Direttissimo Equidimensional Decomposition

Rafael Mohr[1,2]

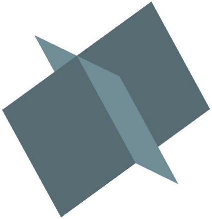
09.03.2023
Journées Nationales de Calcul Formel 2023
joint work with Christian Eder[1], Pierre Lairez[3] and Mohab Safey El Din[2]

[1]RPTU Kaiserslautern-Landau [2]Sorbonne Université Paris [3]Inria Saclay

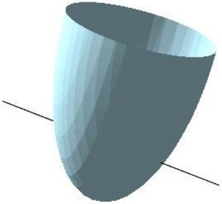
Problem Statement & State of the Art

Definition

Let $\mathbf{K}$ be an algebraically closed field. Algebraic set $X \subset \mathbf{K}^n$ *equidimensional* if all irreducible components of X have the same dimension.



equidimensional



not equidimensional

Definition

Let $\mathbf{K}$ be an algebraically closed field. Algebraic set $X \subset \mathbf{K}^n$ *equidimensional* if all irreducible components of X have the same dimension.

Given X as $X = V(f_1, \ldots, f_r)$ with $f_1, \ldots, f_r \in \mathbf{K}[x_1, \ldots, x_n]$,

compute a representation $X = \bigcup_{i=1}^{m} X_i$, each X_i equidimensional.

Definition

Let $\mathbf{K}$ be an algebraically closed field. Algebraic set $X \subset \mathbf{K}^n$ *equidimensional* if all irreducible components of X have the same dimension.

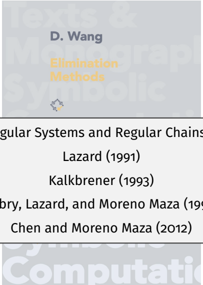
Given X as $X = V(f_1, \ldots, f_r)$ with $f_1, \ldots, f_r \in \mathbf{K}[x_1, \ldots, x_n]$,

compute a representation $X = \bigcup_{i=1}^{m} X_i$, each X_i equidimensional.

Applications in

- Prime and primary decomposition (Gianni, Trager, and Zacharias 1988)
- Robotics (e.g. Pascual-Escudero, Nayak, Briot, Kermorgant, Martinet, Safey El Din, and Chaumette 2021; García Fontán, Nayak, Briot, and Safey El Din 2022, and references therein)
- Real algebraic geometry (e.g. Aubry, Rouillier, and Safey El Din 2002; Safey El Din and Schost 2004)
- Automated theorem proving and geometry (e.g. Wu and Gao 2007; Yang, Hou, and Xia 2001; Yang, Hou, and Xia 1998; Chen, Corless, Moreno Maza, Yu, and Zhang 2013)

State of the art



Triangular Systems and Regular Chains, e.g.

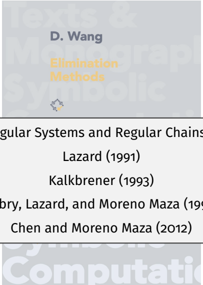
Lazard (1991)

Kalkbrener (1993)

Aubry, Lazard, and Moreno Maza (1999)

Chen and Moreno Maza (2012)

State of the art



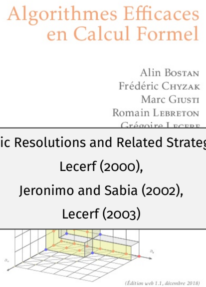
Triangular Systems and Regular Chains, e.g.

Lazard (1991)

Kalkbrener (1993)

Aubry, Lazard, and Moreno Maza (1999)

Chen and Moreno Maza (2012)



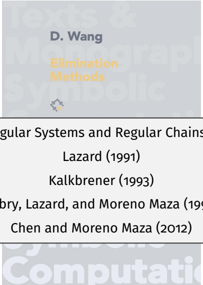
Geometric Resolutions and Related Strategies, e.g.

Lecerf (2000),

Jeronimo and Sabia (2002),

Lecerf (2003)

State of the art



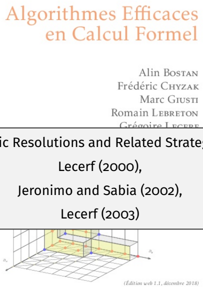
Triangular Systems and Regular Chains, e.g.

Lazard (1991)

Kalkbrener (1993)

Aubry, Lazard, and Moreno Maza (1999)

Chen and Moreno Maza (2012)



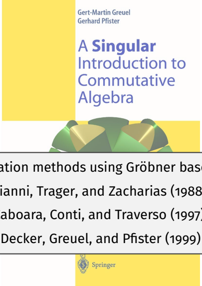
Geometric Resolutions and Related Strategies, e.g.

Lecerf (2000),

Jeronimo and Sabia (2002),

Lecerf (2003)

State of the art



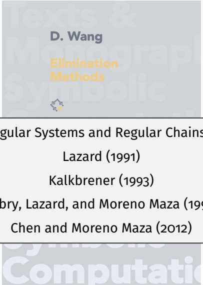
Elimination methods using Gröbner bases, e.g.

Gianni, Trager, and Zacharias (1988),

Caboara, Conti, and Traverso (1997),

Decker, Greuel, and Pfister (1999)

State of the art



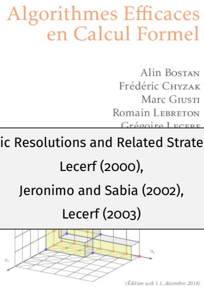
Triangular Systems and Regular Chains, e.g.

Lazard (1991)

Kalkbrener (1993)

Aubry, Lazard, and Moreno Maza (1999)

Chen and Moreno Maza (2012)

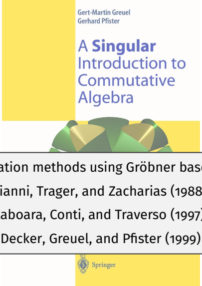


Geometric Resolutions and Related Strategies, e.g.

