Algebraic Geometry C^{2}odes in The sum-rank metric

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Linear codes and codes in the Hamming metric

k a field (think about $k = \mathbb{F}_q$), $\mathcal{H}$ a k-linear vector space endowed with a metric

Linear code $\mathcal{C}$: k-vector subspace of $\mathcal{H}$

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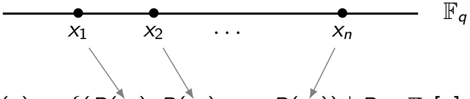

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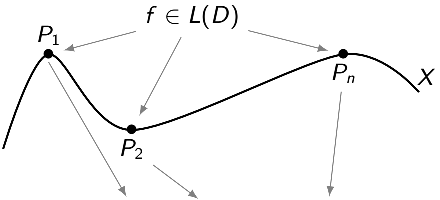

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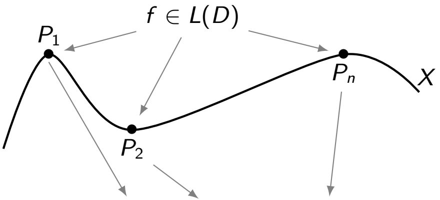

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✔ **Good parameters:** $n+1-g \leq \delta + d \leq n+1$

✔ **Longer codes**

General definitions

$\underline{V} = (V_1, \ldots, V_s)$ s-uple of k-vector spaces $(n_i = \dim_k V_i)$

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$n_i = 1 \ \forall i \quad \rightsquigarrow \quad$ codes of length s in the Hamming metric

Particular case and Singleton bound

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Singleton bound

The ℓ-parameters of $\mathcal{C}$ satisfy

$$d + \delta_\ell \leq n_\ell + 1.$$

Codes with parameters attaining this bound are called Maximum Sum-Rank Distance (MSRD).

Ore polynomials and Linearized Reed–Solomon codes

ℓ field, $\Phi : \ell \to \ell$ a ring homomorphism, $\ell^{\Phi=1} = k$, $[\ell : k] = r$,

$$\begin{array}{c} \ell \\ \Big\downarrow \Phi \\ k \end{array}$$

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As in the Hamming case, we can **try to overcome the problem using algebraic curves**
Main idea: consider Ore polynomials with coefficients in the function field of a curve

Divisors and Riemann–Roch spaces: classical theory

Definition

Let X be a nice curve, K its function field. A divisor on X is a formal finite sum

$$D = \sum_{\mathfrak{p} \in X} n_{\mathfrak{p}} \mathfrak{p} \quad \text{with } n_{\mathfrak{p}} \in \mathbb{Z} \text{ almost all zero.}$$

The group of divisors on X is denoted by $Div(X)$.
$D \in Div(X)$ is positive, $D \geq 0$, if $n_{\mathfrak{p}} \geq 0\ \forall \mathfrak{p}$. The degree of D is $\deg_X(D) = \sum_{\mathfrak{p} \in X} n_{\mathfrak{p}} \deg_X(\mathfrak{p})$.

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Riemann–Roch theorem

Let K_X denotes a canonical divisor on X. For any divisor $D \in \mathrm{Div}(X)$ we have

$$\dim_k L_X(D) = \deg_X(D) + 1 - g_X + \underbrace{\dim_k L_X(K_X - D)}_{=0 \text{ when } \deg_X(D) > 2g_X - 2.},$$

Our setting

$$\begin{array}{c} Y \\ \downarrow \pi \\ X \end{array}$$

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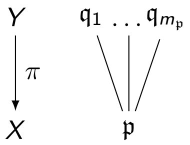

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- ☐ principal divisors associated to $f \in D_{L,x}$
  $\rightsquigarrow$ need to define a valuation
- ☐ Riemann–Roch spaces of $D_{L,x}$
- ☐ a Riemann–Roch theorem
- ☐ equivalent of "evaluate at a rational point"

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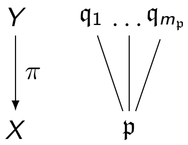

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Define the valuation map $w_{\mathfrak{q}_j,x} : D_{L_\mathfrak{p},x} \to \frac{1}{r}\mathbb{Z} \sqcup \{\infty\}_{(1 \le j \le m_\mathfrak{p})}$: for $f = f_0 + f_1 T + \cdots + f_{r-1} T^{r-1}$,

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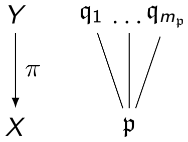
where $e_\mathfrak{q}$ denotes the ramification index of $\mathfrak{q}$.

