The Cones of Non-Negative Polynomials and Measures on Genus One Curves

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Based on a joint work with Greg Blekherman and Rainer Sinn

The Moment Problem and Non-Negative Polynomials

Basic setting: 2-dimensional case in degree $2d = 2$

[1] K. Schmüdgen, The Moment Problem, 2017

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Linear Functionals on Polynomials

- Let $\mathbf{y} = (y_{00}, y_{10}, y_{01}, y_{20}, y_{11}, y_{02}) \in \mathbb{R}^6$
- $L_{\mathbf{y}} \in \mathbb{R}[x_1, x_2]^*_{\leq 2}$: for $f = \sum f_\alpha \mathbf{x}^\alpha \in \mathbb{R}[x_1, x_2]_{\leq 2} = \mathbb{R}[\mathbf{x}]_{\leq 2}$

$$L_{\mathbf{y}}(f) := \sum f_\alpha y_\alpha \quad (\textbf{Riesz} \text{ linear functional})$$

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The (Truncated) Moment Problem[1]

- For $\mathbf{y} \in \mathbb{R}^N$, $\exists \mu \in \mathcal{M}(\mathbb{R}^2)$ s.t. $y_\alpha = \int \mathbf{x}^\alpha \, \mathrm{d}\mu$ for all α?
- For $\mathbf{y} \in \mathbb{R}^N$, $\exists \mu \in \mathcal{M}(\mathbb{R}^2)$ s.t. $L_\mathbf{y}(f) = L_\mu(f) := \int f \, \mathrm{d}\mu \ \forall f$?

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☞ **Haviland's Theorem**: Given $\mathbf{y} \in \mathbb{R}^N$, $L_{\mathbf{y}} = L_\mu$ for some $\mu \in \mathcal{M}(D)$, $D \subset \mathbb{R}^2$ closed, iff $L_{\mathbf{y}}(f) \geq 0$ for all $f \geq 0$ on D

☞ How to **effectively represent** $f \in \mathbb{R}[\mathbf{x}]_{\leq 2}$, $f \geq 0$ on $\mathbb{R}^2$?

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Example: bivariate quadratic polynomials

$$f = \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} f_{00} & \frac{f_{10}}{2} & \frac{f_{01}}{2} \\ \frac{f_{10}}{2} & f_{20} & \frac{f_{11}}{2} \\ \frac{f_{01}}{2} & \frac{f_{11}}{2} & f_{02} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \mathbf{b}^t G \mathbf{b}$$

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☞ $f \geq 0$ on $\mathbb{R}^2$ **iff**:

- $G \succcurlyeq 0$ (**pos. semidefinite, PSD**), $G = U^t U$ for $U \in \mathbb{R}^{3\times 3}$
- $f = \mathbf{b}^t U^t U \mathbf{b} = (U\mathbf{b})^t (U\mathbf{b}) = (\mathbf{u}_1 \cdot \mathbf{b})^2 + (\mathbf{u}_2 \cdot \mathbf{b})^2 + (\mathbf{u}_3 \cdot \mathbf{b})^2$ is a **Sum of Squares** polynomial (**SoS**, $\Sigma_{\mathbb{R}^2, \leq 2} \subset \mathbb{R}[\mathbf{x}]_{\leq 2}$)

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Given $\mathbf{y} \in \mathbb{R}^6$, $L_{\mathbf{y}} = L_\mu$ for some $\mu \in \mathcal{M}(\mathbb{R}^2)$ if and only if the **moment matrix** $M(\mathbf{y})$ is PSD: $M(\mathbf{y}) = \begin{pmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{pmatrix} \succcurlyeq 0$

☞ V. Magron's lectures (and T. Metzlaff's talk) for more about moments, SoS and applications in polynomial optimization!

Sums of Squares vs Non-Negativity

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Theorem (Hilbert, 1888)

Every non-negative homogeneous polynomial $f \in \mathbb{R}[x_0, \ldots, x_n]_{2d}$ of degree $2d$ is an SoS ($\Sigma_{\mathbb{P}^n,2d} = P_{\mathbb{P}^n,2d}$) if and only if:

- $2d = 2$ (quadratic forms)
- $n = 2$ (binary forms)
- $n = 3$ and $2d = 4$ (ternary quartics)

- What if we **restrict** the forms to **varieties** $X \subset \mathbb{P}^n$?
  - ☞ Algebraically, we consider the question above mod $\mathcal{I}(X)$
- Equivalently, what if we restrict the **support** of μ to X?

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Theorem (Blekherman, Smith, Sinn, Velasco, 2016; 2017)

Let $X \subset \mathbb{P}^n$ be a (non-degenerate, totally real) variety. Then $\Sigma_{X,2} = P_{X,2}$ iff X is a 2-regular variety. If X is irreducible, $\Sigma_{X,2} = P_{X,2}$ iff $\deg X = \operatorname{codim} X + 1$ (**minimal degree**).