Lecerf (2000),

Jeronimo and Sabia (2002),

Lecerf (2003)

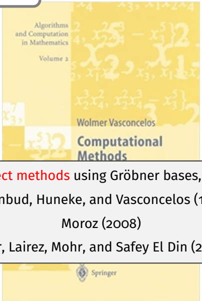


Elimination methods using Gröbner bases, e.g.

Gianni, Trager, and Zacharias (1988),

Caboara, Conti, and Traverso (1997),

Decker, Greuel, and Pfister (1999)



Direct methods using Gröbner bases, e.g.

Eisenbud, Huneke, and Vasconcelos (1992)

Moroz (2008)

Eder, Lairez, Mohr, and Safey El Din (2022)

The Incremental Strategy

To decompose $V(f_1, \ldots, f_r)$:

To decompose $V(f_1, \ldots, f_r)$:

$$\boxed{V(f_1)} \xrightarrow{\text{Dec. } V(f_1, f_2)} \boxed{V(f_1, f_2) = \bigcup_i X_i} \xrightarrow{\text{Dec. each } X_i \cap V(f_3)} \boxed{V(f_1, f_2, f_3) = \bigcup_i \bigcup_j X_{ij}}$$

To decompose $V(f_1, \ldots, f_r)$:

$$\boxed{V(f_1)} \xrightarrow{\text{Dec. } V(f_1, f_2)} \boxed{V(f_1, f_2) = \bigcup_i X_i} \xrightarrow{\text{Dec. each } X_i \cap V(f_3)} \boxed{V(f_1, f_2, f_3) = \bigcup_i \bigcup_j X_{ij}}$$

To decompose $V(f_1, \ldots, f_r)$:

$$\boxed{V(f_1)} \xrightarrow{\text{Dec. } V(f_1, f_2)} \boxed{V(f_1, f_2) = \bigcup_i X_i} \xrightarrow{\text{Dec. each } X_i \cap V(f_3)} \boxed{V(f_1, f_2, f_3) = \bigcup_i \bigcup_j X_{ij}}$$

To decompose $V(f_1, \ldots, f_r)$:

$$\boxed{V(f_1)} \xrightarrow{\text{Dec. } V(f_1, f_2)} \boxed{V(f_1, f_2) = \bigcup_i X_i} \xrightarrow{\text{Dec. each } X_i \cap V(f_3)} \boxed{V(f_1, f_2, f_3) = \bigcup_i \bigcup_j X_{ij}}$$

Have reduced to:

Given X equidimensional, $f \in \mathbf{K}[x_1, \ldots, x_n]$

Compute equidimensional decomposition of $X \cap V(f)$

$X \subset \mathbf{K}^n$ equidimensional, $f \in \mathbf{K}[x_1, \ldots, x_n]$.

Want: Equidimensional decomposition of $X \cap V(f)$.

$X \subset \mathbf{K}^n$ equidimensional, $f \in \mathbf{K}[x_1, \ldots, x_n]$.

Want: Equidimensional decomposition of $X \cap V(f)$.

$$V \text{ irreducible } \Rightarrow \begin{cases} f|_V = 0 \textbf{ or} \\ f|_V \text{ generically } \neq 0 \end{cases}$$

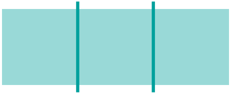
$X \subset \mathbf{K}^n$ equidimensional, $f \in \mathbf{K}[x_1, \ldots, x_n]$.

Want: Equidimensional decomposition of $X \cap V(f)$.

$$V \text{ irreducible} \Rightarrow \begin{cases} f|_V = 0 \textbf{ or} \\ f|_V \text{ generically } \neq 0 \end{cases}$$

Apply this alternative to every irreducible component of X:

$$X = V_1 \cup V_2 \cup V_3$$



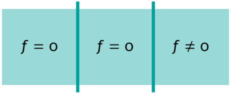
$X \subset \mathbf{K}^n$ equidimensional, $f \in \mathbf{K}[x_1, \ldots, x_n]$.

Want: Equidimensional decomposition of $X \cap V(f)$.

$$V \text{ irreducible } \Rightarrow \begin{cases} f|_V = 0 \textbf{ or} \\ f|_V \text{ generically } \neq 0 \end{cases}$$

Apply this alternative to every irreducible component of X:

$$X = V_1 \cup V_2 \cup V_3$$



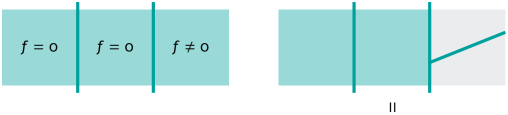
$X \subset \mathbf{K}^n$ equidimensional, $f \in \mathbf{K}[x_1, \ldots, x_n]$.

Want: Equidimensional decomposition of $X \cap V(f)$.

$$V \text{ irreducible } \Rightarrow \begin{cases} f|_V = 0 \textbf{ or} \\ f|_V \text{ generically } \neq 0 \end{cases}$$

Apply this alternative to every irreducible component of X:

$X = V_1 \cup V_2 \cup V_3$ $\qquad$ $X \cap V(f)$

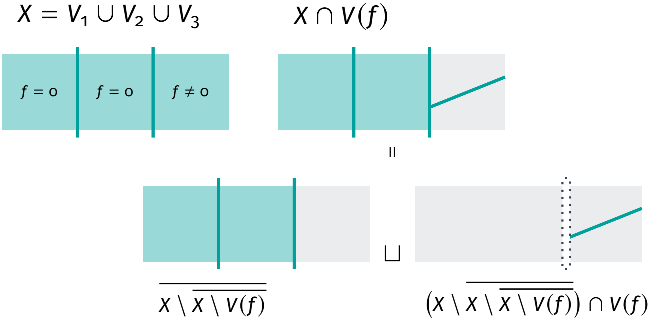


$X \subset \mathbf{K}^n$ equidimensional, $f \in \mathbf{K}[x_1, \ldots, x_n]$.

