Refined F_5 algorithms for ideals of minors of square matrices

Sriram Gopalakrishnan, Vincent Neiger, Mohab Safey El Din

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Sorbonne Université

Determinantal systems

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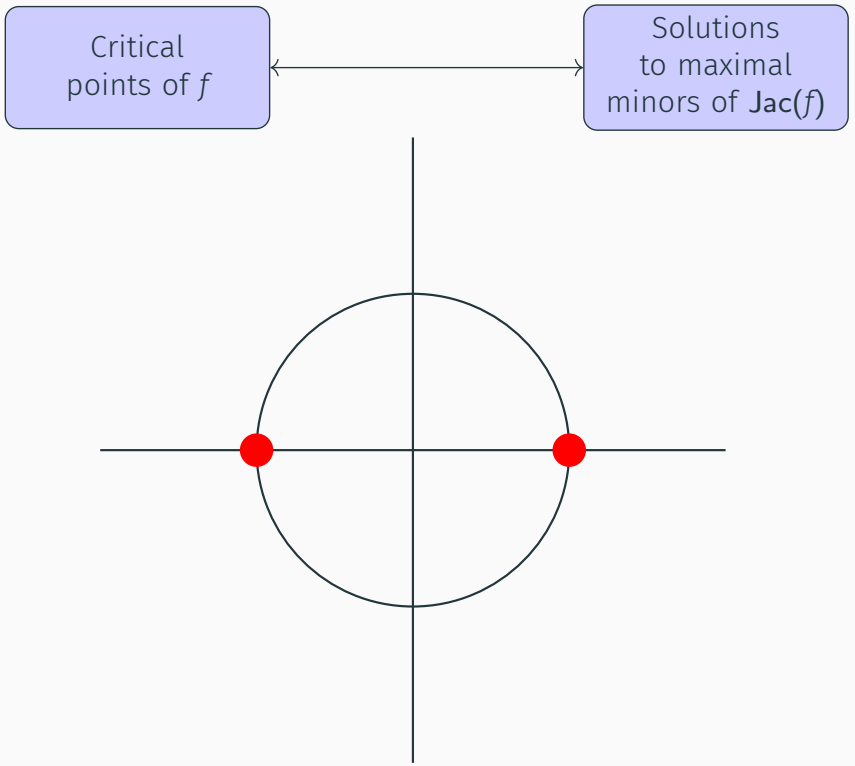

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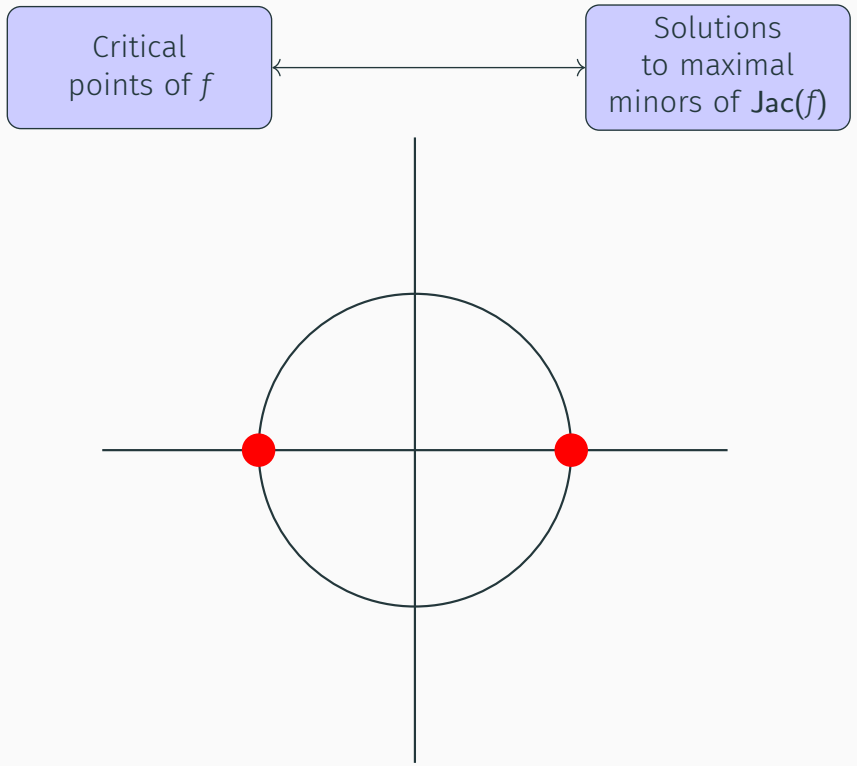
Post-quantum crypto
(multivariate and code-based cryptography)



The MinRank Problem

$f_{i,j}$ are **linear forms** in $\mathbb{k}[x_1, \ldots, x_k]$.
Find $a \in \overline{\mathbb{k}}^k$ with $\mathbf{rank}(M(a)) \leq r$.

Gröbner bases

$F \subseteq \mathbb{k}[x_1, \ldots, x_k]$ a polynomial system.

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Definition (Gröbner bases)

A $\succ$-*Gröbner basis* is a finite generating set G for $\langle F \rangle$ such that $\langle \mathsf{LM}_\succ(G) \rangle = \mathsf{LM}_\succ(\langle F \rangle)$.

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Theorem (Buchberger's criterion, [Buchberger, 1976])

$g_1, \ldots, g_m$ is a $\succ$-Gröbner basis for $\langle g_1, \ldots, g_m \rangle$ if and only if all S-pairs reduce to zero upon division by $g_1, \ldots, g_m$.

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Complexity

Worst case is doubly exponential in the number of variables. [Mayr, Mayer, 1982]

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For zero-dimensional systems:

$$\max_{f \in F} \{\deg f\}^{O(\#\text{ of variables})} \qquad \text{[Lazard, 1983]}$$

Macaulay matrices

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$$\begin{cases} 2x^2 + 11xy - y^2 \\ 4x^2 + xy - 2y^2 \\ -6x^2 - xy + y^2 \end{cases}$$

$$\begin{array}{c} \begin{array}{cccc} x^3 & x^2y & xy^2 & y^3 \end{array} \\ \begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

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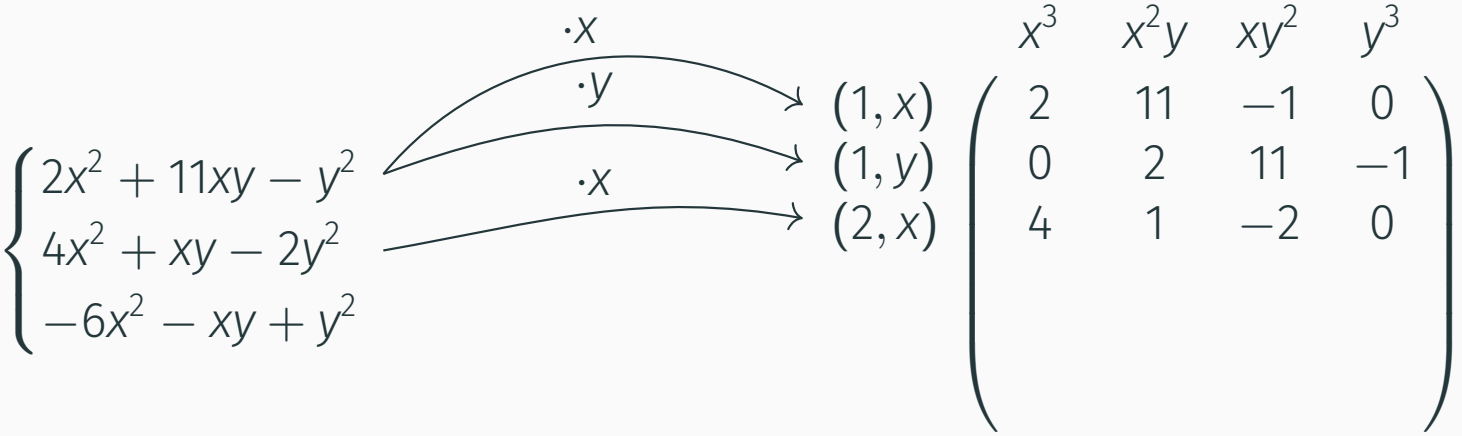

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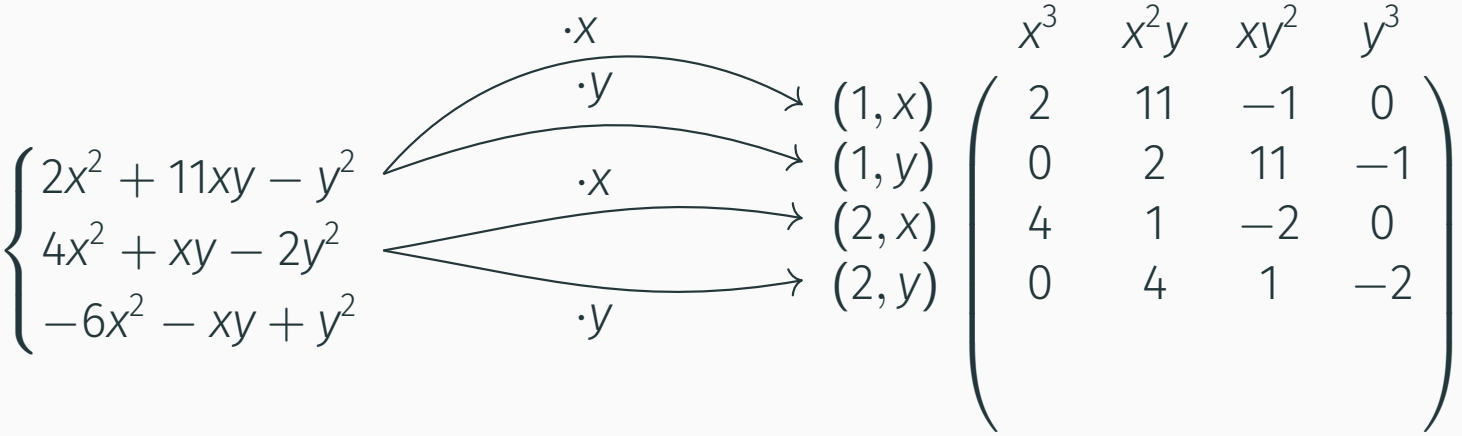

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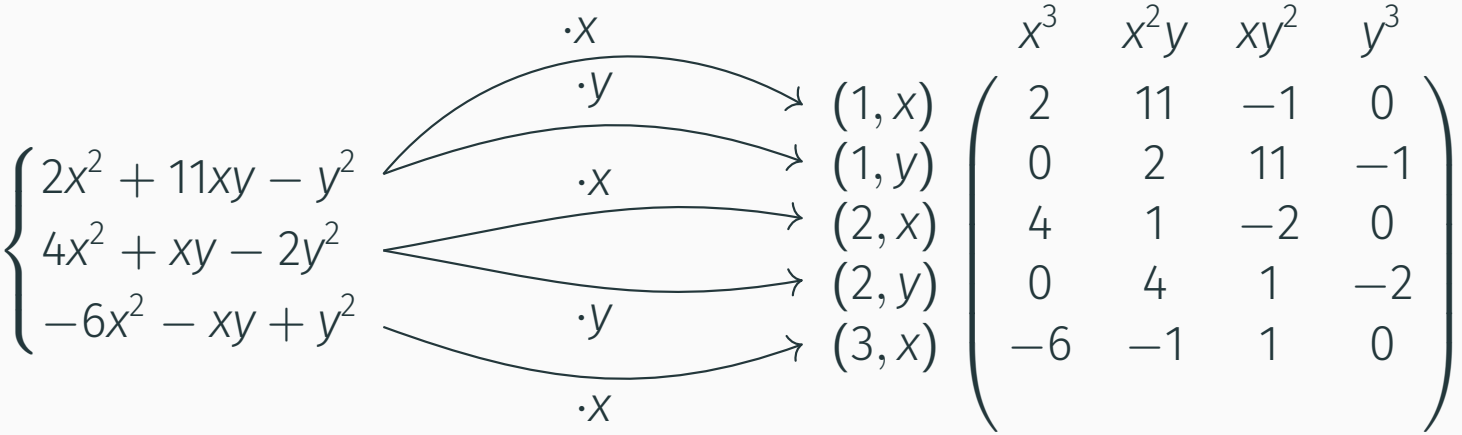

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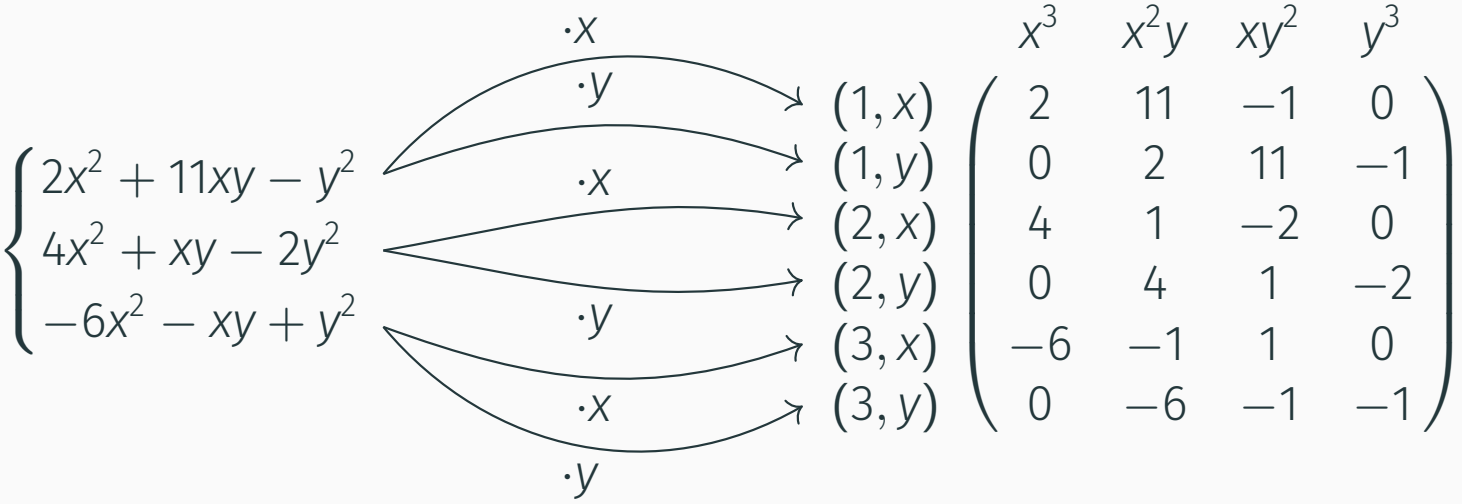