Literature: Decision vs Certificates

Deciding Non-Negativity

- $f \geq 0$ on $\mathbb{R}^n$ iff $\{ x \in \mathbb{R}^n : f(x) < 0 \} = \emptyset$
- **Decision** of non-negativity: **algorithm** that outputs
  - `TRUE` if $f \geq 0$ on $\mathbb{R}^n$
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  - Collins, 1975; Grigor'ev, Vorobjov, 1988;
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Certificates of Non-Negativity

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- **More complex** than decision! **Degree bounds**?

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Krivine-Stengle Positivstellensatz (Lombardi, Perrucci, Roy, 2020), **Putinar/Schmüdgen's P.** (Fang, Fawzi; Laurent, Slot; B., Mourrain, Parusiński; …); **SoS modulo gradient ideal** (Nie, Demmel, Sturmfels, 2005; Magron, Safey El Din, Trung-Hieu Vu 2023), …

Our Focus: Certificates for Cubic Curves

Question: Easiest cases when $\Sigma_{X,2} \subsetneq P_{X,2}$?

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Plane Cubics

Let $C \subset \mathbb{P}^2$ be a (smooth, non-degenerate, totally real) curve defined by a form of degree 3.

- C has genus $g = \frac{(d-1)(d-2)}{2} = 1$
- $C(\mathbb{R})$ has 2 possible topologies (Harnack's curve theorem)
- Group law $\oplus, \ominus$ on C
- Weierstrass form ($a_1 \in \mathbb{R}$, $a_2 < a_3 \in \mathbb{R}$ or $a_3 = \overline{a_2}$):
  $\mathcal{I}(C) = \langle x_0 x_2^2 - (x_1 - a_1 x_0)(x_1 - a_2 x_0)(x_1 - a_3 x_0) \rangle$

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☞ **Goal**: describe **effectively** $P_{C,2}$ (as a **convex cone**: if $p, q \geq 0$ on $C(\mathbb{R})$, and $a, b \in \mathbb{R}_{\geq 0}$, then $ap + bq \geq 0$ on $C(\mathbb{R})$)

Extremal points of a convex set K: the $f \in K$ s.t. $f = tp + (1-t)q$ for $p, q \in K$ and $t \in [0,1]$, implies $p = q = f$.

Minkowski's Theorem

If $K \subset \mathbb{R}^N$ is a compact convex set, then K is equal to convex hull of its extremal points.

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Example: Univariate Quadratic Polynomials

- if $f \in P_{\mathbb{R},\leq 2}$ is **strictly positive**, then for all $p \in \mathbb{R}[x_1]_{\leq 2}$, $f + \varepsilon p \in P_{\mathbb{R},\leq 2}$ if $\varepsilon > 0$ is small enough. Then f is not extremal: e.g for $\delta < 1$, $x_1^2 + 1 = \frac{(x_1^2+1-\delta)}{2} + \frac{(x_1^2+1+\delta)}{2}$
- if $f \in \mathbb{R}[x_1]_{\leq 2}$ has a **simple root**, then f changes sign
- if $f \in P_{\mathbb{R},\leq 2}$ has a **double root** at ξ and $f = tp + (1-t)q$, then p, q vanish twice at $\xi \Rightarrow p, q \propto f$.

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☞ Extremal: we have the **max. number** of **real zeros**!

Intersection of Curves[2]

Let $C \subset \mathbb{P}^2$ given by
$f = x_0x_2^2 - (x_1 - a_1x_0)(x_1 - a_2x_0)(x_1 - a_3x_0)$, and
$q \in \mathbb{R}[x_0, x_1, x_2]_2$

- **Bezout's theorem**: $q = 0$ and C intersect in 6 (real or complex) points, with multiplicities: $q.C = A_1 + \cdots + A_6$
- **Example**: if $a_1 < a_2 < a_3 \in \mathbb{R}$, $q = x_2^2$, $q.C = 2(T_1 + T_2 + T_3)$, where $T_i = (0 : 0 : a_i)$ for $i = 1, 2, 3$

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Lemma[2]

There exist q such that $q.C = A_1 + \cdots + A_6$ if and only if $A_1 \oplus \cdots \oplus A_6 = O$ (solution to an **overdetermined** system).

In particular, $q.C = 2(A_1 + A_2 + A_3)$ if and only if $A_1 \oplus A_2 \oplus A_3$ is a 2-torsion point for $(C, \oplus)$.

☞ If $q.C = 2(A_1 + A_2 + A_3)$, $A_i \in C(\mathbb{R})$, is $q \geq 0$ on $C(\mathbb{R})$?

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Let $q \in \mathbb{R}[x_0, x_1, x_2]_2$

First case: $C(\mathbb{R})$ is connected

- $a_1 \in \mathbb{R}$, $a_2 = \overline{a_3}$
- Two real 2-torsion points: O and T_1:
    - if $A_1 \oplus A_2 \oplus A_3 = O$, $\ell^2_{A_1,A_2}.C = 2(A_1 + A_2 + A_3)$ and $q = \ell^2_{A_1,A_2} \geq 0$ on $\mathbb{P}^2(\mathbb{R})$
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Second case: $C(\mathbb{R})$ has two connected components

- $a_1 < a_2 < a_3 \in \mathbb{R}$
- Four real 2-torsion points: O and T_1, T_2, T_3:
  - if $A_1 \oplus A_2 \oplus A_3 = O$, $\ell^2_{A_1,A_2}.C = 2(A_1 + A_2 + A_3)$ and $q = \ell^2_{A_1,A_2} \geq 0$ on $\mathbb{P}^2(\mathbb{R})$

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☞ What if $A_1 \oplus A_2 \oplus A_3 = T_i$ for $i = 1, 2, 3$?