Want: Equidimensional decomposition of $X \cap V(f)$.

$$V \text{ irreducible} \implies \begin{cases} f|_V = 0 \textbf{ or} \\ f|_V \text{ generically } \neq 0 \end{cases}$$

Apply this alternative to every irreducible component of X:

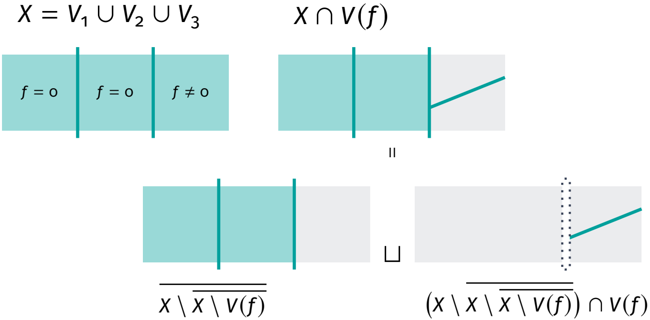


$X \subset \mathbf{K}^n$ equidimensional, $f \in \mathbf{K}[x_1, \ldots, x_n]$.

Want: Equidimensional decomposition of $X \cap V(f)$.

$$V \text{ irreducible } \Rightarrow \begin{cases} f|_V = 0 \textbf{ or} \\ f|_V \text{ generically } \neq 0 \end{cases}$$

Apply this alternative to every irreducible component of X:

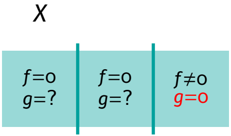


2000; Lecerf 2003; Moroz 2008; Eder, Lairez, Mohr, and Safey El Din 2022; Lazard 1991)

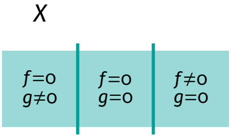
Our Algorithm

Black box function $\mathrm{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.

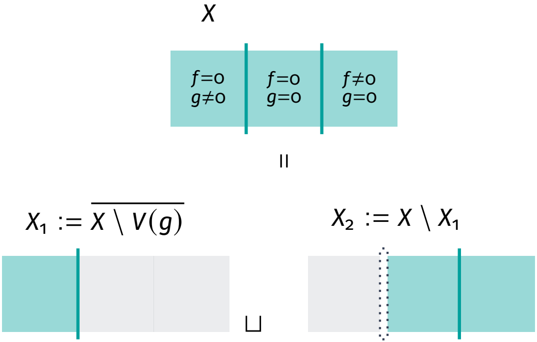
Black box function $\text{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.



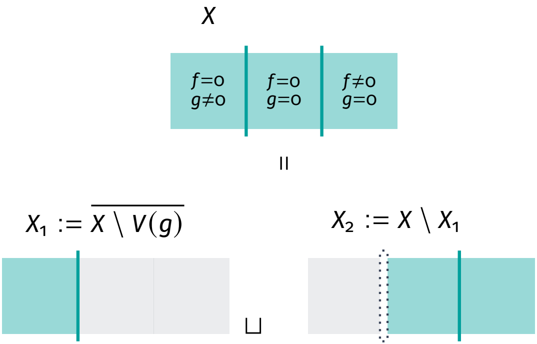
Black box function $\mathrm{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.



Black box function $\mathrm{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.

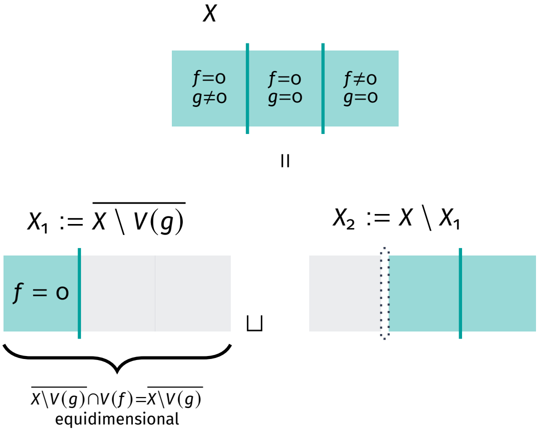


Black box function $\mathrm{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.



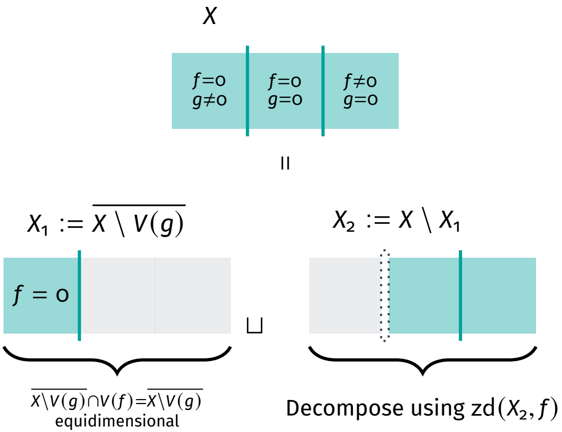
Decompose $X \cap V(f)$ by decomposing $X_1 \cap V(f), X_2 \cap V(f)$

Black box function $\mathrm{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.



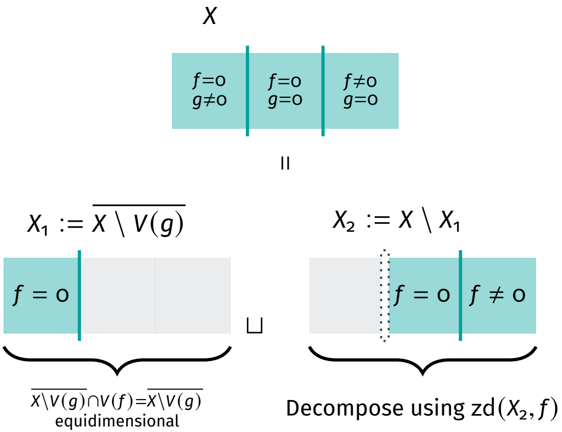
Decompose $X \cap V(f)$ by decomposing $X_1 \cap V(f), X_2 \cap V(f)$

Black box function $\mathrm{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.