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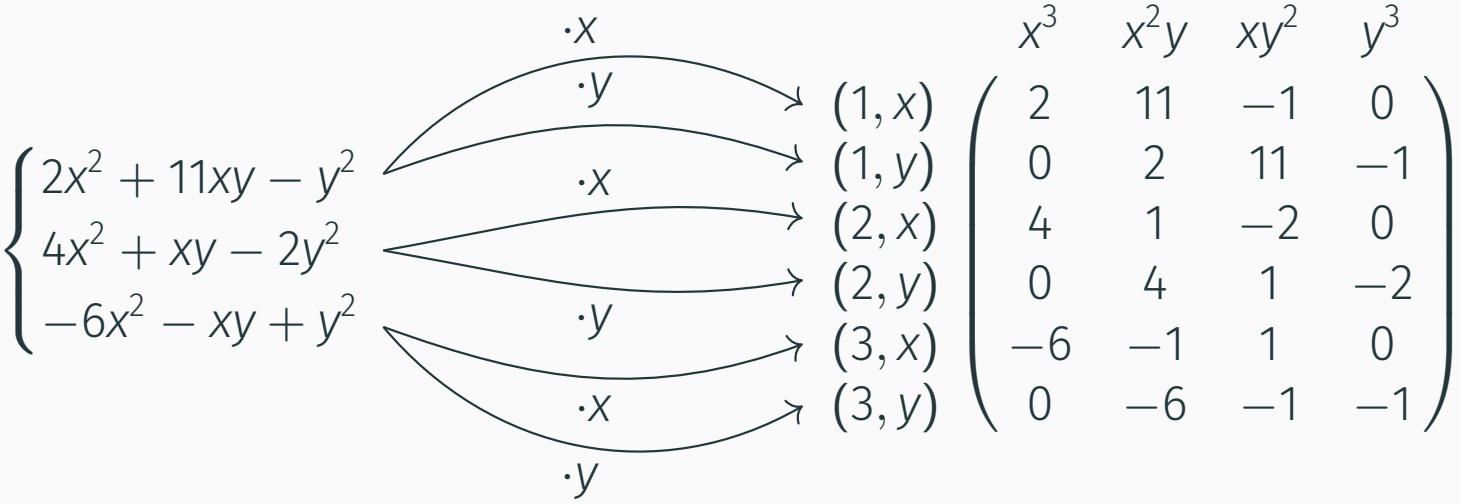

The rows of the echelonization of the Macaulay matrix of F in degree d form the elements of degree d of a Gröbner basis for F.

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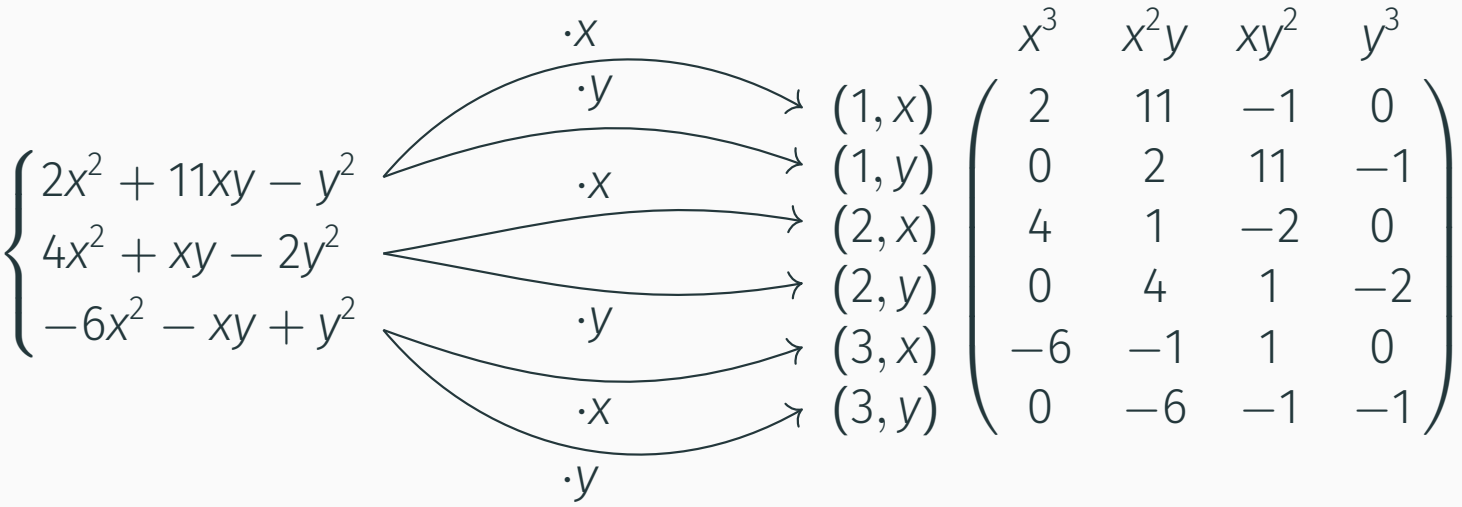

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Theorem (Macaulay bound, [Lazard, 1983])

The maximum degree of a polynomial in the grevlex Gröbner basis of a generic polynomial system $f_1, \ldots, f_m$ is

$$\left(\sum_{i=1}^{m} \deg(f_i) - 1 \right) + 1.$$

Gröbner bases - linear algebra

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The F_4 algorithm ([Faugère, 1999])

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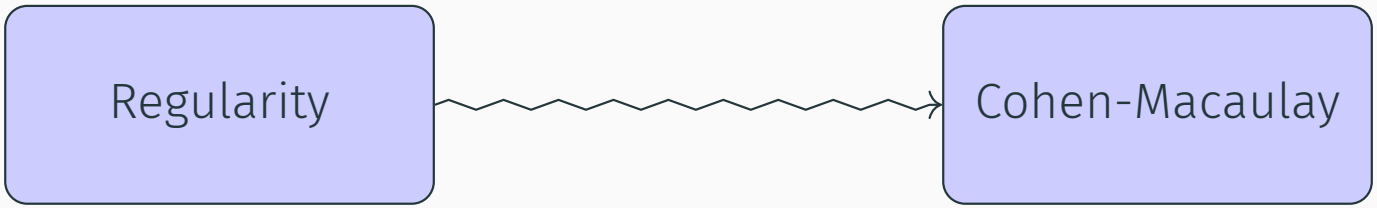

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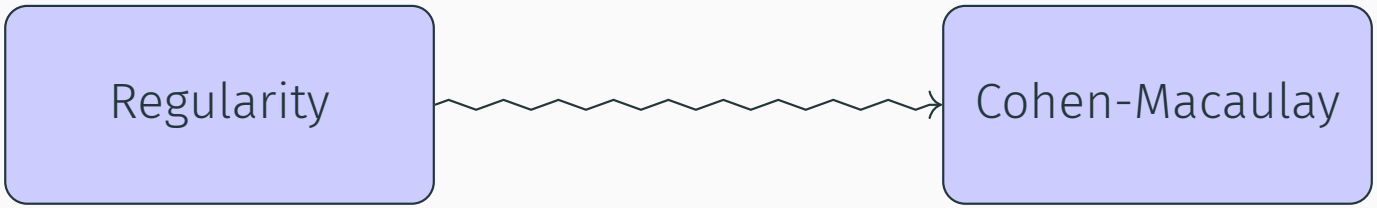

$\rightsquigarrow$ initial complexity analysis of generalized MinRank:

Theorem ([Faugère, Safey, Spaenlehauer, 2013])

Computing a grevlex Gröbner basis of the $(r+1)$-minors of an $n \times m$ matrix of generic polynomials of degree D in k variables has arithmetic complexity

$$O\left(\binom{n}{r+1}\binom{m}{r+1}\binom{Dr(m-r)+(D-1)k+1+k}{k}^{\omega}\right).$$

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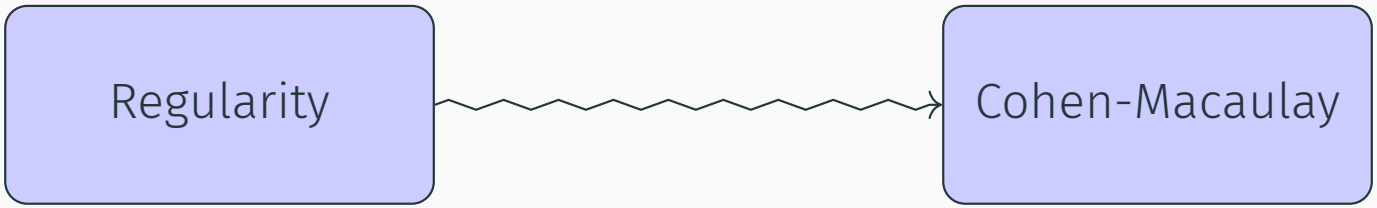

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Matrices computed by F_4 are generically rank-deficient

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$$\widetilde{\mathcal{M}_2}$$

$$\begin{array}{c} \\ (1,1) \\ (2,1) \\ (3,1) \end{array} \begin{array}{c} \begin{array}{cccccc} x^2 & xy & y^2 & xz & yz & z^2 \end{array} \\ \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 & 0 & 4 \end{pmatrix} \end{array}$$

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$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
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$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
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$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
$$f_2 = 2x^2 + xy + 4y^2 + 2xz + 4z^2$$
$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}}_2$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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Let $\mathbb{k} = \mathbb{F}_7$, $\succ =$ grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$(f_1, \ldots, f_m)$ regular sequence

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$(f_1, \ldots, f_m)$ regular sequence

$\Downarrow$

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

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$$f_3 = 4x^2 + xy + 4y^2 + 3xz + 5yz + 2z^2$$

$(f_1, \ldots, f_m)$ regular sequence

$\Downarrow$

No reductions to zero.

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
$(3,y^2)$	0	0	4	1	4	0	0	3	5	0	0	2	0	0	0
$(3,xz)$	0	0	0	0	0	4	1	4	0	3	5	0	2	0	0
$(3,yz)$	0	0	0	0	0	0	4	1	4	0	3	5	0	2	0
$(3,z^2)$	0	0	0	0	0	0	0	0	0	4	1	4	3	5	2

The F_5 algorithm ([Faugère, 2002])

Let $\mathbb{k} = \mathbb{F}_7$, $\succ$ = grevlex.

$$f_1 = 5x^2 + 5xy + 3y^2 + 5xz + 5yz + 6z^2$$
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$(f_1, \ldots, f_m)$ regular sequence

$\Downarrow$

- **No** reductions to zero.
- Precise complexity analysis [1]

$\widetilde{\mathcal{M}_2}$

	x^2	xy	y^2	xz	yz	z^2
$(1,1)$	1	1	2	1	1	4
$(2,1)$	0	1	0	0	2	4
$(3,1)$	0	0	1	2	0	4

Lazard: $\mathcal{M}_4$ is 18×15.

F5: $\mathcal{M}_4$ is 15×15 and full rank!

$\mathcal{M}_4$

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,x^2)$	5	5	3	0	0	5	5	0	0	6	0	0	0	0	0
$(1,xy)$	0	5	5	3	0	0	5	5	0	0	6	0	0	0	0
$(1,y^2)$	0	0	5	5	3	0	0	5	5	0	0	6	0	0	0
$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
$(1,yz)$	0	0	0	0	0	0	5	5	3	0	5	5	0	6	0
$(1,z^2)$	0	0	0	0	0	0	0	0	0	5	5	3	5	5	6
$(2,x^2)$	2	1	4	0	0	2	0	0	0	4	0	0	0	0	0
$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
$(2,yz)$	0	0	0	0	0	0	2	1	4	0	2	0	0	4	0
$(2,z^2)$	0	0	0	0	0	0	0	0	0	2	1	4	2	0	4
$(3,x^2)$	4	1	4	0	0	3	5	0	0	2	0	0	0	0	0
$(3,xy)$	0	4	1	4	0	0	3	5	0	0	2	0	0	0	0
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[1] [Bardet, Faugère, Salvy, 2015]

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Determinantal systems are not regular sequences!