Our setting



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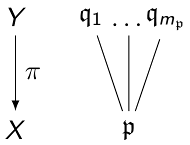

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Divisors and Riemann–Roch spaces over Ore polynomial rings

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Riemann's inequality for $\Lambda_{L,x}(E)$

For a divisor $E = \sum_{\mathfrak{q}\in Y} n_\mathfrak{q}\mathfrak{q} \in \mathrm{Div}_{\mathbb{Q}}(Y)$ the space $\Lambda_{L,x}(E)$ is finite dimensional over k and

$$\dim_k \Lambda_{L,x}(E) \geq r\cdot\deg_Y(E) - r\cdot(g_Y-1) - \frac{r^2}{2}\sum_{\mathfrak{p}\in X}\frac{b_\mathfrak{p}-1}{b_\mathfrak{p} e_\mathfrak{p}}\deg_X(\mathfrak{p}).$$

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Linearized Algebraic Geometry codes (B., Caruso, 2023)

Let $E = \sum_{\mathfrak{q}\in Y} n_\mathfrak{q}\mathfrak{q} \in \mathrm{Div}_\mathbb{Q}(Y)$. Chose $x \in K$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ rational places on X such that the hypotheses hold. Consider

$$\begin{array}{rcccl} \alpha : & \Lambda_{L,x}(E) & \longrightarrow & \prod_{i=1}^s \mathrm{End}_k(V_{\mathfrak{p}_i}) \\ & f & \mapsto & \left(\bar{\varepsilon}_{\mathfrak{p}_i}(f)\right)_{1\le i\le s}. \end{array}$$

The code $\mathcal{C}(x; E; \mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ is defined as the image of α.

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We study the parameters of the k-linear code $\mathcal{C}$ in $\prod_{i=1}^{s} \mathrm{End}_k(V_{\mathfrak{p}_i})$.

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Theorem (B., Caruso, 2023)

Assume $\deg_Y(E) < sr$. Assume the previous hypotheses and that $D_{L,x}$ contains no nonzero divisors. Then, the dimension δ and the minimum distance d of $\mathcal{C}(x; E; \mathfrak{p}_1, \ldots, \mathfrak{p}_s)$ satisfy

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Singleton bound: $rd + \delta \leq n + r$

We have: $rd + \delta \geq n + r - \left(r \cdot g_Y + \frac{r^2}{2} \sum_{\mathfrak{p} \in X} \frac{b_{\mathfrak{p}} - 1}{b_{\mathfrak{p}} e_{\mathfrak{p}}} \deg_X(\mathfrak{p}) \right)$

The case of isotrivial covers

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$X = \mathbb{P}^1_k$, $Y = \mathbb{P}^1_\ell$, $E = \frac{\delta}{r} \cdot \infty \in \operatorname{Div}_\mathbb{Q}(Y)$ $\rightsquigarrow$ linearized Reed–Solomon codes!
Our lower bounds $\Rightarrow$ MSRD codes

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AGC²T '17 AGC²T '19

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Merci de votre attention !

Questions?
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Remarks on the hypotheses

(H1) the algebra $D_{L,x}$ has no nonzero zero divisor

(H2) for all places $\mathfrak{q}$ above $\mathfrak{p}$, there exists $u_{\mathfrak{q}} \in L_{\mathfrak{q}}^{\times}$ such that $v_{\mathfrak{q}}(u_{\mathfrak{q}}) = \frac{e_{\mathfrak{p}}}{r} \cdot v_{\mathfrak{p}}(x)$ and

$$x = \prod_{\mathfrak{q}|\mathfrak{p}} N_{L_{\mathfrak{q}}/K_{\mathfrak{p}}}(u_{\mathfrak{q}})$$

Lemma

The hypothesis (H1) holds as soon as there exists a place $\mathfrak{p} \in X$ which is inert in Y and at which $v_{\mathfrak{p}}(x)$ is coprime with r.

Lemma

We assume that k is a finite field. Let $\mathfrak{p}$ be a place of X. If $\mathfrak{p}$ is unramified in Y and $v_{\mathfrak{p}}(x)$ is divisible by r, then (H2) holds.