Decompose $X \cap V(f)$ by decomposing $X_1 \cap V(f), X_2 \cap V(f)$

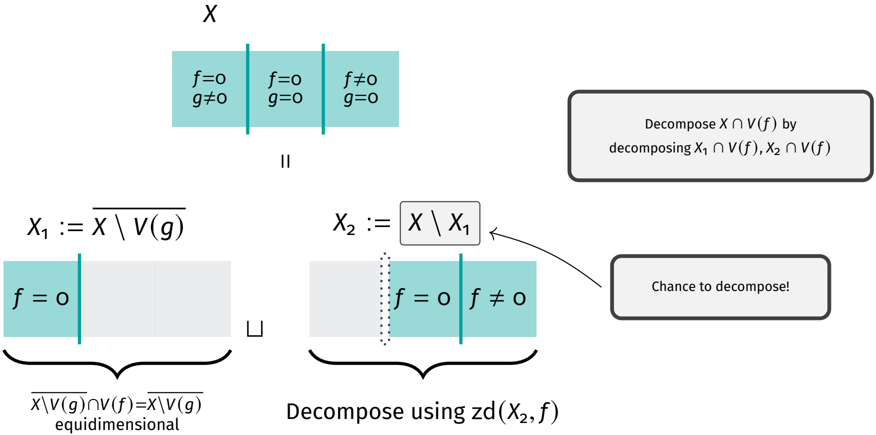
Black box function $\mathrm{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.



Decompose $X \cap V(f)$ by decomposing $X_1 \cap V(f), X_2 \cap V(f)$

Split as often as possible
as irredundantly as possible

Black box function $\mathrm{zd}(X, f)$: returns g s.t. $gf|_X = 0$, $g|_X \neq 0$.



Split as often as possible
as irredundantly as possible

Given algebraic sets X, Z, how to represent $X \setminus Z$?

Given algebraic sets X, Z, how to represent $X \setminus Z$?
Break down into sets $Y \setminus V(h)$, Y algebraic, h a singleton.

Given algebraic sets X, Z, how to represent $X \setminus Z$?
Break down into sets $Y \setminus V(h)$, Y algebraic, h a singleton.

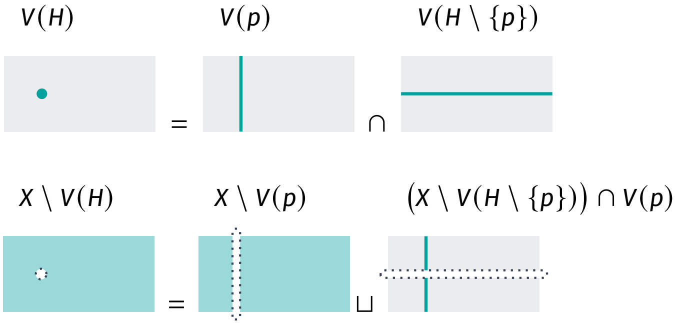
Say $Z = V(H)$, $H \subset \mathbf{K}[\mathbf{x}]$ finite set. For $p \in H$,

Given algebraic sets X, Z, how to represent $X \setminus Z$?
Break down into sets $Y \setminus V(h)$, Y algebraic, h a singleton.

Say $Z = V(H)$, $H \subset \mathbf{K}[\mathbf{x}]$ finite set. For $p \in H$,

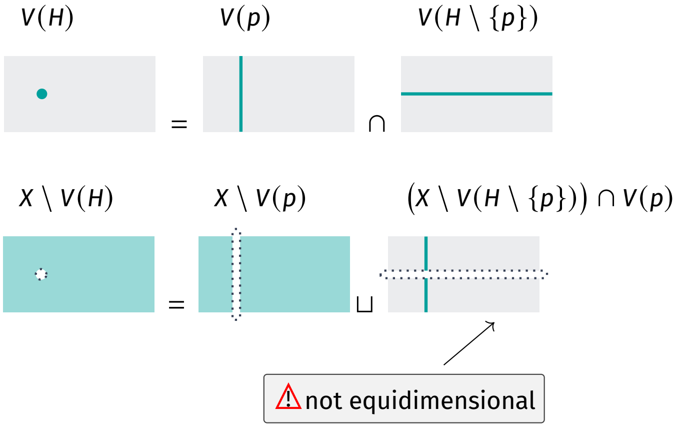
Given algebraic sets X, Z, how to represent $X \setminus Z$?
Break down into sets $Y \setminus V(h)$, Y algebraic, h a singleton.

Say $Z = V(H)$, $H \subset \mathbf{K}[\mathbf{x}]$ finite set. For $p \in H$,



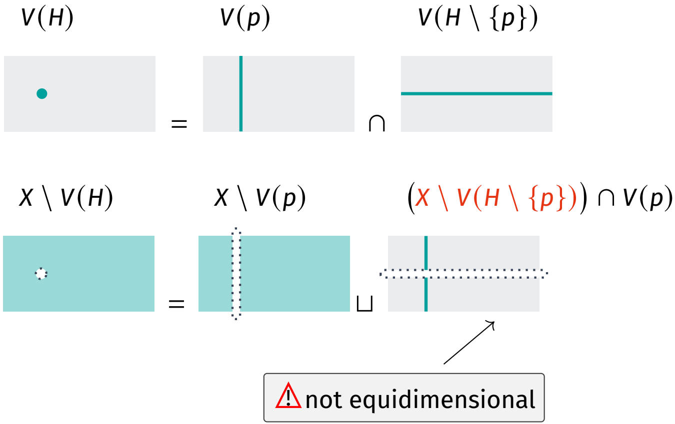
Given algebraic sets X, Z, how to represent $X \setminus Z$?
Break down into sets $Y \setminus V(h)$, Y algebraic, h a singleton.