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$(3,1)$	0	0	1	2	0	4

	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
$(1,xy)$	6	0	0	0	0	0
$(1,y^2)$	0	6	0	0	0	0
$(1,xz)$	0	0	6	0	0	0
$(1,yz)$	5	5	0	6	0	0
$(1,z^2)$	5	5	3	5	5	6
$(2,x^2)$	0	0	4	0	0	0
$(2,xy)$	0	0	0	4	0	0
$(2,y^2)$	0	0	0	0	4	0
$(2,xz)$	2	0	0	4	0	0
$(2,yz)$	1	4	0	2	0	4
$(2,z^2)$	4	1	4	2	0	4
$(3,x^2)$	0	0	3	0	0	0
$(3,xy)$	0	0	2	0	0	0
$(3,y^2)$	0	0	0	2	0	0
$(3,xz)$	3	5	0	0	2	0
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How do we remove reductions to zero?

	x^4	x^3y	x^2y^2	xy^3	y^4	x^3z	x^2yz	xy^2z	y^3z	x^2z^2	xyz^2	y^2z^2	xz^3	yz^3	z^4
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$(1,xz)$	0	0	0	0	0	5	5	3	0	5	5	0	6	0	0
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$(2,xy)$	0	2	1	4	0	0	2	0	0	0	4	0	0	0	0
$(2,y^2)$	0	0	2	1	4	0	0	2	0	0	0	4	0	0	0
$(2,xz)$	0	0	0	0	0	2	1	4	0	2	0	0	4	0	0
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M is an $n \times n$ matrix of generic linear forms over $\mathbb{k}[x_1, \ldots, x_k]$, $r \leq n - 1$. Let $F_r(M)$ be the system of $(r+1)$-minors of M. Suppose $F_r(M)$ is zero-dimensional.

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New F_5-type criteria

- Allows us to avoid all reductions to zero in degree $r+2$

$$\rightsquigarrow \binom{n}{r+2}^2 \left(\frac{2(r+2)(r+1)}{n-r-1} + 2r + 2 \right) \text{ reductions avoided.}$$

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Theorem ([G., Neiger, Safey, 2023])

The complexity of computing a grevlex Gröbner basis for the system of $(n-1)$-minors of M is in

Homogeneous	*Affine*
$O(n^{4\omega-1})$	$O(n^{4\omega})$

Syzygies

Definition (Syzygy)

A sequence $(a_1, \ldots, a_m) \in \mathbb{k}[x_1, \ldots, x_k]^m$ such that

$$a_1 f_1 + \cdots + a_m f_m = 0$$

is called a **syzygy** of $f_1, \ldots, f_m$.

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Syzygies of F $\longleftrightarrow$ Reductions to zero in F_5

Theorem ([Hilbert, 1890])

Free resolution $0 \to \mathcal{E}_\ell \xrightarrow{d_\ell} \mathcal{E}_{\ell-1} \xrightarrow{d_{\ell-1}} \cdots \to \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{\epsilon} \langle F \rangle \to 0 \implies$

$$\mathsf{Syz}_k(F) = \ker(d_k) = \mathsf{im}(d_{k+1}).$$

The Gulliksen-Negård complex

Theorem ([Gulliksen, Negård, 1972])

$r = n - 2$, entries of M sufficiently generic $\rightsquigarrow$ free resolution:

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$$\begin{pmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{pmatrix}$$

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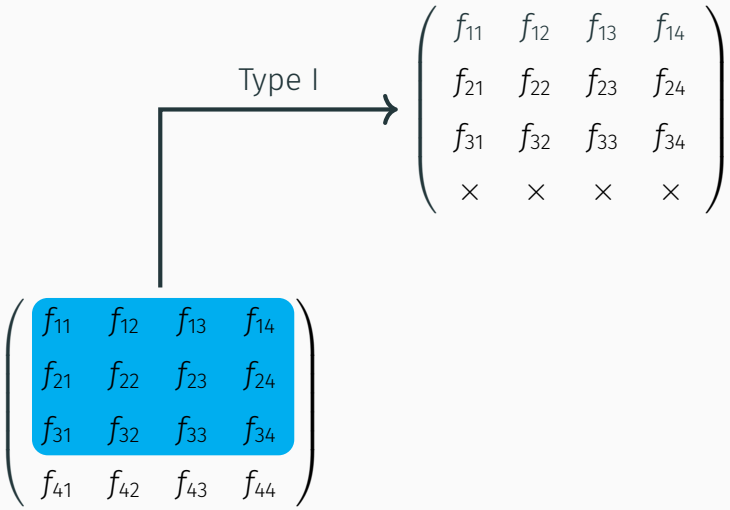

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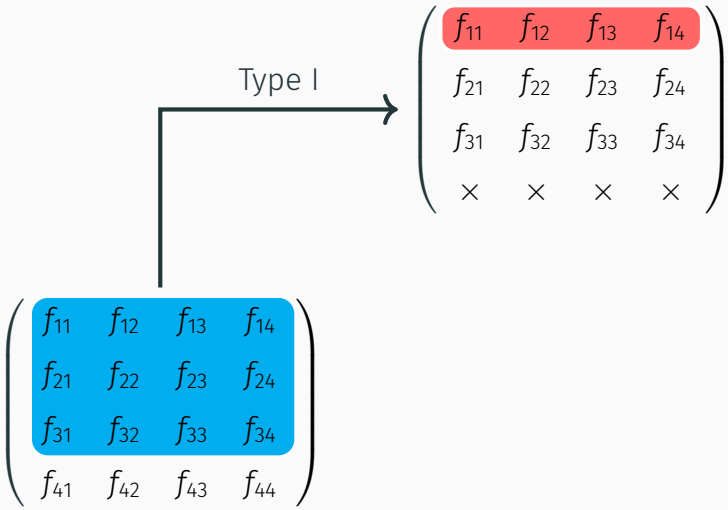

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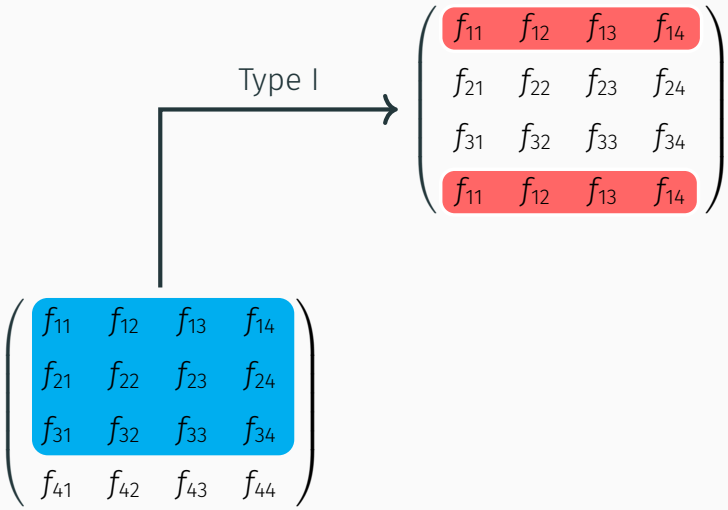

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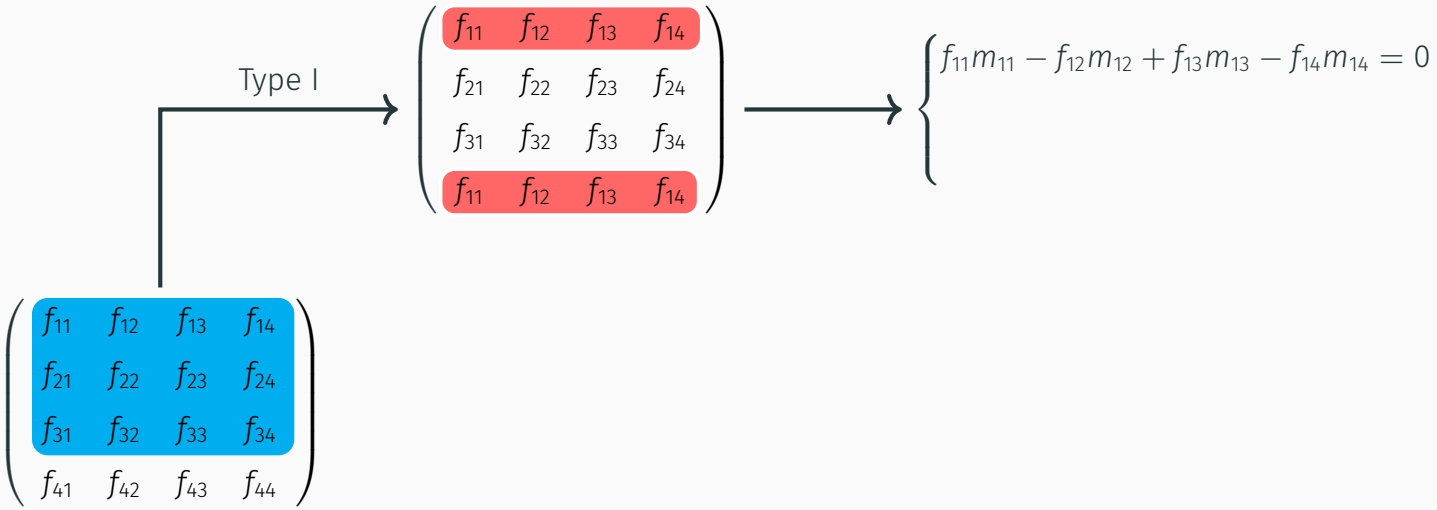

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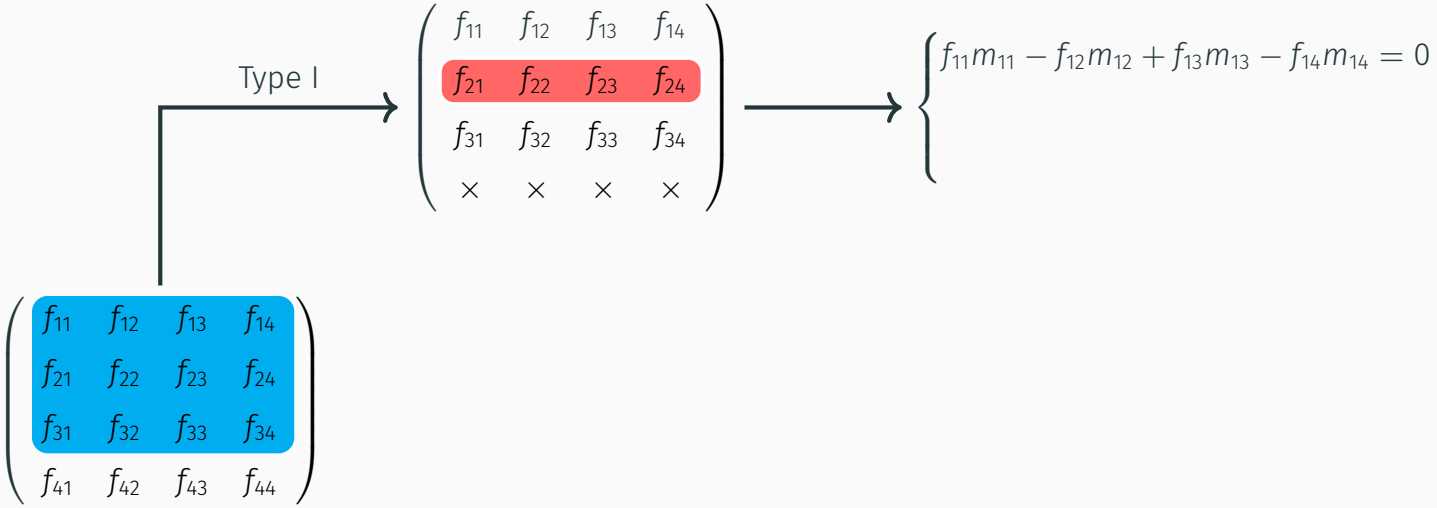

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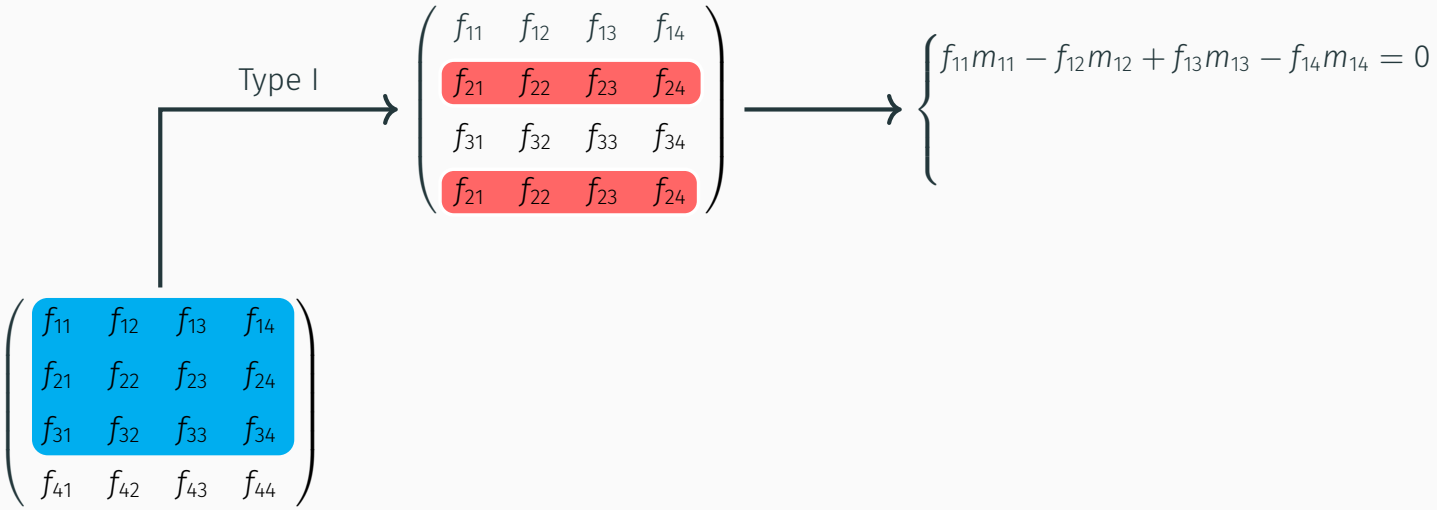

