Geometric Iwasawa Theory

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- Iwasawa theory for $\mathbf{Z}_p$-towers of number fields. Example:

$$\# \operatorname{Cl}(\mathbf{Q}(\zeta_{p^n}))[p^\infty] = p^{\lambda n + \nu}, \quad n \gg 0$$

- Reminder: $\operatorname{Cl}(K)$ is (non-zero) fractional ideals in K modulo principal fractional ideals.

Iwasawa Theory for Number Fields and Function Fields

- Iwasawa theory for $\mathbf{Z}_p$-towers of number fields. Example:

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- Reminder: $\operatorname{Cl}(K)$ is (non-zero) fractional ideals in K modulo principal fractional ideals.
- Iwasawa Theory for $\mathbf{Z}_p$-towers $\{C_n\}$ of curves over k (function fields. Example: for nice towers when $\operatorname{char}(k) = p$,

$$\#\operatorname{Cl}(k(C_n))[p] = \#\operatorname{Jac}(C_n)(k)[p] = p^{\mu p^n+\nu}, \quad n \gg 0$$

- Reminder: $\operatorname{Jac}(C_n)$ moduli spaces for degree zero divisors modulo principle divisors

The "Physical" and "Motivic" Class Group

Let C be a curve over a finite field k with function field K.

- "physical" class group of C, $\mathrm{Cl}(K) = \mathrm{Jac}(C)(k)$ and Tate module studied most often (Abelian group, $\mathbf{Z}_\ell$-module)
- $\mathrm{Jac}(C)$ is an Abelian variety, and $\mathrm{Jac}(C)[p]$ is a group scheme (and $\mathrm{Jac}(C)[p^\infty]$ p-divisible group).

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- $\mathrm{Jac}(C)$ is an Abelian variety, and $\mathrm{Jac}(C)[p]$ is a group scheme (and $\mathrm{Jac}(C)[p^\infty]$ p-divisible group).
- $\mathrm{Jac}(C)[p](k)$ loses information if $\mathrm{char}(k) = p$.

Example

E an elliptic curve over $\mathbf{F}_p$. $E \simeq \mathrm{Jac}(E)$. The group scheme $E[p]$ has order p^2. But if E ordinary, $\#E[p](\overline{\mathbf{F}}_p) = p$ and

$$1 \to \mu_p \to E[p] \to \mathbf{Z}/p\mathbf{Z} \to 0 \quad \text{(split)}$$

The "Physical" and "Motivic" Class Group

Perspectives on $\text{Jac}(C)[p]$ in characteristic p:

1. Group scheme of order $p^{2g(C)}$ (built out of $\mathbf{Z}/p\mathbf{Z}$, μ_p, α_p ...)
2. Dieudonné module $\mathbf{D}(\text{Jac}(C)[p])$ ($k[F, V]$-module...)
3. de Rham cohomology $H^1_{dR}(C)$ (+Frobenius and Verschiebung)

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Something more concrete:

- $H^0(\Omega^1_C)$ with Cartier operator V_C; this vector space is isomorphic to $H^1_{dR}(C)[F]$ and has dimension the genus $g(C)$
- Study $H^0(\Omega^1_C)$ as $k[V_C]$-module:

Definition (p-rank and higher a-numbers)

$$f(C) := \dim_k H^0(\Omega^1_C)^{ss}, \qquad a^r(C) := \dim_k \ker V_C^r$$

$\mathbf{Z}_p$-towers of Curves

k finite field of characteristic p. Branched covers of curves over k

$$\cdots \to C_n \to \cdots \to C_2 \to C_1 \to C_0$$

with $\mathrm{Gal}(C_n/C_0) \simeq \mathbf{Z}/p^n\mathbf{Z}$

Example

- $C_0 = \mathbf{P}^1_k$, with function field $k(x)$
- Function field of C_1 splitting field of $y^p - y = f$ with $f \in k(x)$
- Function field of C_n via Artin-Schreier-Witt theory
- Poles of f are the branch points: orders control ramification.

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From now on, our $\mathbf{Z}_p$-towers will be totally ramified over a finite set $S \subset C_0(k)$, unramified elsewhere, and:

Definition (strictly monodromy stable)

for $Q \in S$, the n-th break in upper ram. filt. over Q is $d_Q p^{n-1}$.

Iwasawa Theory for Function Fields

Philosophy

Properties of the p-part of the motivic class group of curves in "reasonable" $\mathbf{Z}_p$-towers "behave regularly."

Example (consequence of Riemann-Hurwitz)

Genus: in a strictly monodromy stable $\mathbf{Z}_p$-tower

$$g(C_n) = ap^{2n} + bp^n + c$$

with $a, b, c \in \mathbf{Q}$ depending on the ramification $\{d_Q\}_{Q \in S}$.

Example

Results about p-part of "physical" class group $\mathrm{Jac}(C_n)(k)$ and Tate modules. (Mazur-Wiles, Gold-Kiselevsky, ...)

Philosophy

Properties of the p-part of the motivic class group of curves in "reasonable" $\mathbf{Z}_p$-towers "behave regularly."

Example (Davis-Wan-Xiao)

The slopes of curves of "basic" $\mathbf{Z}_p$-towers $\{C_n\}$ come in arithmetic progressions (strictly monodromy stable, $\#S = 1$ and $C_0 = \mathbf{P}^1$).