Say $Z = V(H)$, $H \subset \mathbf{K}[\mathbf{x}]$ finite set. For $p \in H$,



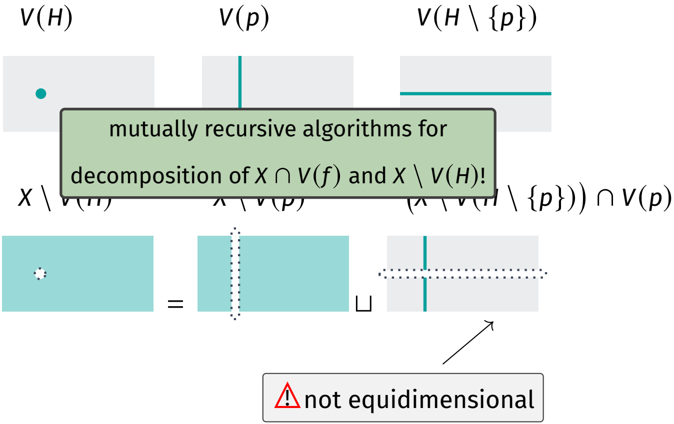
Given algebraic sets X, Z, how to represent $X \setminus Z$?
Break down into sets $Y \setminus V(h)$, Y algebraic, h a singleton.

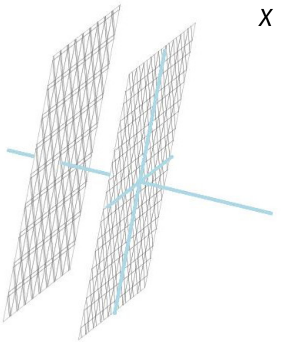
Say $Z = V(H)$, $H \subset \mathbf{K}[\mathbf{x}]$ finite set. For $p \in H$,



Given algebraic sets X, Z, how to represent $X \setminus Z$?
Break down into sets $Y \setminus V(h)$, Y algebraic, h a singleton.

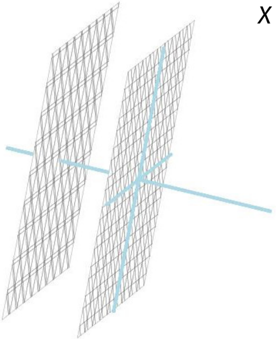
Say $Z = V(H)$, $H \subset \mathbf{K}[\mathbf{x}]$ finite set. For $p \in H$,





$X = V(xy, xz, yz), f = x(x-1)$

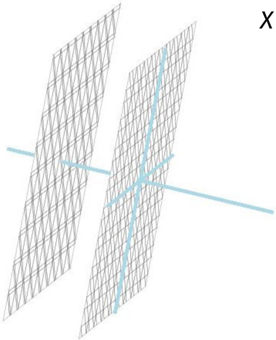
$yf = yx(x-1) = 0$ on X



$X = V(xy, xz, yz), f = x(x-1)$

$yf = yx(x-1) = 0$ on X

$X \xrightarrow{zd(X,f) = y} \overline{X \setminus V(y)}$

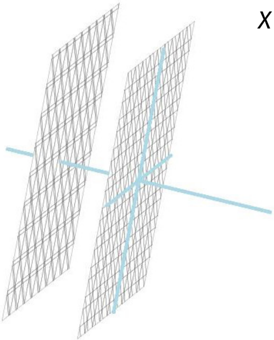


$X = V(xy, xz, yz), f = x(x-1)$

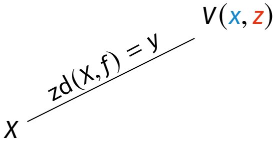
$yf = yx(x-1) = 0$ on X

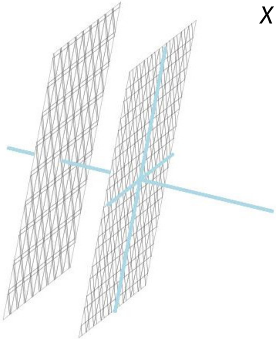
$\overline{V(xy, xz, yz) \setminus V(y)}$

