciani curves with potentially good reduction

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Invariants of plane quartics

Y a smooth plane quartic over $K = \mathbb{Q}_p^{\mathrm{nr}}$, where $p > 3$.
Dixmier–Ohno invariants $\underline{DO}(Y) = (I_3 : \ldots : I_{27}) \in \mathbb{P}_w(K)$ satisfy

$$Y \simeq_{\overline{K}} Y' \quad \Leftrightarrow \quad \underline{DO}(Y) = \underline{DO}(Y') \in \mathbb{P}_w(\overline{K})$$

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Geometric properties of the curve Y are determined by the invariants.

Assume: invariants are normalized in $\mathbb{P}_w(\overline{K})$ and $\Delta = I_{27} \neq 0$.

Exa Y has potentially good quartic reduction iff $p \nmid \Delta := I_{27}$.
([LLLR]) Similar condition for potentially good hyperelliptic reduction.

Arithmetic questions

Proposition([LRRS]) Assume that $|\mathrm{Aut}_{\overline{K}}(Y)| > 2$.
There exists a K-curve Y with $\underline{DO}(Y) = \underline{DO} \in \mathbb{P}_w(\overline{K})$.

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We restrict to:
A *Ciani curve* is a plane quartic with $C_2^2 \simeq V \subset \mathrm{Aut}_{\overline{K}}(Y)$ and $g(Y/V) = 0$.

These form a 3-dimensional family, there are invariants $\underline{I} = (I_3 : I_3' : I_3'' : I_6) \in \mathbb{P}^3_{1,1,1,2}(K)$ classifying $\{(Y, V)\}/_{\simeq_{\overline{K}}}$.

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Question: Given $\underline{I} \in \mathbb{P}^3_{1,1,1,2}(K)$ s.t. $Y_{\underline{I}}$ has pot. good reduction.
Does there exists a K-model Y with good reduction over K?

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Answer: No!
The corresponding statements for elliptic curves holds.

Ciani curves with given invariants

Let $\underline{I} \in \mathbb{P}^1_{1,1,1,2}(K)$ be normalized invariants. Define

$$\mathcal{P}(T) = T^3 - S_1 T^2 + S_2 T - S_3 \in K[T],$$

where $P = 8I_3 + I_3' - I_3''$, $S_3 = I_3 P$, $S_2 = \big(P^2 + 16 I_3(P + I_3'') - I_6\big)/4$, $S_1 = I_3' + 12 I_3$.
Assume $P \neq 0$ (for simplicity). Let L/K be the splitting field of $\mathcal{P}$, and A, B, C its roots.

Proposition Then

$$Y/L: \ A_1 x^4 + B_1 y^4 + C_1 z^4 + P(x^2y^2 + x^2z^2 + y^2z^2) = 0$$

satisfies $\underline{I}(Y) = \underline{I} \in \mathbb{P}^3_{1,1,1,2}(\overline{K})$.

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Interpretation of L: The three elliptic curves $E/\langle\sigma\rangle$ for $0 \neq \sigma \in V$ are each defined over L.

$$Q := \mathrm{discriminant}(\mathcal{P}) \neq 0 \quad \Leftrightarrow \quad \mathrm{Aut}_{\overline{K}}(Y_{\underline{I}}) = V.$$

The main result

Assume that $Y_{\underline{I}}$ has pot. good reduction, and that $Q \neq 0$.

Proposition Assume pot. good **quartic** reduction,i.e. $\nu(\Delta(\underline{I})) = 0$.

(a) There exists a K-model of $Y_{\underline{I}}$ with good reduction iff $\nu(Q) = 0$, i.e. $\mathrm{Aut}_k(\overline{Y}) = \mathrm{Aut}_{\overline{K}}(Y_{\underline{I}}) = V$.

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(b) Otherwise, $\exists$ a K-model Y with stable reduction over $[L : K] \in \{2, 3, 4\}$. (Cases determined by NP of $\mathcal{P}$.)

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"Reason": $\mathrm{Gal}(L/K) \hookrightarrow \mathrm{Aut}_k(\overline{Y})$ if L minimal.

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(b) Otherwise, $\exists$ a K-model Y with stable reduction over $[L : K] \in \{2, 3, 4\}$. (Cases determined by NP of $\mathcal{P}$.)

Proposition Assume pot. good **hyperelliptic** reduction, i.e. $0 < 3\nu(I_3'') = 2\nu(I_6) \leq 6\nu(I_3')$. Set $e = \nu(I_6/I_3'')$.

(a) $Y_{\underline{I}}$ admits a K-model with good reduction iff e is even.
(b) Otherwise, $\exists$ a K-model with good reduction over $[L : K] = 2$.

Conductor exponent

Let Y/K be a smooth plane curve.
Conductor exponent $f_p(Y) \neq 0 \;\Rightarrow\; Y$ has bad reduction at p.
Converse is true for Ciani curves.

General question: $f_p(Y) \leq \nu(\Delta(Y))$?

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Our results imply for Ciani curves with potentially good reduction

$$\min f_p(Y) = \begin{cases} 0 & \text{if } \exists\ K\text{-model with good reduction,} \\ 4 & \text{otherwise if pot. good quartic reduction,} \\ 6 & \text{otherwise if pot. good hyperelliptic reduction.} \end{cases}$$

Here the minimum runs over all K-models of $Y_{\underline{I}}$.

Hence in general

$$\min f_p(Y) \not\leq \nu(\Delta(\underline{I})).$$

Example

Choose $\underline{I} = (1 : -6 : 1 : 1)$. Then

$$\mathcal{P}(T) = T^3 - 6T^2 + 8T - 1, \qquad \text{discriminant}(\mathcal{P}) = 229.$$

Over the splitting field L we find the model

$$Y_1 \; Ax^4 + By^4 + C^4 + x^2y^2 + x^2z^2 + y^2z^2 = 0, \qquad \Delta(Y) = \Delta(\underline{I})2^{20}.$$

There exists a $\mathbb{Q}_p^{\mathrm{nr}}$-model Y with good reduction for all odd prime $p \neq 229$.

For $p = 229$ there exists a $K = \mathbb{Q}_p^{\mathrm{nr}}$-model Y with $f_p(Y) = 4$. (Best possible.)
The concrete K-model Y has $\nu(\Delta(Y)) > 0$ at $p = 229$.