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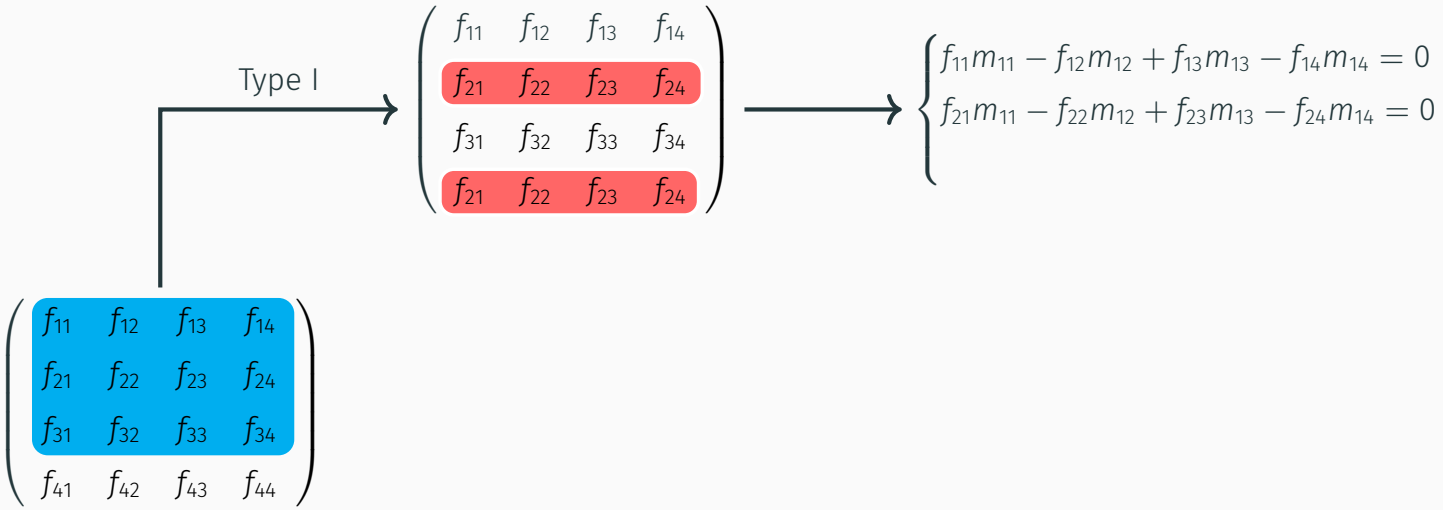

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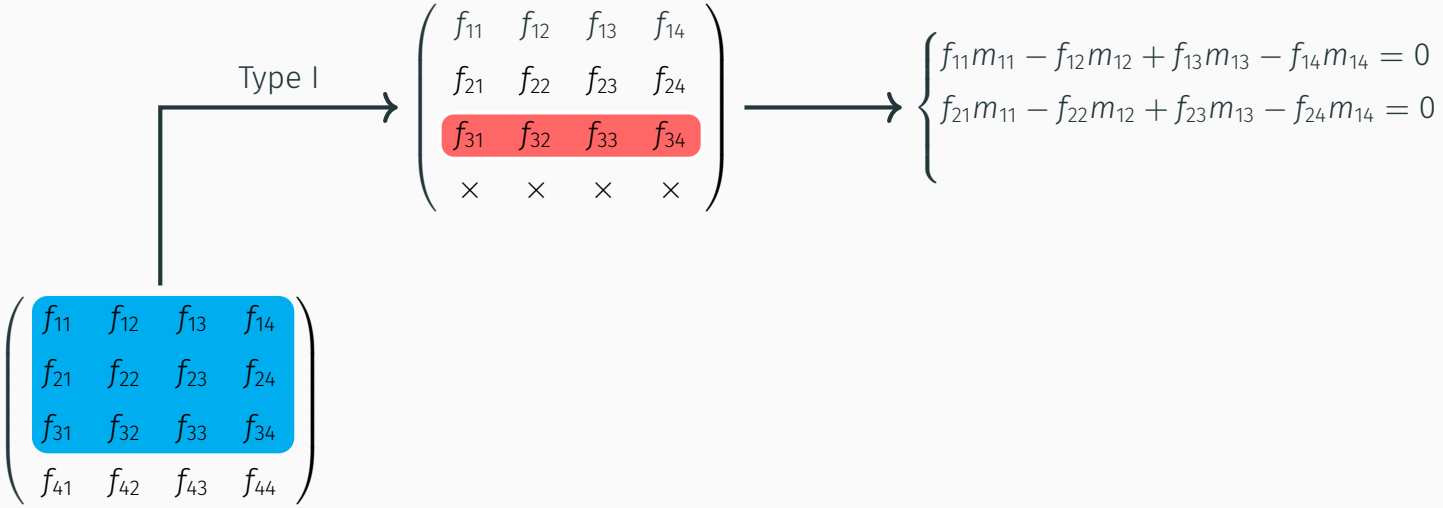

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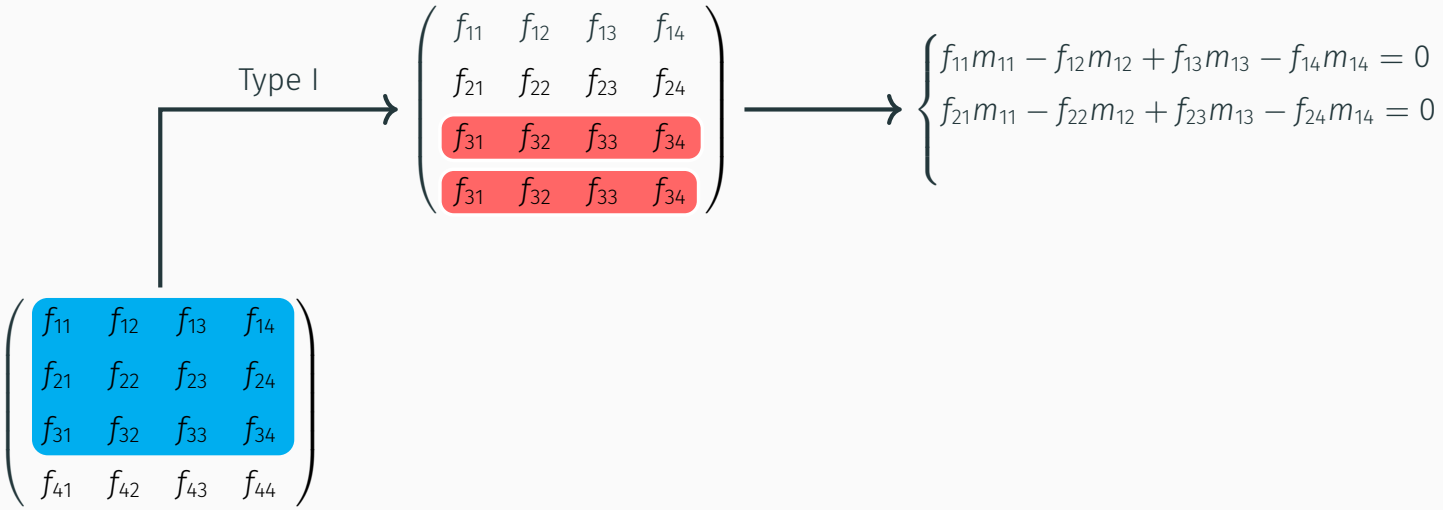

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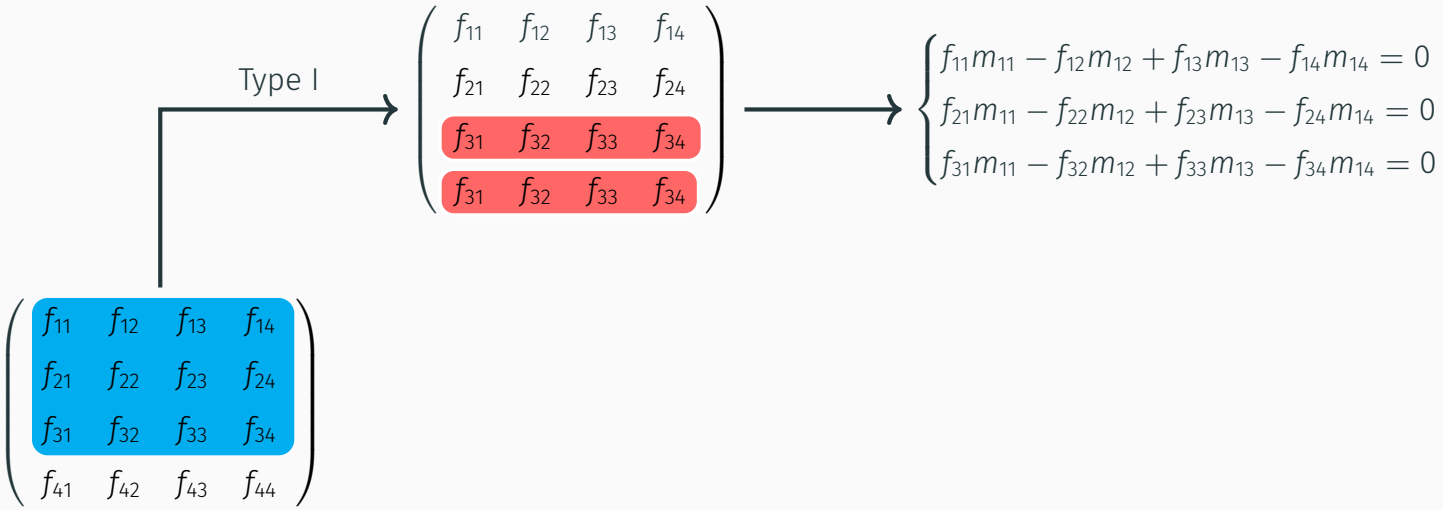

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Theorem ([Gulliksen, Negård, 1972])

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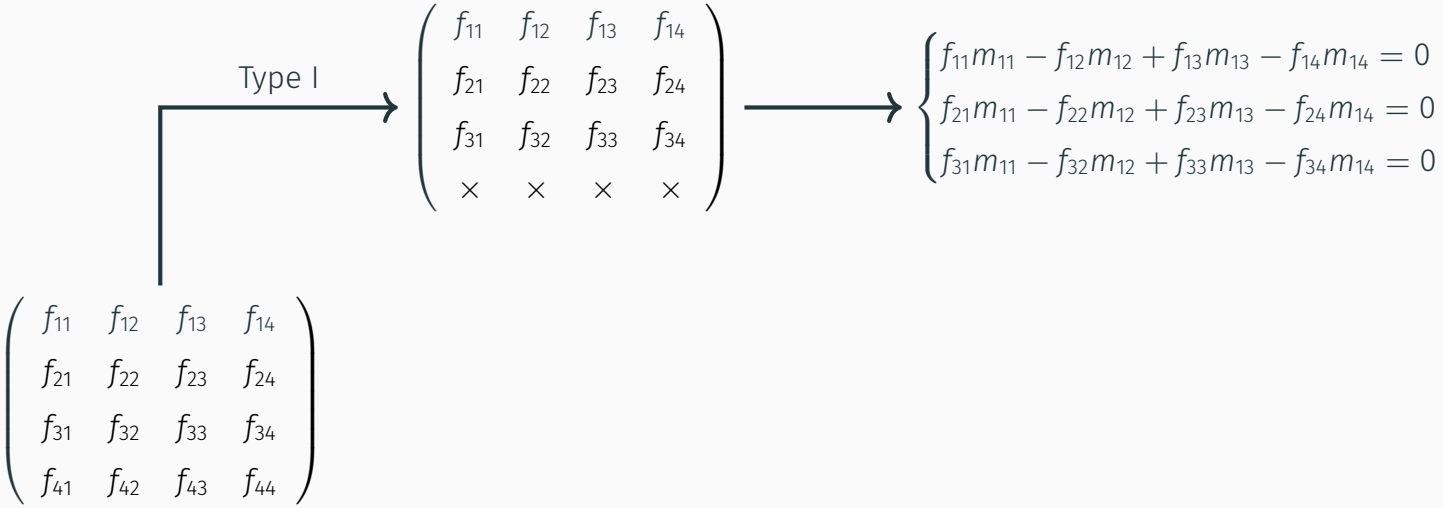