$X \xrightarrow{\ zd(X,f) = y\ }$

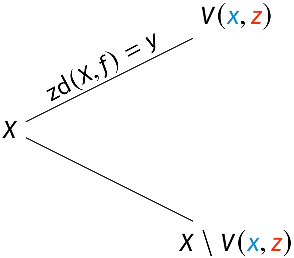


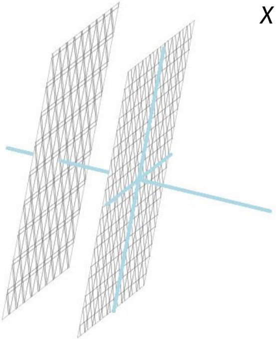
$X = V(xy, xz, yz), f = x(x-1)$



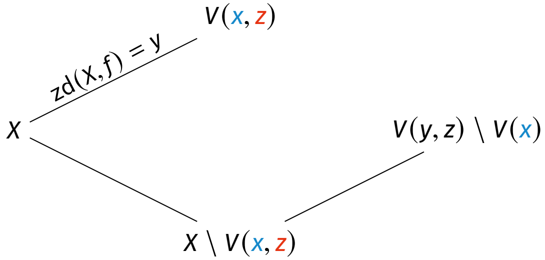


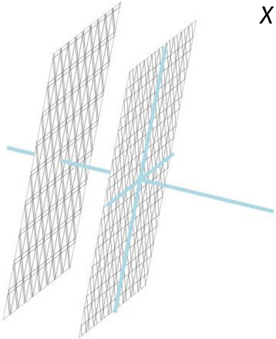
$X = V(xy, xz, yz), f = x(x-1)$



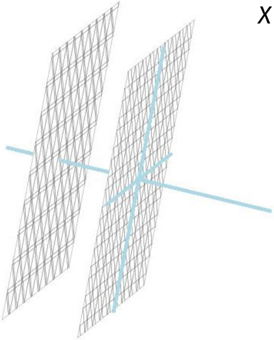


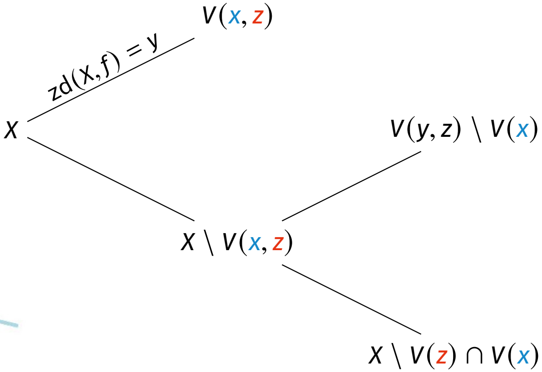
$X = V(xy, xz, yz), f = x(x-1)$



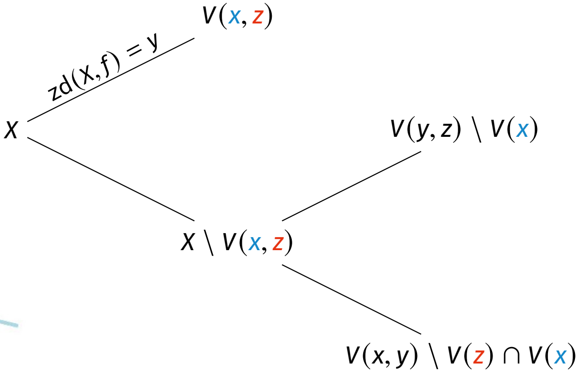


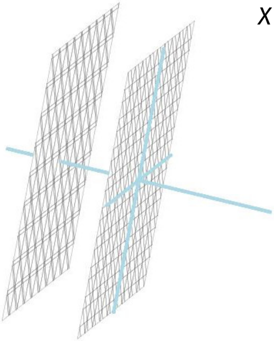
$X = V(xy, xz, yz), f = x(x-1)$



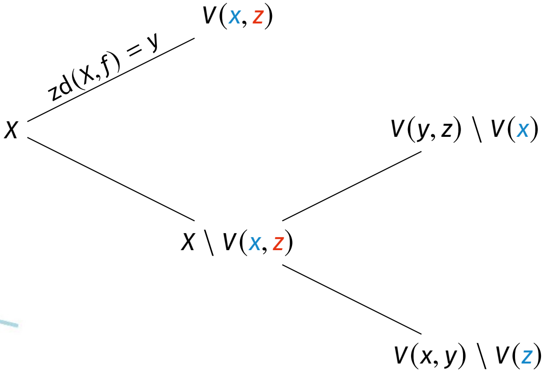


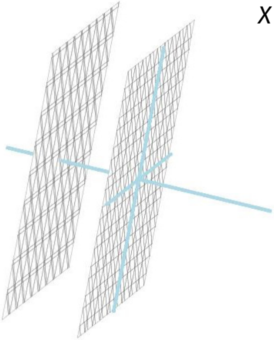
$X = V(xy, xz, yz), f = x(x-1)$



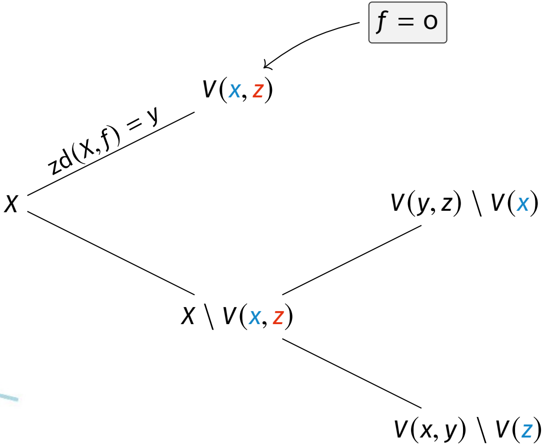


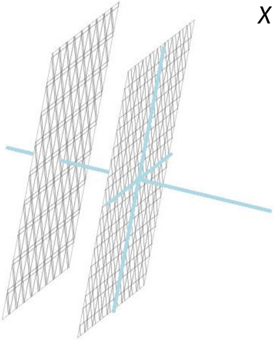
$X = V(xy, xz, yz), f = x(x-1)$



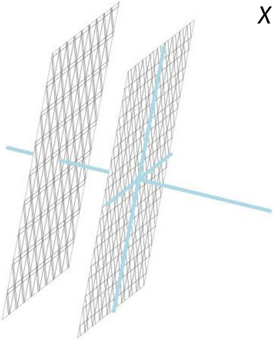


$X = V(xy, xz, yz), f = x(x-1)$

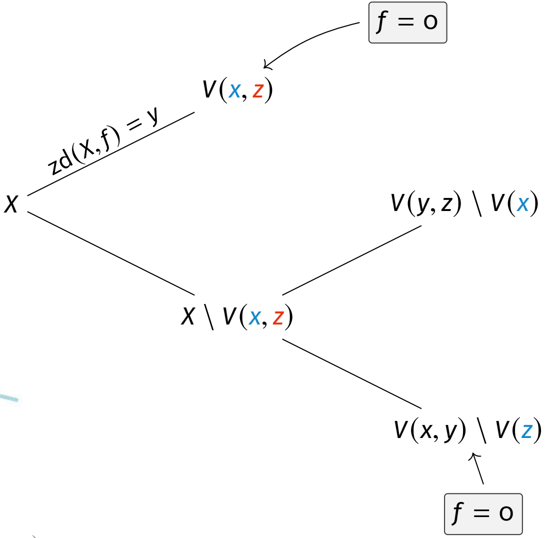


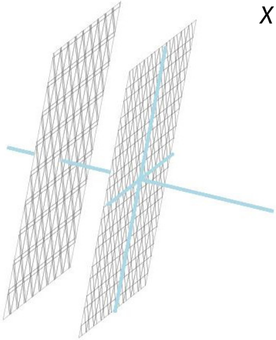


$X = V(xy, xz, yz), f = x(x-1)$

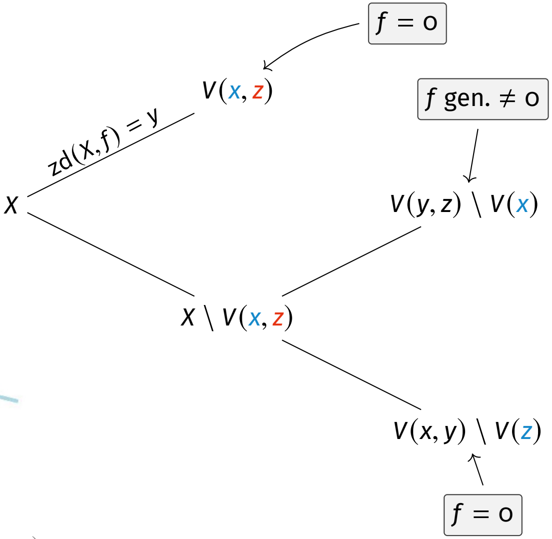


$X = V(xy, xz, yz), f = x(x-1)$





$X = V(xy, xz, yz), f = x(x-1)$



$f = 0$

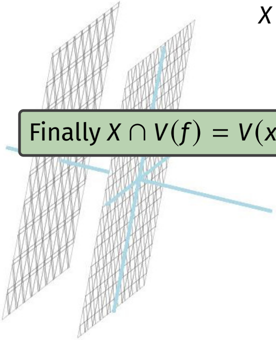
$V(x, z)$

f gen. $\neq 0$

$\text{zd}(X, f) = y$

X

$V(y, z) \setminus V(x)$



Finally $X \cap V(f) = V(x, z) \sqcup V(x, y) \setminus V(z) \sqcup V(y, z, x - 1) \setminus V(x)$

$V(x, y) \setminus V(z)$

$f = 0$

$X = V(xy, xz, yz), f = x(x - 1)$

“Naive” data structure:

- (F, h) modelling $X := V(F) \setminus V(h)$ ($F \subset \mathbf{K}[\mathbf{x}]$ finite, $h \in \mathbf{K}[\mathbf{x}]$).
- Gröbner basis of $I(X) := \{f \in \mathbf{K}[\mathbf{x}] \mid f|_X = 0\} = (\langle F \rangle : g^{\infty})$.

"Naive" data structure:

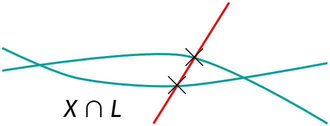
- (F, h) modelling $X := V(F) \setminus V(h)$ ($F \subset \mathbf{K}[\mathbf{x}]$ finite, $h \in \mathbf{K}[\mathbf{x}]$).
- Gröbner basis of $I(X) := \{f \in \mathbf{K}[\mathbf{x}] \mid f|_X = 0\} = (\langle F \rangle : g^\infty)$.

$X \cap V(f)$ *already equidimensional* if

$f|_X = 0$ or $f \neq 0$ generically on X

"Naive" data structure:

- (F, h) modelling $X := V(F) \setminus V(h)$ ($F \subset \mathbf{K}[\mathbf{x}]$ finite, $h \in \mathbf{K}[\mathbf{x}]$).
- Gröbner basis of $I(X) := \{f \in \mathbf{K}[\mathbf{x}] \mid f|_X = 0\} = (\langle F \rangle : g^\infty)$.



$X \cap V(f)$ *already equidimensional* if

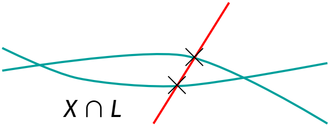