Reminder: the slopes are equivalent to the isogeny class of the p-divisible group $\mathrm{Jac}(C_n)[p^\infty]$. Defined using Newton polygon of numerator of the zeta function $Z(C_n, T)$.

a-numbers and Geometric Iwasawa Theory

For a strictly monodromy stable $\mathbf{Z}_p$-tower $\{C_n\}$ with $C_0 = \mathbf{P}^1$, $\#S = 1$, and ramification breaks dp^{n-1}:

Conjecture (with Bryden Cais)

- (asymptotic) $\lim_{n\to\infty} \frac{a(C_n)}{p^{2n}} = \lim_{n\to\infty} \frac{\dim_k \ker V_{C_n}}{p^{2n}} = \frac{(p-1)d}{4(p+1)p}$.
- (exact) there exists $b, c \in \mathbf{Q}$ (depending on the tower) s.t.

$$a(C_n) = \dim_k \ker V_{C_n} = \frac{(p-1)d}{4(p+1)p} p^{2n} + bp^n + c \text{ for } n \gg 0.$$

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- We conjecture similar behavior for $a^r(C_n) = \dim \ker V^r_{C_n}$ so $H^0(\Omega^1_{C_n})$ is well-behaved as $k[V_{C_n}]$-module.
- In this situation, $\# \mathrm{Cl}(C_n)[p] = 1$ and p-rank is always zero.
- We conjecture similar behavior for more complicated towers.

a-numbers and Artin-Schreier covers

$a(C)$	8	9	10
Number	6633	2988	379

Table: 10000 Artin-Schreier covers of $\mathbf{P}^1$, $p = 3$ and $d = 17$.

$a(C)$	10	11	12	13	14
Number	8021	1901	64	10	4

Table: 10000 Artin-Schreier covers of $\mathbf{P}^1$, $p = 5$ and $d = 11$.

- No analog of the Riemann-Hurwitz formula for a-numbers.
- Conjecture for a-numbers is deeper than regularity of genus.

a-numbers in $\mathbf{Z}_p$-Towers

Level:	1	2	3	4	5
$g(C_n) = g(C'_n)$	4	46	442	4060	36784
$a(C_n)$	2	19	154	1369	12304
$a(C'_n)$	2	18	153	1368	12303
$\delta(C_n)$	2	4	4	4	4
$\delta(C'_n)$	2	3	3	3	3

Table: $\mathbf{Z}_p$-towers with $p = 3$ and ramification invariant $d = 5$

$\delta(C_n)$ is the difference between our predicted main term and the a-number (i.e. basically the constant term c in the conjecture)

a-numbers and $\mathbf{Z}_p$-Towers

Work in progress for strictly monodromy stable $\mathbf{Z}_p$-towers $\{C_n\}$ with $C_0 = \mathbf{P}^1$, $\#S = 1$, and ramification breaks dp^{n-1}:

Theorem (with Bryden Cais, Joe Kramer-Miller, and James Upton)

$$\lim_{n\to\infty} \frac{a(C_n)}{p^{2n}} = \frac{(p-1)d}{4(p+1)p}.$$

Similar statement for higher a-numbers. Don't believe necessary that $C_0 = \mathbf{P}^1$, $\#S = 1$.

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Theorem (with Bryden Cais, Joe Kramer-Miller, and James Upton)

If p odd and $d|p-1$,

$$a(C_n) = \frac{(p-1)d}{4(p+1)p}\left(p^{2n-1}+1\right) - \begin{cases} 0 & d \text{ even} \\ (p-1)/(4d) & d \text{ odd} \end{cases}$$

Dwork Theory and L-functions

- The proof is inspired by work of Kramer-Miller and Upton on Newton polygons in more general $\mathbf{Z}_p$-towers. (i.e. isogeny class of $\mathrm{Jac}(C_n)[p^\infty]$)
- **Chasm**: the a-number and $\mathrm{Jac}(C_n)[p]$ are not isogeny invariant.

Dwork Theory and L-functions

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- **Chasm**: the a-number and $\mathrm{Jac}(C_n)[p]$ are not isogeny invariant.
- The main tool is information from Dwork theory about an equicharacteristic L-function. The tower determines

$$\rho_0 : \pi_1^{\text{ét}}(C_0 - S) \twoheadrightarrow \mathbf{Z}_p \to \mathbf{F}_p[[T]]^\times$$
$$a \mapsto (1+T)^a$$

Definition

The equicharacteristic L-function for the tower is $L(\rho_0, s)$.

The Role of the Regular Differentials

$$\begin{array}{ccc} \operatorname{Spec} R_n & \longrightarrow & C_n \\ \downarrow & & \downarrow {\scriptstyle \mathbf{Z}/p^n\mathbf{Z}} \\ \operatorname{Spec} k[x, x^{-1}] & \longrightarrow & \mathbf{P}^1 \end{array}$$

- R_n is $k[[T]]$-module with $1 + T$ acting as $1 \in \mathbf{Z}_p/p^n\mathbf{Z}_p$.
- $\Omega_n := H^0(\Omega^1_{C_n}) \hookrightarrow \Omega^1_{R_n} = R_n \frac{dx}{x}$
- Ω_∞ and R_∞ limiting $k[[T]]$-modules using trace maps.
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- Dwork trace formula relates L-function to operator Θ on R_∞ which is very close to Cartier operator.
- upper bound on Newton polygon of Θ from Dwork theory.
- lower bound on Hodge polygon of Θ from strictly stable monodromy and $k[[T]]$-module structure of R_∞.
- Newton polygon over Hodge polygon, and very close.
- (crossing chasm) Ω_n is isomorphic to image of

$$\Omega_\infty \hookrightarrow R_\infty \frac{dx}{x} \to R_\infty \frac{dx}{x} \otimes k[[T]]/(T^{p^n}).$$

A Picture Proof