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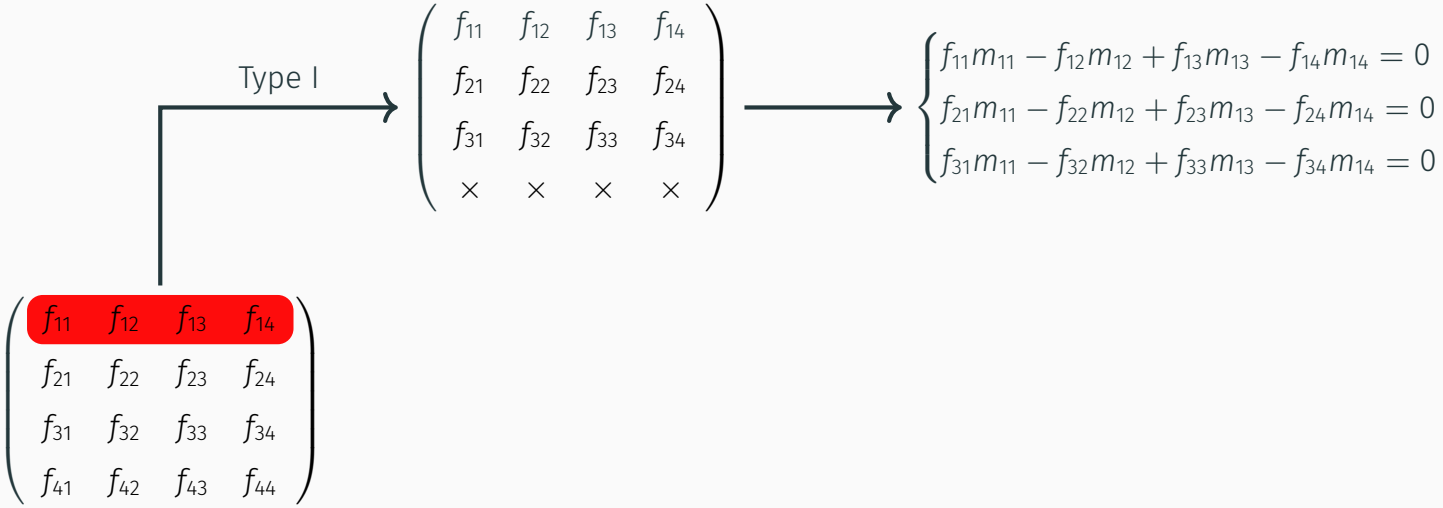

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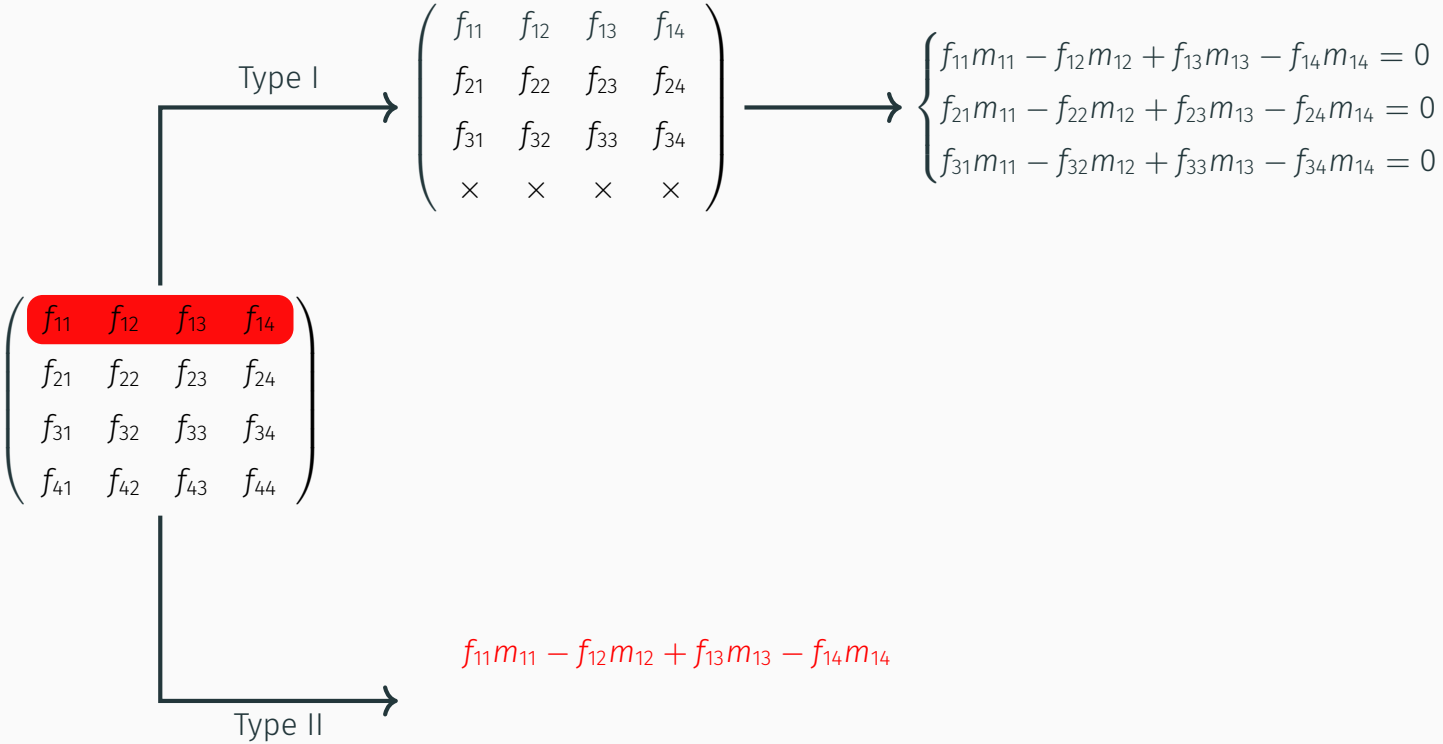

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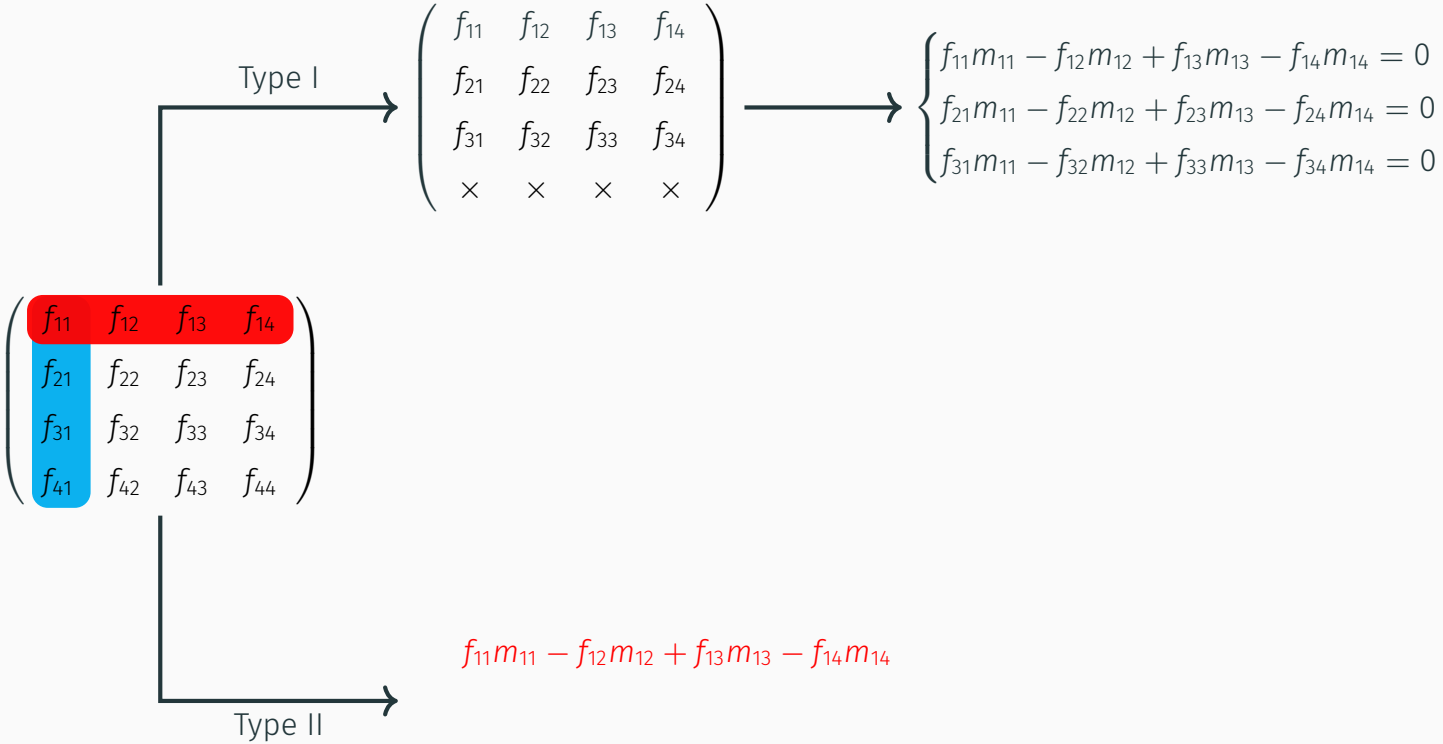

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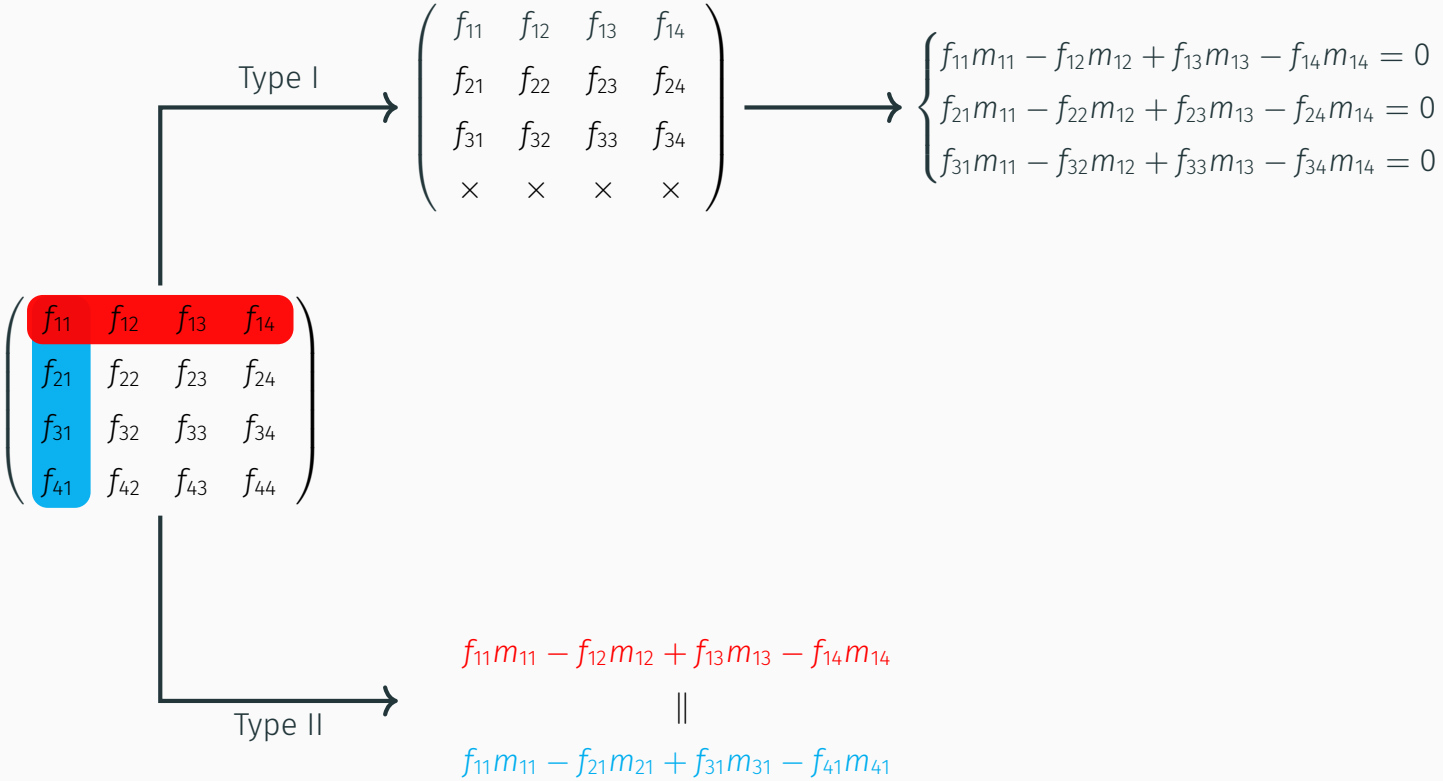