$f|_X = 0$ or $f \neq 0$ generically on $X \Leftrightarrow$

this happens at randomly selected

points on X

"Naive" data structure:

- (F, h) modelling $X := V(F) \setminus V(h)$ ($F \subset \mathbf{K}[\mathbf{x}]$ finite, $h \in \mathbf{K}[\mathbf{x}]$).
- Gröbner basis of $I(X) := \{f \in \mathbf{K}[\mathbf{x}] \mid f|_X = 0\} = (\langle F \rangle : g^\infty)$.



L general affine subspace of complementary dimension

$X \cap V(f)$ *already equidimensional* if

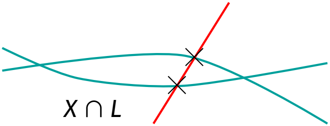
$f|_X = 0$ or $f \neq 0$ generically on X

$f|_X = 0$ or gen. $\neq 0$ iff
$f|_{X \cap L} = 0$ or gen. $\neq 0$

$\Longrightarrow$ Avoid knowing a GB of $I(X)$!

“Naive” data structure:

- (F, h) modelling $X := V(F) \setminus V(h)$ ($F \subset \mathbf{K}[\mathbf{x}]$ finite, $h \in \mathbf{K}[\mathbf{x}]$).
- Gröbner basis of $I(X) := \{f \in \mathbf{K}[\mathbf{x}] \mid f|_X = 0\} = (\langle F \rangle : g^\infty)$.



L general affine subspace of complementary dimension

$X \cap V(f)$ *already equidimensional* if

$f|_X = 0$ or $f \neq 0$ generically on X

$f|_X = 0$ or gen. $\neq 0$ iff
$f|_{X \cap L} = 0$ or gen. $\neq 0$

$\Longrightarrow$ Avoid knowing a GB of $I(X)$!

$X \cap L$ called *lifting fiber* or *witness set* (Lecerf 2003; Sommese, Verschelde, and Wampler 2005)

name	nb. comp.	Our Alg.	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✖
C1	4	129	> 1h	> 1h	> 1h	> 1h	✖
C2	4	0.3	100	152	> 1h	2.0	✖
C3	13	10	55	7	0.3	1.5	✖
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	< 0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	< 0.1	4	1.7	1.2	0.8	6
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✖
sys2161	33	8	29 ×3.8	> 1h	> 1h	8 ×1.0	1196 ×159
sys2874	5	✋ 0.3	202	1.9	8	10	> 1h
sys2880	50	4 ×2.4	144 ×80	1.8	3.4 ×1.9	4 ×2.2	⚠ 324 ×180
sys2882		> 1h	39	> 1h	> 1h	> 1h	✖

name	nb. comp.	Our Alg.	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✖
C1	4	129	> 1h	> 1h	> 1h	> 1h	✖
C2	4	0.3	100	152	> 1h	2.0	✖
C3	13	10	55	7	0.3	1.5	✖
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	< 0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	< 0.1	4	1.7	1.2	0.8	6
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✖
sys2161	33	8	29 ×3.8	> 1h	> 1h	8 ×1.0	1196 ×159
sys2874	5	✋ 0.3	202	1.9	8	10	> 1h
sys2880	50	4 ×2.4	144 ×80	1.8	3.4 ×1.9	4 ×2.2	⚠ 324 ×180
sys2882		> 1h	39	> 1h	> 1h	> 1h	✖