☞ How to **distinguish** T_1, T_2, T_3?

Lemma

If ℓ_{T_i} is s.t. $\ell_{T_i}.C = 2T_i + O$, $\ell_{T_i}\ell_O \geq 0$ on $C(\mathbb{R})$ iff $i = 1$.

Proof

For $i = 1, 2$, $\ell_{T_i}\ell_O$ changes sign on $C(\mathbb{R})$. For $i = 1$, write:

$$(\ell_{T_1}^2 + (a_2 - a_1)(a_3 - a_1)\ell_O^2)\,\ell_O\ell_{T_1} = \\ = x_2^2\ell_O^2 + (a_2 + a_3 - 2a_1)\ell_O^2\ell_{T_1}^2 \mod I(C)$$

a Krivine-Stengle certificate [3] of non-negativity for $\ell_O\ell_{T_1}$.

[3] Blekherman, Smith, Velasco, 2019

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Theorem (B., Blekherman, Sinn)

If q is s.t. $q.C = 2(A_1 + A_2 + A_3)$, $A_i \in C(\mathbb{R})$, $q \in P_{C,2}$ iff:

- $A_1 \oplus A_2 \oplus A_3 = O$ ($q = \ell^2$ is a double line); or
- $A_1 \oplus A_2 \oplus A_3 = T_1$.

These are the **extremal rays** of $P_{C,2}$.

- $(\ell^2_{\ominus A_1 \ominus A_2, A_1 \oplus A_2}\ell^2_{T_i})\,q = (\ell^2_{A_1,A_2}\ell^2_{A_1\oplus A_2, A_3})\,\ell_O\ell_{T_i} \mod I(C)$

[3] Blekherman, Smith, Velasco, 2019

Faces of $P_{C,2}$

- **Explicit, Krivine-Stengle** certificates of deg. 8 $\forall q \in P_{C,2}$!
- Extremal rays are **faces** of $P_{C,2}$ of dimension 1
- $P_{C,2} \subset \mathbb{R}[x_0, x_1, x_2]_2$ has dimension 6

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☞ What about other faces?

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- **Explicit, Krivine-Stengle** certificates of deg. 8 $\forall q \in P_{C,2}$!
- Extremal rays are **faces** of $P_{C,2}$ of dimension 1
- $P_{C,2} \subset \mathbb{R}[x_0, x_1, x_2]_2$ has dimension 6

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- Let $A_1, A_2 \in C(\mathbb{R})$. The dimension of the face $\mathcal{F}_{A_1,A_2} = \{q \in P_{C,2} \colon q.C \geq 2(A_1 + A_2)\} \subset P_{C,2}$ is 2.
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- Fixing **one** point: dimension drops by **2**
- Fixing **two** points: dimension dimension drops by **4**
- Fixing **three special** points: dimension drops by **5**

Extension: Higher Degrees

We consider now $P_{C,2d} \subset \mathbb{R}[x_0, x_1, x_2]_{2d}$ and $q \in \mathbb{R}[x_0, x_1, x_2]_{2d}$

Theorem (B., Blekherman, Sinn)

Let q be s.t. $q.C = 2(A_1 + \cdots + A_{3d})$, $A_i \in C(\mathbb{R})$. Then $q \in P_{C,2d}$ iff:

- $A_1 \oplus \cdots \oplus A_{3d} = O$ ($q = g^2 \in \Sigma_{C,2d}$); or
- $A_1 \oplus \cdots \oplus A_{3d} = T_1$.

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Theorem (B., Blekherman, Sinn)

Let $A_1, \ldots, A_k \in C(\mathbb{R})$, $k \leq 3d - 1$. The dimension of the face $\mathcal{F}_{A_1,\ldots,A_k} = \{q \in P_{C,2d} \colon q.C \geq 2(A_1 + \cdots + A_k)\} \subset P_{C,2d}$ is $6d - 2k$

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- Dual cone of $P_{C,2d}$: convex sums of Dirac measures supported on $C(\mathbb{R})$ (cubature rules)
- How many atoms are needed? (**Carathéodory numbers**)
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Perspectives

- What if we replace $\mathbb{R}$ with any real closed field R?
- **Geometry** of Krivine-Stengle certificates, **effectivity**
- Extension to **higher degree curves** $C \subset \mathbb{P}^2$ and **higher dimensional varieties** $X \subset \mathbb{P}^n$
- How the **topology** of $X(\mathbb{R})$ affect $P_{X,2}$?

Survey: G. Blekherman, R. Sinn, G. G. Smith, M. Velasco, Sums of Squares: A Real Projective Story, 2021

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Thank you for your attention!