

Fast Algorithms for Discrete Differential Equations

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Motivation: Enumerating discrete structures...

Counting walks in $\mathbb{N}$ with steps in $\{+1, -2\}$

$c_n := \#\{n \text{ steps walks starting at } 0 \text{ and ending at height } 0\}$



$G(t) := \sum_{n=0}^{\infty} c_n t^n$ generating function



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Enumeration refinement

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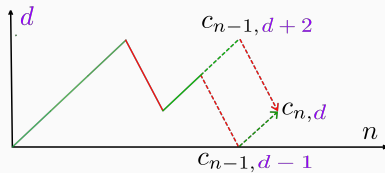
$$F(t, u) := \sum_{n=0}^{\infty} \sum_{d=0}^n c_{n,d} u^d t^n \quad \text{complete generating function}$$

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DDE of order 2

$$c_{n,d} = c_{n-1,d-1} + c_{n-1,d+2}$$

$$F(t, u) = 1 + t \cdot u \cdot F(t, u) + t \cdot \frac{F(t, u) - F(t, 0) - u \cdot \partial_u F(t, 0)}{u^2}$$

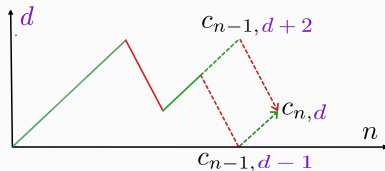
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$$F(t, u) = 1 + t \cdot u \cdot F(t, u) + t \cdot \frac{F(t, u) - F(t, 0) - u \cdot \partial_u F(t, 0)}{u^2}$$

$$c_{n,0} = c_n \implies F(t, 0) = G(t)$$

... yields **challenging** computational problems

$\mathbb{K}$ effective field of characteristic 0.

$\mathbb{K} = \mathbb{Q}, \mathbb{Q}(y), \dots$

Starting point: $F \in \mathbb{K}[\textcolor{red}{u}][[t]]$, solution of the discrete differential equation of order **2**

$$F(t, \textcolor{red}{u}) = 1 + t \cdot \textcolor{red}{u} \cdot F(t, \textcolor{red}{u}) + t \cdot \Delta^{(2)} F(t, \textcolor{red}{u}),$$

where $\Delta F(t, \textcolor{red}{u}) := \frac{(F(t, \textcolor{red}{u}) - F(t, 0))}{\textcolor{red}{u}}$ and $\Delta^{(2)} F(t, \textcolor{red}{u}) = \frac{F(t, \textcolor{red}{u}) - F(t, 0) - \textcolor{red}{u} \cdot \partial_u F(t, 0)}{\textcolor{red}{u}^2}$.

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Classical: F , $F(t, 0)$ and $\partial_{\textcolor{red}{u}} F(t, 0)$ are **algebraic**. [Bousquet-Mélou, Jehanne '06]

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Goals:

- Compute a polynomial $R \in \mathbb{K}[t, z_0] \setminus \{0\}$ such that $R(t, F(t, 0)) = 0$.
- Estimate the size of R for such DDEs.
- Complexity estimates (ops. in $\mathbb{K}$) for the computation of R .

Let $k \geq 1$, $f \in \mathbb{K}[u]$ and $Q \in \mathbb{K}[x, y_1, \dots, y_k, t, u]$. For $F \in \mathbb{K}[u][[t]]$, define $\Delta(F) := (F - F(t, 0))/u \in \mathbb{K}[u][[t]]$ and $\Delta^{(i)}(F) := \Delta \circ \Delta^{(i-1)}(F)$.

Theorem [Bousquet-Mélou, Jehanne '06]

There exists a unique solution $F \in \mathbb{K}[u][[t]]$ to

$$F(t, u) = f(u) + t \cdot Q(F, \Delta(F), \dots, \Delta^{(k)}(F), t, u), \quad (\text{DDE})$$

and moreover $F(t, u)$ is **algebraic** over $\mathbb{K}(t, u)$.

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[Tutte, Brown 60's], [Zeilberger '92]:

Guess-and-prove

[Gessel, Zeilberger '14]:

Guess-and-prove

[Brown '65], [Bender, Canfield '94]:

Quadratic method

[Knuth '68], [Banderier, Flajolet '02],

Kernel method (linear case)

[Bousquet-Mélou, Petkovšek '00]:

[Bousquet-Mélou, Jehanne '06]:

Polynomial elimination

[Bostan, Chyzak,

Notarantonio, Safey El Din '22]:

Polynomial elimination,
Hybrid guess-and-prove

Modelization: from (DDE) of order k to structured polynomial systems

We write $P(u) \equiv P(F(t, u), F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, u)$ and $\overline{\mathbb{K}}[[t^{\frac{1}{\star}}]] \equiv \bigcup_{d \geq 1} \overline{\mathbb{K}}[[t^{\frac{1}{d}}]]$

[Bousquet-Mélou, Jehanne '06]:

Discrete Differential Equation
(DDE)

$\Downarrow$ numer

$$P(u) = 0$$

$\Downarrow \partial_u$

$$\partial_u F(t, u) \cdot \partial_1 P(u) + \partial_u P(u) = 0$$

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General idea:

$$\mathbf{U}(t) \in \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]] \setminus \overline{\mathbb{K}} \text{ solution in } u \text{ of } \partial_1 P(u) = 0$$

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$$\begin{aligned} \mathbf{U}_1(t), \dots, \mathbf{U}_\ell(t) \in \\ \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]] \setminus \overline{\mathbb{K}} \text{ distinct} \\ \text{solutions in } u \text{ of} \\ \partial_1 P(u) = 0 \end{aligned}$$

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The points

$(x_1, \mathbf{u}_1) = (F(t, \mathbf{U}_1), \mathbf{U}_1), \dots, (x_\ell, \mathbf{u}_\ell) = (F(t, \mathbf{U}_\ell), \mathbf{U}_\ell) \in \overline{\mathbb{K}}[[t^{\frac{1}{\star}}]]^2$
are solutions of the conditions:

$$\forall 1 \leq i \leq \ell, \quad \begin{cases} P(x_i, F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, \mathbf{u}_i) = 0, \\ \partial_1 P(x_i, F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, \mathbf{u}_i) = 0, \\ \partial_u P(x_i, F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, \mathbf{u}_i) = 0. \end{cases}$$

and $\prod_{i \neq j} (\mathbf{u}_i - \mathbf{u}_j) \neq 0$.