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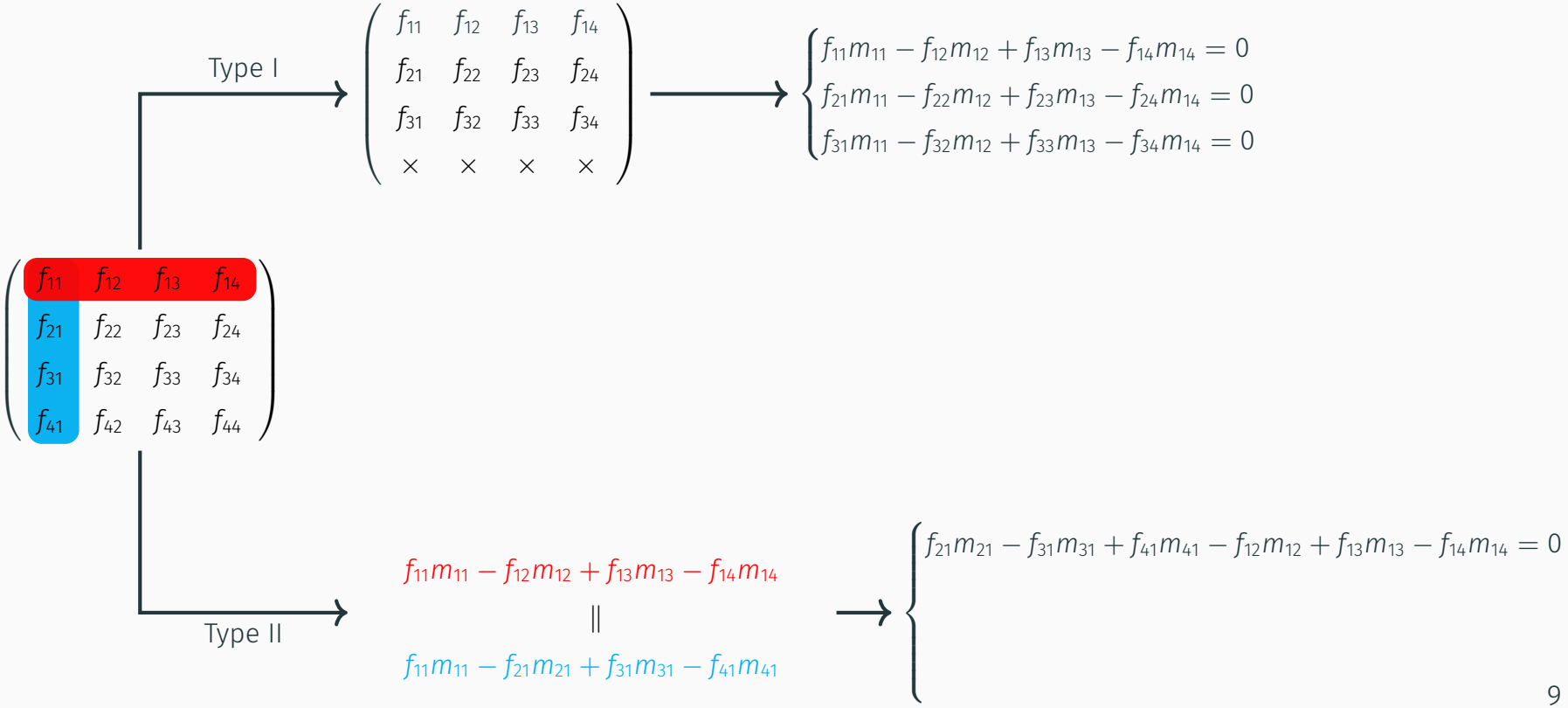

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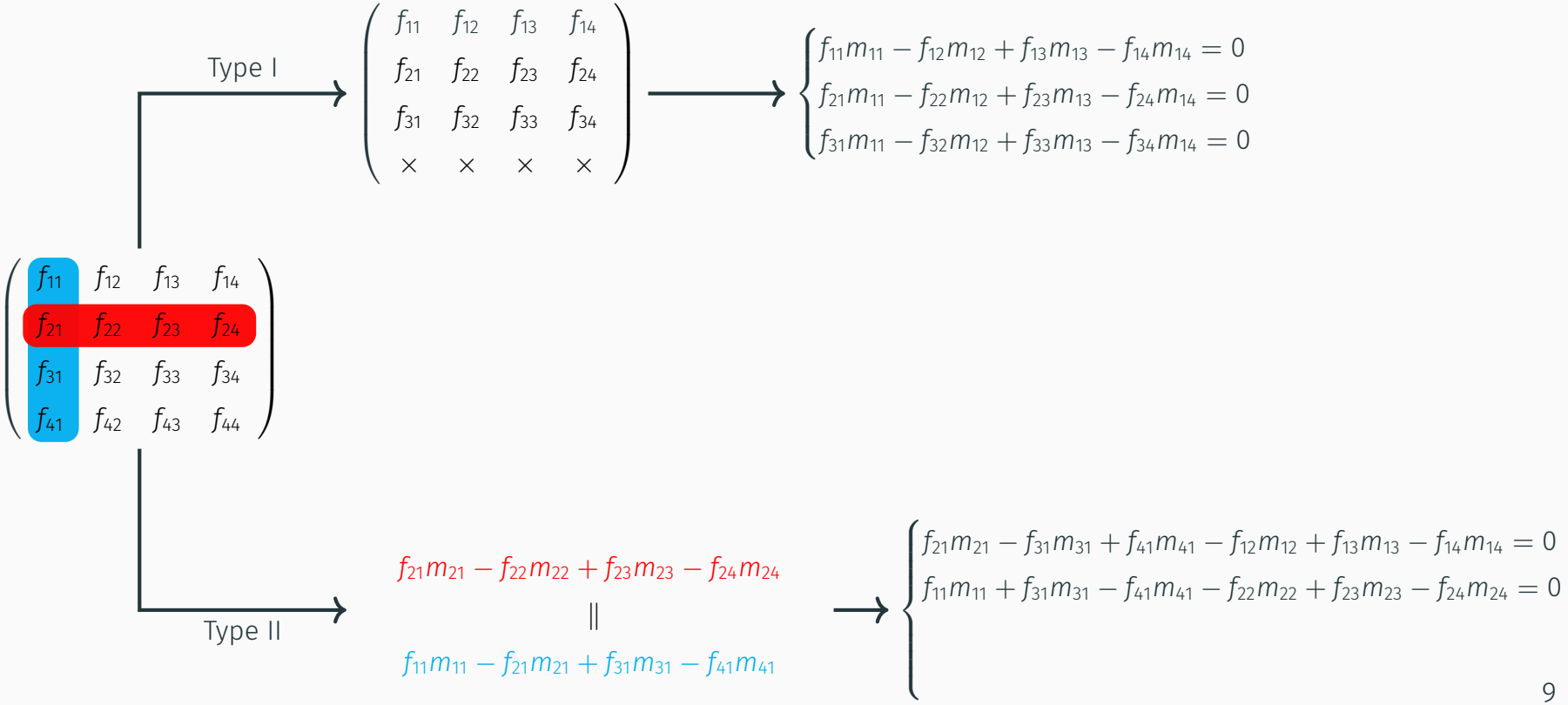

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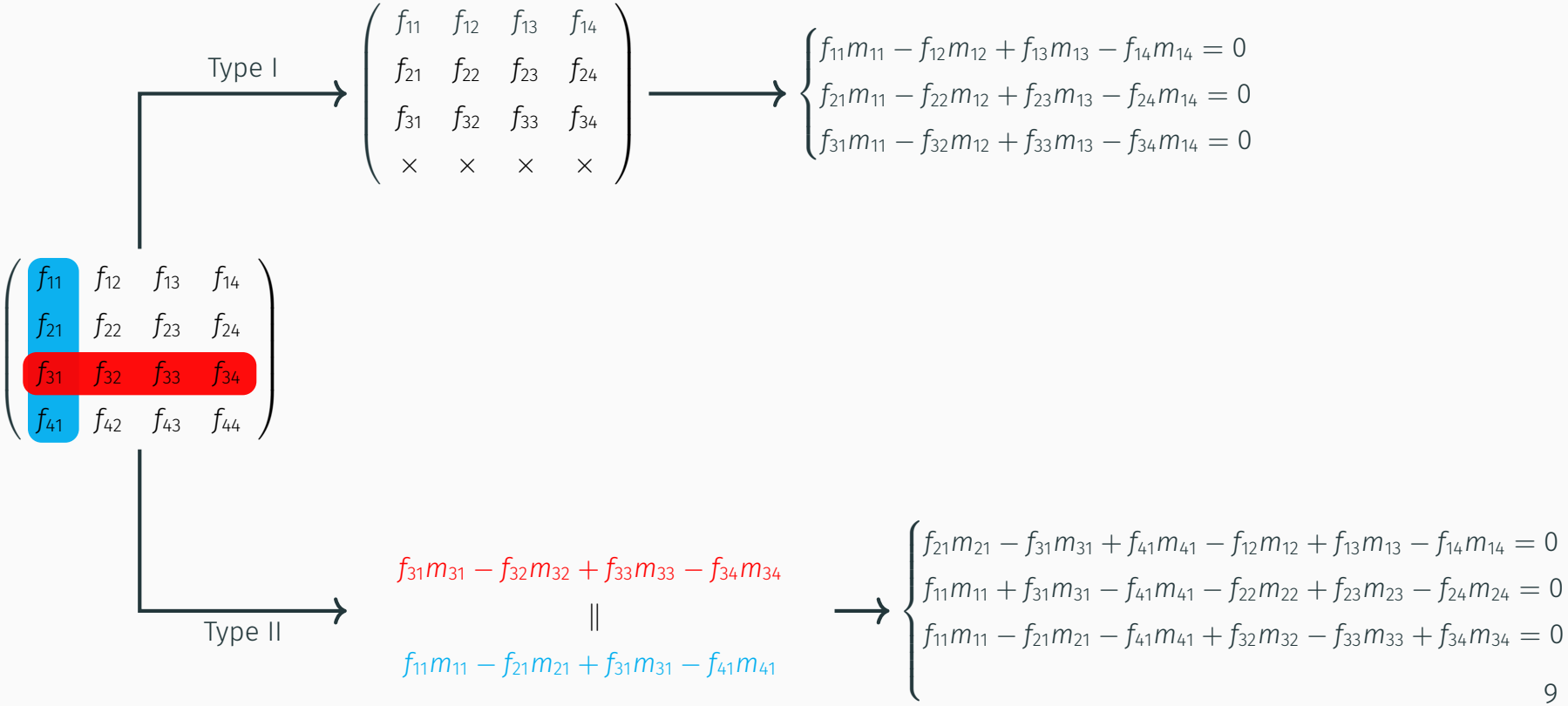

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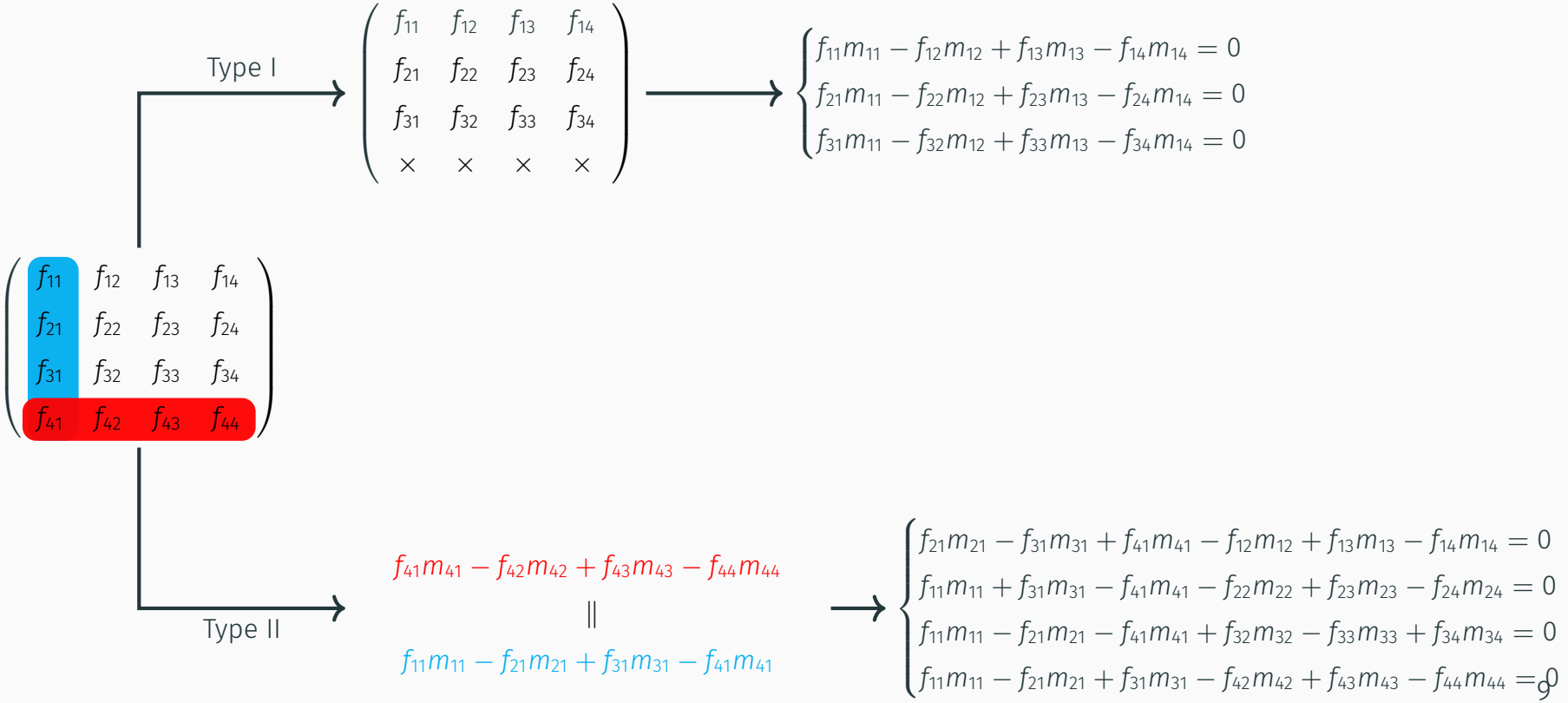

