Invariants of genus 4 curves

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Invariants of curves?

A natural question when it comes to curves is the parametrization of isomorphism classes of curves of genus g. Often, this parametrization involves a linear action of a group $\rightsquigarrow$ classical invariant theory.

Definition

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Example

Let $V = \{ax^2 + bxy + cy^2$ *with* $a, b, c \in K\}$. *The natural* $\mathrm{SL}_2(K)$*-action on* K^2 *induces an* $\mathrm{SL}_2(K)$*-action on* $K[V]$.

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We can prove that $K[V]^G = K[b^2 - 4ac]$.

Why genus 4 curves?

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Let K be an algebraically closed field of characteristic 0. We have the following results:

- Invariants of genus 2 curves [Bolza, 1887], [Igusa, 1960]
- Invariants of hyperelliptic genus 3 curves [Sylvester and Franklin, 1879], [Shioda, 1967]
- Invariants of non-hyperelliptic curves of genus 3 [Dixmier, 1987] and [Ohno, 2007]
- Invariants of hyperelliptic curves of genus 4 [Brouwer, Popoviciu, 2009]

Main theorem

Theorem (B.)

The algebra $K[V_{3,3}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)\rtimes\mathbb{Z}/2\mathbb{Z}}$ *is generated by* 65 *homogeneous invariants, with* 10 *of them algebraically independant.*

Geometry of the problem

Let K be an algebraically closed field of characteristic 0.
Let $\mathcal{C} = \{Q = 0\} \cap \{E = 0\} \subset \mathbb{P}^3_K$ be the canonical embedding of a non-hyperelliptic curve of genus 4 over K.

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- Q (irreducible quadric) is unique, and of rank 3 or 4.
- E (irreducible cubic) can be changed to $E + \ell Q$, with ℓ a linear form.

This leads directly to the following result on isomorphisms of non-hyperelliptic curves of genus 4:

Isomorphisms between non-hyp genus 4 curves

Let $\mathcal{C}_1, \mathcal{C}_2$ be two non-hyp curves of genus 4 canonically embedded in $\mathbb{P}^3_K$, with $\mathcal{C}_i = \{Q_i = 0\} \cap \{E_i = 0\}$.

Lemma

$\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic if and only if there exist $f \in \mathrm{PGL}_4(K)$, ℓ a linear form such that:

$$\begin{cases} Q_1(f(X,Y,Z,T)) = Q_2 \\ E_1(f(X,Y,Z,T)) = E_2 + \ell Q_2 \end{cases}$$

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Problem: This action seems too complicated to be studied effectively.

Changing the point of view

From now on, we will assume that Q is of rank 4, and that $Q = XT - YZ$.

Let $\mathcal{C} = \{Q = 0\} \cap \{E = 0\}$ a non-hyperelliptic curve of genus 4. We introduce the Segre embedding:

$$\Psi : ([x : y], [u : v]) \in \mathbb{P}^1_K \times \mathbb{P}^1_K \mapsto [xu : xv : yu : yv] \in \mathbb{P}^3_K.$$

Ψ is an isomorphism from $\mathbb{P}^1_K \times \mathbb{P}^1_K$ to $\{Q = XT - YZ = 0\}$.
$\rightsquigarrow$ We consider the pullback of the cubic form E, and we write it as

$$f := \sum_{0 \leq i,j \leq 3} a_{i,j} x^i y^{3-i} u^j v^{3-j}.$$

Action? $\mathrm{Aut}(\mathbb{P}^1_K \times \mathbb{P}^1_K)$

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Action? $\mathrm{Aut}(\mathbb{P}^1_K \times \mathbb{P}^1_K) \simeq \mathrm{PGL}_2(K) \times \mathrm{PGL}_2(K) \rtimes \mathbb{Z}/2\mathbb{Z}$.

The algebra of invariants

Generators of $K[V_{3,3}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)\rtimes\mathbb{Z}/2\mathbb{Z}}$?

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Problem: The $\mathbb{Z}/2\mathbb{Z}$ term.

The algebra of invariants

Generators of $K[V_{3,3}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)\rtimes\mathbb{Z}/2\mathbb{Z}}$?
Problem: The $\mathbb{Z}/2\mathbb{Z}$ term.

Lemma

We can do this study in two steps:

$$K[V_{3,3}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)\rtimes\mathbb{Z}/2\mathbb{Z}} = \left(K[V_{3,3}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)}\right)^{\mathbb{Z}/2\mathbb{Z}}.$$

Both steps are hard, but I will only talk about the study of $K[V_{3,3}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)}$.

First tool: the Hilbert series

Definition

Let A be a finitely generated, graded K-algebra, with $A = \bigoplus_{d=0}^{+\infty} A_d$.

We define

$$H(A,t) := \sum_{d=0}^{+\infty} \dim(A_d) t^d.$$

Hilbert series

Theorem (B.)