Maple: Regular Chains/Triangular Decomposition

name	nb. comp.	Our Alg.	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✗
C1	4	129	> 1h	> 1h	> 1h	> 1h	✗
C2	4	0.3	100	152	> 1h	2.0	✗
C3	13	10	55	7	0.3	1.5	✗
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	< 0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	< 0.1	4	1.7	1.2	0.8	6
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✗
sys2161	33	8	29 ×3.8	> 1h	> 1h	8 ×1.0	1196 ×159
sys2874	5	✋ 0.3	202	1.9	8	10	> 1h
sys2880	50	4 ×2.4	144 ×80	1.8	3.4 ×1.9	4 ×2.2	⚠ 324 ×180
sys2882		> 1h	39	> 1h	> 1h	> 1h	✗

Oscar: Homological/Direct method (Eisenbud, Huneke, and Vasconcelos 1992)

name	nb. comp.	Our Alg.	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✗
C1	4	129	> 1h	> 1h	> 1h	> 1h	✗
C2	4	0.3	100	152	> 1h	2.0	✗
C3	13	10	55	7	0.3	1.5	✗
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	< 0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	< 0.1	4	1.7	1.2	0.8	6
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✗
sys2161	33	8	29 ×3.8	> 1h	> 1h	8 ×1.0	1196 ×159
sys2874	5	✋ 0.3	202	1.9	8	10	> 1h
sys2880	50	4 ×2.4	144 ×80	1.8	3.4 ×1.9	4 ×2.2	⚠ 324 ×180
sys2882		> 1h	39	> 1h	> 1h	> 1h	✗

Magma: Elimination methods

name	nb. comp.	Our Alg.	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✗
C1	4	129	> 1h	> 1h	> 1h	> 1h	✗
C2	4	0.3	100	152	> 1h	2.0	✗
C3	13	10	55	7	0.3	1.5	✗
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	< 0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	< 0.1	4	1.7	1.2	0.8	6
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✗
sys2161	33	8	29 ×3.8	> 1h	> 1h	8 ×1.0	1196 ×159
sys2874	5	✋ 0.3	202	1.9	8	10	> 1h
sys2880	50	4 ×2.4	144 ×80	1.8	3.4 ×1.9	4 ×2.2	⚠ 324 ×180
sys2882		> 1h	39	> 1h	> 1h	> 1h	✗

Bertini: Numerical Solver

name	nb. comp.	Our Alg.	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✖
C1	4	129	> 1h	> 1h	> 1h	> 1h	✖
C2	4	0.3	100	152	> 1h	2.0	✖
C3	13	10	55	7	0.3	1.5	✖
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	< 0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	< 0.1	4	1.7	1.2	0.8	6
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✖
sys2161	33	8	29 ×3.8	> 1h	> 1h	8 ×1.0	1196 ×159
sys2874	5	✋ 0.3	202	1.9	8	10	> 1h
sys2880	50	4 ×2.4	144 ×80	1.8	3.4 ×1.9	4 ×2.2	⚠ 324 ×180
sys2882		> 1h	39	> 1h	> 1h	> 1h	✖

light green ~best timing to dark red ~worst timing

name	nb. comp.	Our Alg.	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✗
C1	4	129	> 1h	> 1h	> 1h	> 1h	✗
C2	4	0.3	100	152	> 1h	2.0	✗
C3	13	10	55	7	0.3	1.5	✗
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	< 0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	< 0.1	4	1.7	1.2	0.8	6
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✗
sys2161	33	8	29 ×3.8	> 1h	> 1h	8 ×1.0	1196 ×159
sys2874	5	✋ 0.3	202	1.9	8	10	> 1h
sys2880	50	4 ×2.4	144 ×80	1.8	3.4 ×1.9	4 ×2.2	⚠ 324 ×180
sys2882		> 1h	39	> 1h	> 1h	> 1h	✗

⚠: Bertini reports different degree per dimension

name	nb. comp.	Our Alg.	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✗
C1	4	129	> 1h	> 1h	> 1h	> 1h	✗
C2	4	0.3	100	152	> 1h	2.0	✗
C3	13	10	55	7	0.3	1.5	✗
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	< 0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	< 0.1	4	1.7	1.2	0.8	6
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✗
sys2161	33	8	29 ×3.8	> 1h	> 1h	8 ×1.0	1196 ×159
sys2874	5	✋ 0.3	202	1.9	8	10	> 1h
sys2880	50	4 ×2.4	144 ×80	1.8	3.4 ×1.9	4 ×2.2	⚠ 324 ×180
sys2882		> 1h	39	> 1h	> 1h	> 1h	✗

✋: Minor preparation of input system

available here:

Christian Eder, Pierre Lairez, Rafael Mohr, Mohab Safey El Din - A Direttissimo Algorithm for Equidimensional Decomposition, arXiv:2302.08174

available here:

Christian Eder, Pierre Lairez, Rafael Mohr, Mohab Safey El Din - A Direttissimo Algorithm for Equidimensional Decomposition, arXiv:2302.08174

implementation:

OSCAR SYMBOLIC TOOLS + msolve

available here:

Christian Eder, Pierre Lairez, Rafael Mohr, Mohab Safey El Din - A Direttissimo Algorithm for Equidimensional Decomposition, arXiv:2302.08174

implementation:

OSCAR SYMBOLIC TOOLS + msolve

Thank you!