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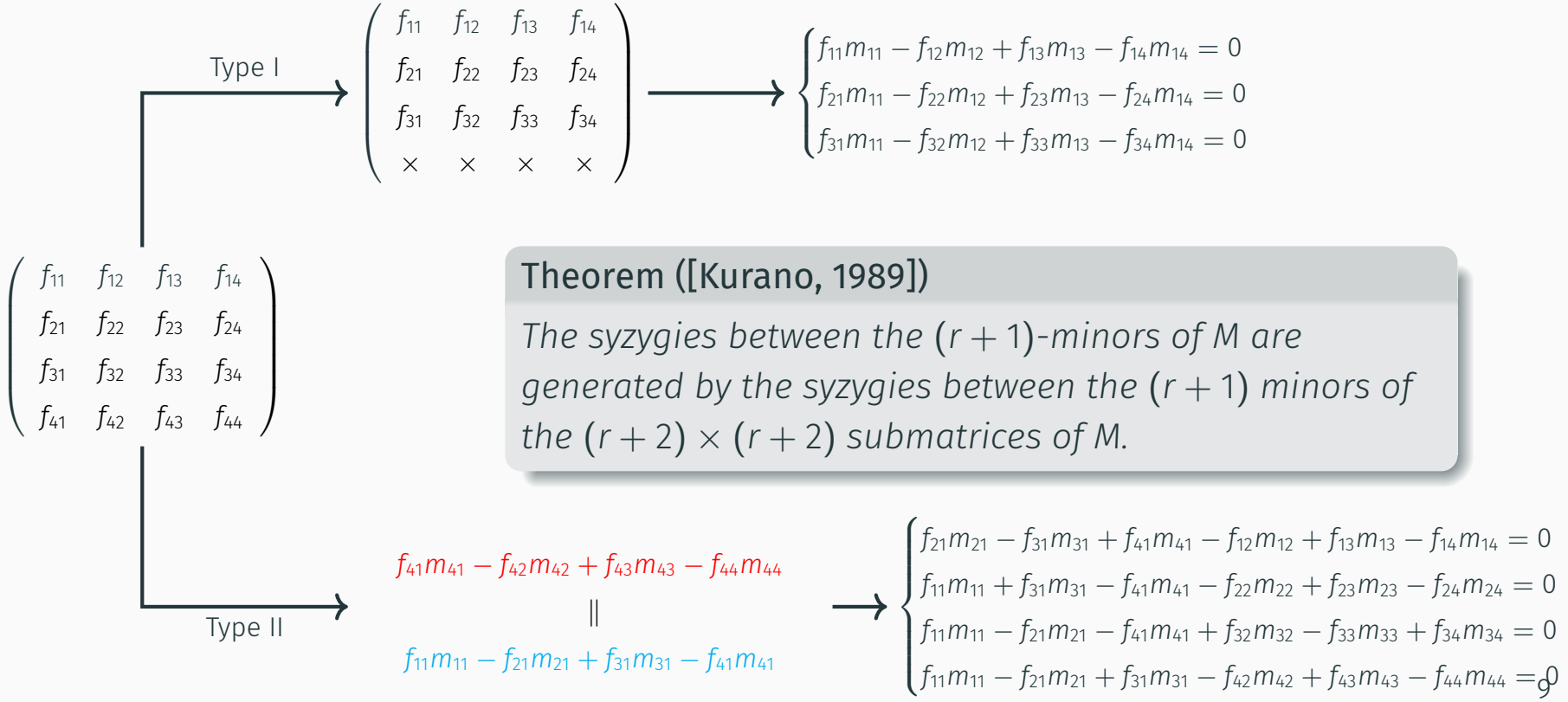

Theorem ([Kurano, 1989])

The syzygies between the $(r+1)$-minors of M are generated by the syzygies between the $(r+1)$ minors of the $(r+2) \times (r+2)$ submatrices of M.

New F_5 algorithms - the general case

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Matrix M of generic linear forms

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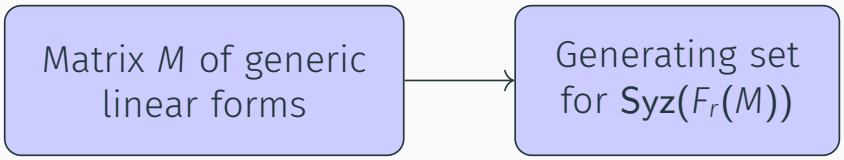

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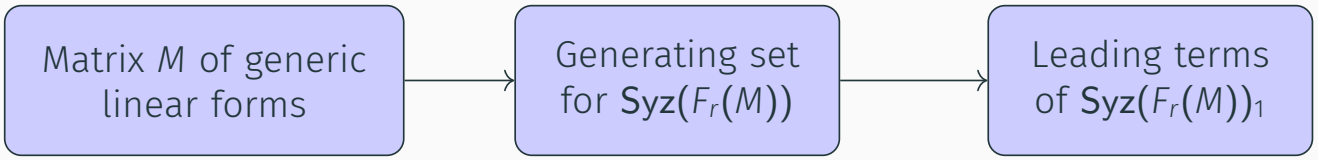

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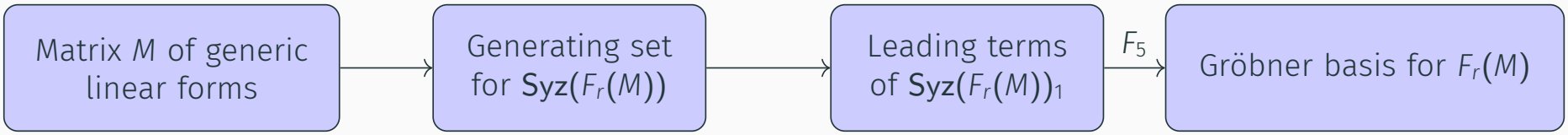

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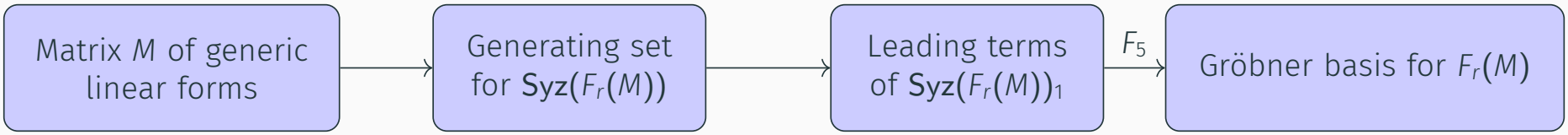

$$\#\,\mathsf{Syz}(F_r(M)) = \binom{n}{r+2}^2 \left(\frac{2(r+2)(r+1)}{n-r-1} + 2r + 2\right).$$

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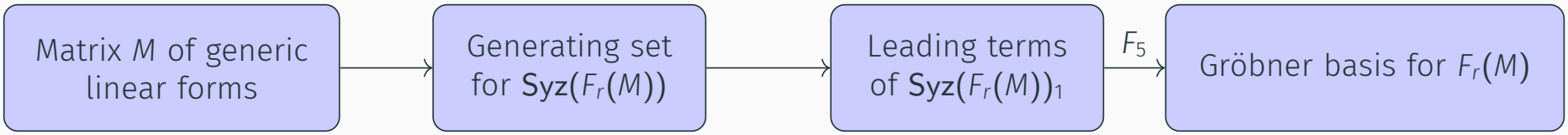

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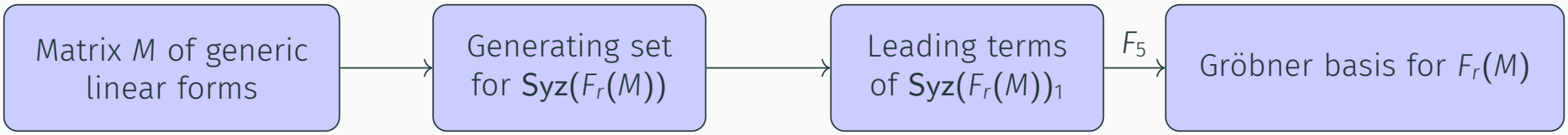

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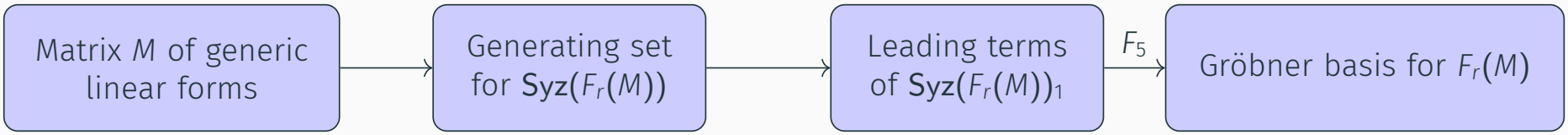

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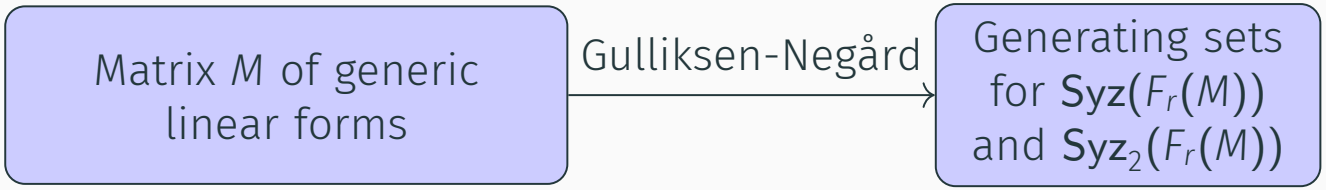

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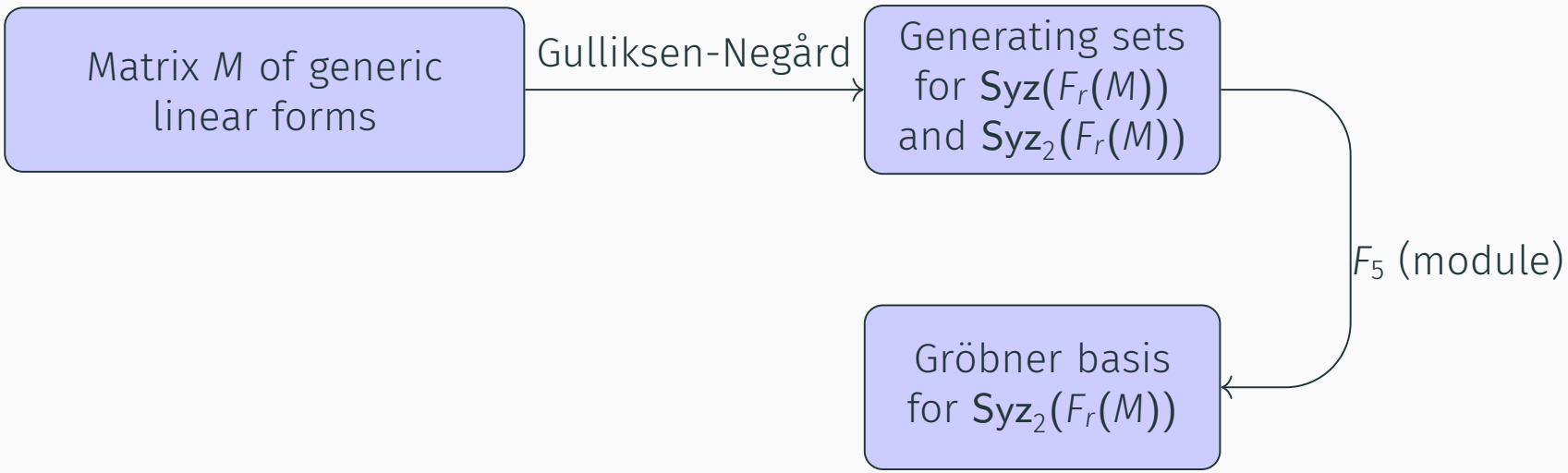

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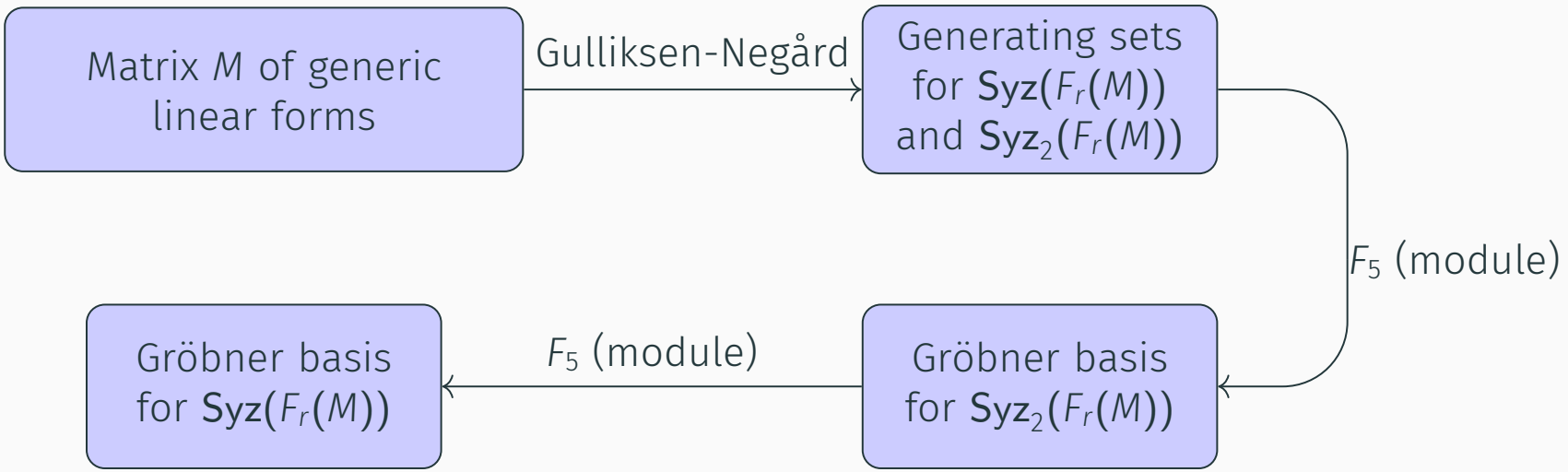