The Hilbert series of $K[V_{3,3}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)}$ is:

$$\frac{P(t)}{(1-t^2)(1-t^4)^2(1-t^6)^2(1-t^8)^2(1-t^{10})(1-t^{12})(1-t^{14})},$$

with P an explicit polynomial of degree 58 with nonnegative integral coefficients.

Second tool: the nullcone

Definition

Let V be a finite-dimensional K-vector space with a G-action. The nullcone of V is defined as:

$$\mathcal{N}_V := \{g \in V \text{ s.t. } \forall I \in K[V]^G_{d>0}, I(g) = 0\}.$$

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But how to actually compute invariants?

Transvectants for binary forms

What are transvectants?

Transvectants for binary forms

What are transvectants?

- Differential operators,
- Very efficient to compute invariants and covariants of binary forms: just a list of differential operations, instead of a huge polynomial.

$$(f,g)_k = C \sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} \frac{\partial^k f}{\partial x^i \partial y^{k-i}} \frac{\partial^k g}{\partial x^{k-i} \partial y^i},$$

where f, g are binary forms, and C is a constant.

Transvectants for binary forms

Theorem

Let V_n be the space of binary forms of degree n. Then $K[V_n]^{\mathrm{SL}_2(K)}$ is generated by transvectants evaluated on a generic element of V_n.

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Proof: Gordan's algorithm.

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Proof: Gordan's algorithm.

Example

Let $f := a_4x^4 + a_3x^3y + a_2x^2y^2 + a_1xy^3 + a_0y^4 \in V_4$. We define $i_2 = (f,f)_4$, $i_3 = ((f,f)_2, f)_4$. Then $K[V_4]^{\mathrm{SL}_2(K)} = K[i_2, i_3]$.

The space of biforms $V_{m,n}$

Definition

Let $V_{m,n}$ be the space of biforms of bidegree (m,n) over K:

$$\begin{aligned} V_{m,n} &= \mathrm{Sym}^m(K^2) \otimes \mathrm{Sym}^n(K^2) \\ &= \left\{ \sum_{i,j} a_{i,j} x^i y^{m-i} u^j v^{n-j} \text{ s.t. } a_{i,j} \in K \right\}. \end{aligned}$$

The bicubic form f is an element of $V_{3,3}$.

Transvectants for biforms

Definition

We define $(f,g)_{k,l}$ as

$$C\sum_{\substack{0\leq i\leq k\\0\leq j\leq l}}(-1)^{i+j}\binom{k}{i}\binom{l}{j}\frac{\partial^{k+l}f}{\partial x^i\partial y^{k-i}\partial u^j v^{l-j}}\frac{\partial^{k+l}g}{\partial x^{k-i}\partial y^i\partial u^{l-j}v^j},$$

where f,g are biforms, and C is a constant.

It is a simultaneous transvectant on both sets of variables.

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where f, g are biforms, and C is a constant.

It is a simultaneous transvectant on both sets of variables. Previously known in very few cases by P. Olver (in a 1986 preprint) and H.W. Turnbull (several articles around 1925).

Transvectants for biforms

Theorem (Turnbull, 1923)

The transvectants for biforms evaluated on a generic element of $V_{m,n}$ generate the algebra of invariants $K[V_{m,n}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)}$ for all $m, n \geq 1$.

Proof: Gordan's algorithm for biforms.

The invariant of degree 2

Let $f = \sum_{0 \leq i,j \leq 3} a_{ij} x^i y^{3-i} u^j v^{3-j}$. We have:

$$(f,f)_{3,3} = \frac{2}{3} a_{20} a_{13} - \frac{2}{9} a_{21} a_{12} + \frac{2}{9} a_{22} a_{11} - \frac{2}{3} a_{23} a_{10} - 2 a_{30} a_{03} + \frac{2}{3} a_{31} a_{02} - \frac{2}{3} a_{32} a_{01} + 2 a_{33} a_{00}.$$

Main theorem

Theorem (B.)

The algebra $K[V_{3,3}]^{\mathrm{SL}_2(K)\times\mathrm{SL}_2(K)\rtimes\mathbb{Z}/2\mathbb{Z}}$ *is generated by* 65 *homogeneous invariants, with* 10 *of them algebraically independant.*

Sketch of the proof of the theorem

- We select 10 homogeneous invariants (of degrees (2,4,4,6,6,8,8,10,12,14)) such that they define $\mathcal{N}_{V_{3,3}}$ (Hard, need a lot of tricks).

Sketch of the proof of the theorem

- We select 10 homogeneous invariants (of degrees (2,4,4,6,6,8,8,10,12,14)) such that they define $\mathcal{N}_{V_{3,3}}$ (Hard, need a lot of tricks).
- Using the Hilbert series, we add invariants degree by degree until we find the right dimension for each degree. (evaluation of invariants on many forms, and computation of the rank).

Implementation

Implementation in Magma: given a quadric of rank 4 and a cubic, computes the 65 invariants (instantly).

Thank you for your attention!