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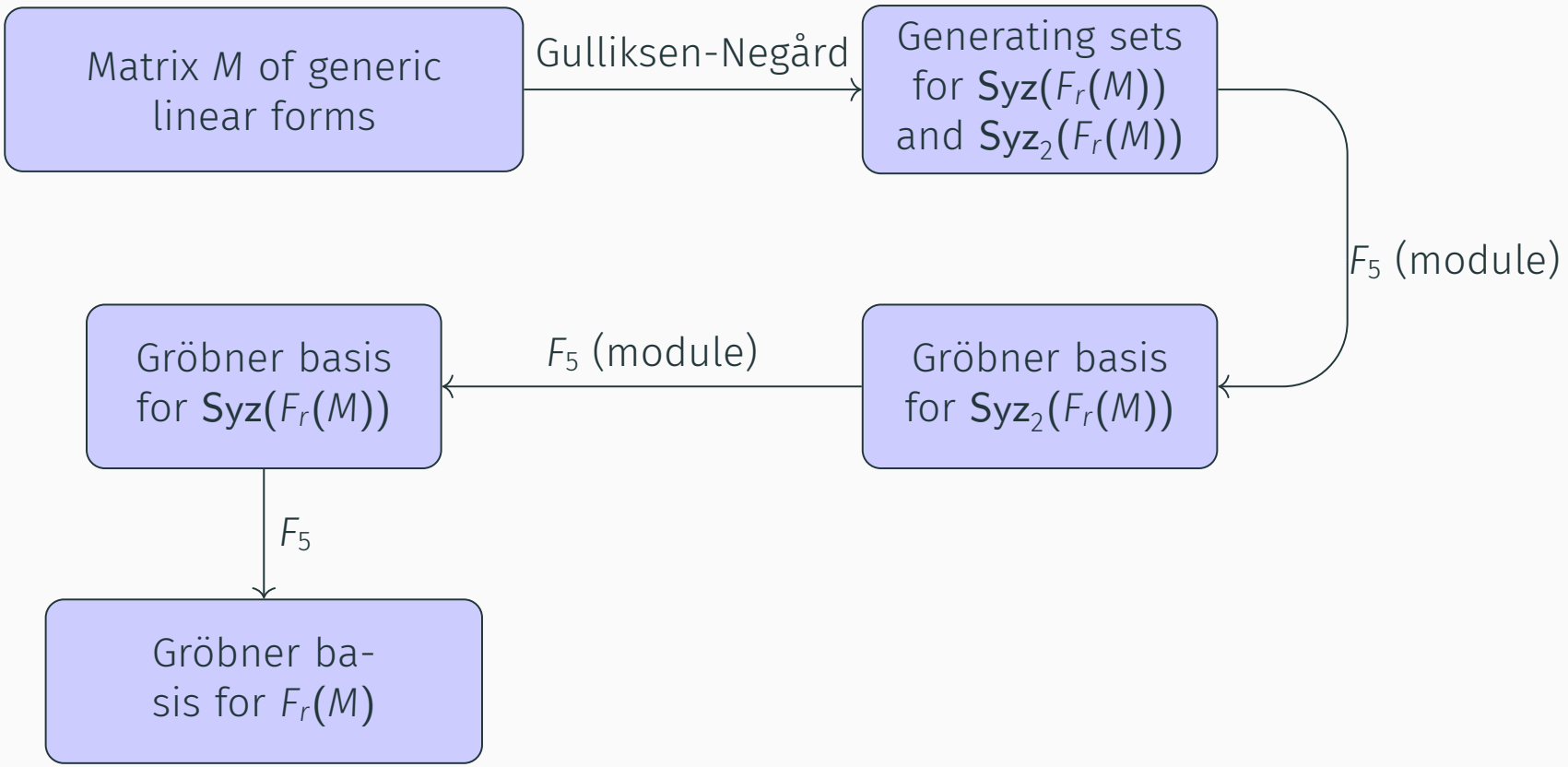

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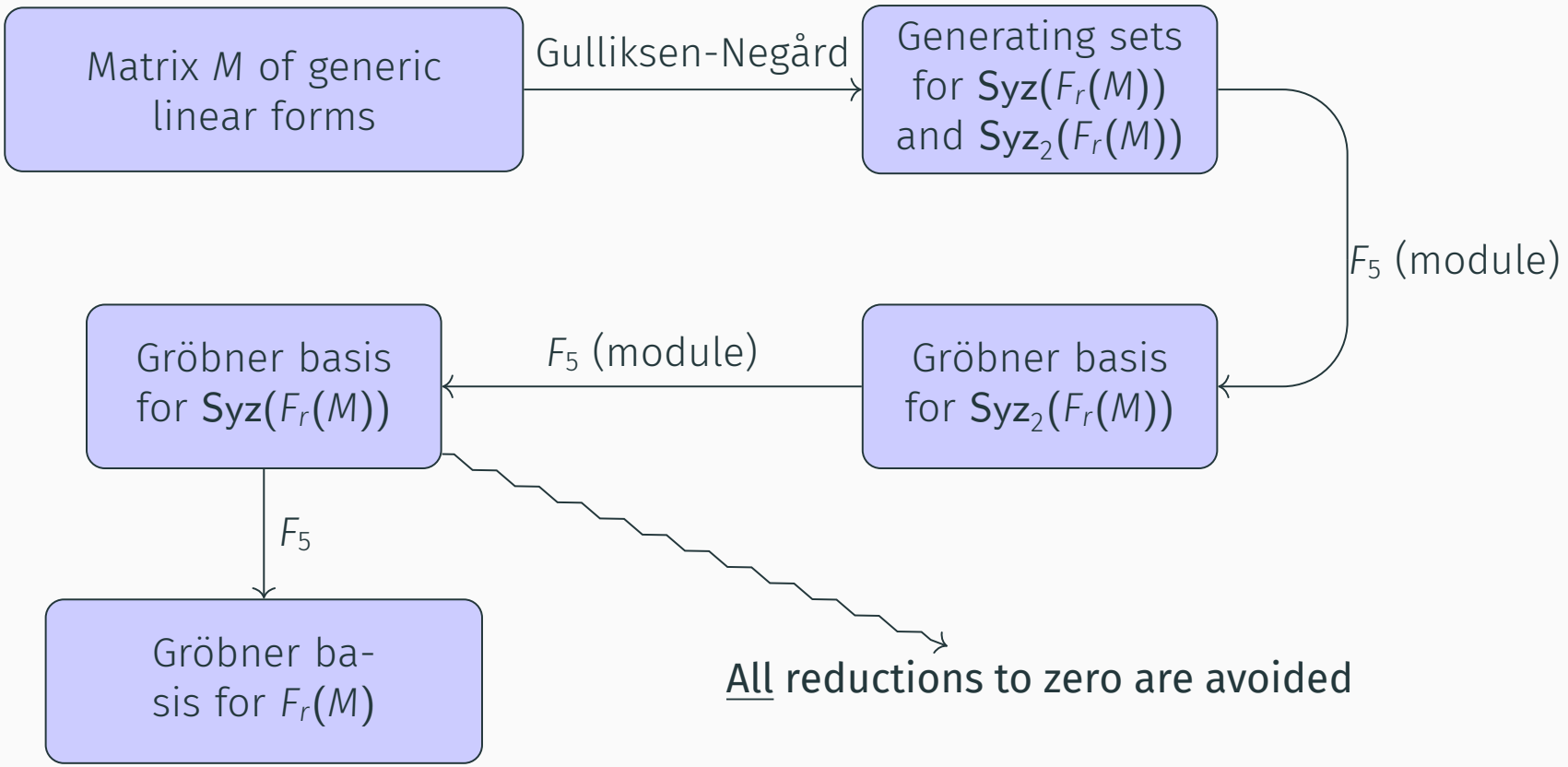

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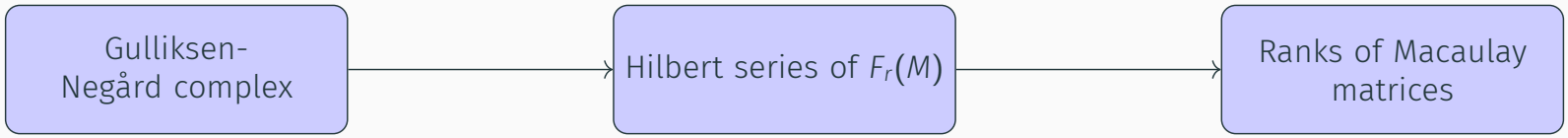

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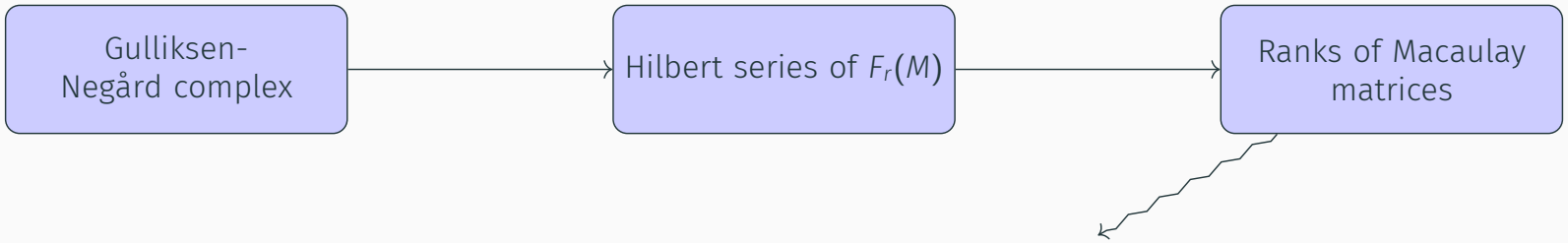

Theorem ([G., Neiger, Safey, 2023])

Let M be a matrix of generic linear forms in four variables. The complexity of computing a grevlex-Gröbner basis for $F_r(M)$ is in

$$O\left(\left(\sum_{d=n-1}^{2n-3} n^2\binom{d-n+4}{3} - (2n^2-2)\binom{d-n+3}{3} + n^2\binom{d-n+2}{3}\right)^{\omega-1}\binom{2n+1}{5}\right).$$

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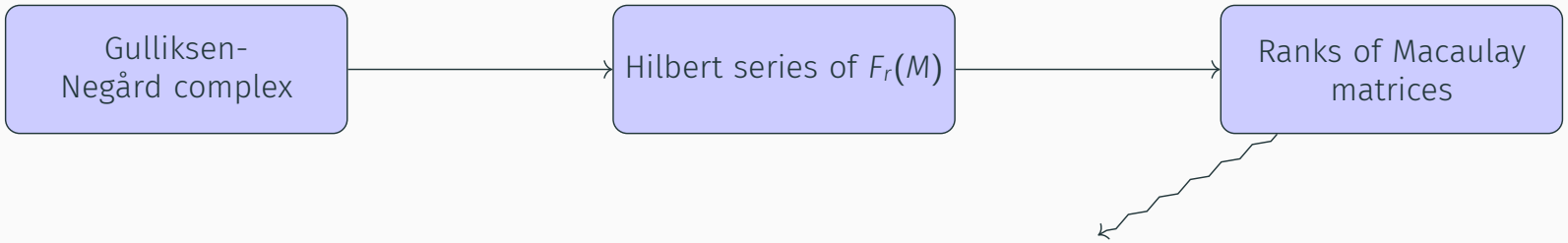

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Asymptotically:

[Faugère, Safey, Spaenlehauer, 2013]

$$O\left(n^{5\omega+2}\right)$$

[G., Neiger, Safey El Din, 2023]

$$O\left(n^{4\omega-1}\right)$$

Experimental results

n	r	k	D	d	rank	Std. F_5	Det. F_5
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				8	130	256	130
				9	200	322	200
				10	276	385	276
				11	360	471	360
				12	454	559	454
				13	560	650	560
9	7	4	15	8	81	81	81
				9	164	324	164
				10	251	401	251
				11	344	486	344
				12	445	584	445
				13	556	675	556
				14	679	813	679
				15	816	931	816

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				4	450	900	450
				5	1278	1956	1956
				6	3002	3546	3546
				7	6435	6685	6685
6	3	9	6	4	225	225	225
				5	1017	2025	1017
				6	2838	4715	4715
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				6	2009	3969	2009

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				3	800	1600	800
				4	3875	4662	4662
6	2	16	4	3	400	400	400
				4	3250	6400	3250

$k =$ number of variables.
$D =$ highest degree appearing in the (reduced) grevlex Gröbner basis for $F_r(M)$.

- When $r = n - 2$, all Macaulay matrices are full rank.
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$\sim 30\%$ of reductions to zero removed in general case

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[a] [Ma, 1994]

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[a] [Ma, 1994]

[b] [Lascoux, 1978], [Eagon, Northcott, 1962],...

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- Second syzygies in the general case.[a]
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- Sharper complexity analyses of new algorithms and implications for cryptography schemes.

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[a] [Ma, 1994]

[b] [Lascoux, 1978], [Eagon, Northcott, 1962],...

Conclusion and perspectives

Summary

- New F_5-type criteria to identify and avoid reductions to zero for determinantal systems.
- In the case $r = n - 2$:
  - New algorithm which avoids **all** reductions to zero.
  - Explicit Hilbert series leading to new complexity bound.
- Experimental data suggests improved complexity in general.

Future works

- Second syzygies in the general case.[a]
- Other free resolutions of determinantal ideals.[b]
- Sharper complexity analyses of new algorithms and implications for cryptography schemes.
- Efficient implementations of new algorithms.

[a] [Ma, 1994]

[b] [Lascoux, 1978], [Eagon, Northcott, 1962],...

Thanks. Questions?