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and $\prod_{i \neq j} (\mathbf{u}_i - \mathbf{u}_j) \neq 0$. $\ell = k \implies 3k \text{ equations and } 3k \text{ unknowns!}$

$$F(t, u) = f(u) + t \cdot Q(F, \Delta(F), \dots, \Delta^{(k)}(F), t, u), \quad (\text{DDE})$$

where $\Delta(F) := \frac{F(t, u) - F(t, 0)}{u}$.

Input: $P := \text{numerator}(\text{DDE})$,

Goal: Compute $R \in \mathbb{K}[t, z_0] \setminus \{0\}$ s.t. $R(t, F(t, 0)) = 0$.

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1. **Geometric analysis** of Bousquet-Mélou and Jehanne's algorithm yielding:
 - Theoretical estimate for the degree of $R \in \mathbb{K}[t, z_0]$ s.t. $R(t, F(t, 0)) = 0$,
 - Arithmetic complexity.
2. **New algorithm** based on **algebraic elimination** + Gröbner bases,
3. **Implementations** yielding practical improvements.

Example: (walks in $\mathbb{N}$ with steps in $\{+1, -2\}$)

We consider $P(F(t, u), F(t, 0), \partial_u F(t, 0), t, u) = 0$

$$u^2 - (u^2 - t(1 + u^3)) \cdot F(t, u) - t \cdot F(t, 0) - t \cdot u \cdot \partial_u F(t, 0) = 0$$

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Study the solutions of $\partial_1 P(F(t, u), F(t, 0), \partial_u F(t, 0), t, u) = 0$

$$u^2 = t(1 + u^3) \implies u = U_1(t), U_2(t) \in \{\sqrt{t} + O(t), -\sqrt{t} + O(t)\}$$

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Hence $(x_1, u_1, x_2, u_2) = (F(t, U_1), U_1, F(t, U_2), U_2)$ is a solution of the constraints $\mathcal{T}$

$$\mathcal{T}: \quad \text{For } 1 \leq i \leq 2, \quad \begin{cases} P(x_i, F(t, 0), \partial_u F(t, 0), t, u_i) = 0, \\ \partial_1 P(x_i, F(t, 0), \partial_u F(t, 0), t, u_i) = 0, \\ \partial_u P(x_i, F(t, 0), \partial_u F(t, 0), t, u_i) = 0. \end{cases} \quad m \cdot (u_1 - u_2) = 1,$$

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Permuting (x_1, u_1) and (x_2, u_2) does not change the solution set
 $\implies \mathfrak{S}_2$ acts on $V(\mathcal{T})$ and preserves the $\{z_0, z_1, t\}$ -coordinate space.

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Impact of this group action?

Denote by $\mathcal{I}$ the ideal generated by the k duplications of $(P, \partial_1 P, \partial_u P)$ and $m \cdot \prod_{i \neq j} (u_i - u_j) - 1 = 0$.

Assume that:

- there exist k distinct solutions $u = U_1, \dots, U_k \in \overline{\mathbb{K}}[[t^{\frac{1}{*}}]]$ of $\partial_1 P(u) = 0$,
- $\mathcal{I}$ is radical and of dimension 0 over $\mathbb{K}(t)$. (3k equations and 3k unknowns)

Denote by $\mathcal{I}$ the ideal generated by the k duplications of $(P, \partial_1 P, \partial_u P)$ and $m \cdot \prod_{i \neq j} (u_i - u_j) - 1 = 0$.

Assume that:

- there exist k distinct solutions $u = U_1, \dots, U_k \in \overline{\mathbb{K}}[[t^{\frac{1}{k}}]]$ of $\partial_1 P(u) = 0$,
- $\mathcal{I}$ is radical and of dimension 0 over $\mathbb{K}(t)$. (3k equations and 3k unknowns)

Theorem [Bostan, N., Safey El Din '23]

- Let $\delta := \deg(P)$. There exists a nonzero polynomial $R \in \mathbb{K}[t, z_0]$ whose partial degrees are bounded by $\delta^k(\delta - 1)^{2k}/k!$ and such that $R(t, F(t, 0)) = 0$.
- There exists an algorithm computing R in $\tilde{O}(\delta^{6k}(k^2\delta^{k+3} + \delta^{1.89k}/k!))$ ops. in $\mathbb{K}$.

Summary of the initial problem: $\underline{z} \equiv z_0, \dots, z_{k-1}; P = \text{"numer"}(DDE) \in \mathbb{K}(t)[x, \mathbf{u}, \underline{z}]$

There exist k solutions $(x, \mathbf{u}) \in \overline{\mathbb{K}(t)}^2$ with **distinct** $\mathbf{u}$ -coordinates to

$$\begin{cases} P(x, \mathbf{u}, F(t, 0), \dots, \partial_u^{k-1} F(t, 0)) = 0, \\ \partial_1 P(x, \mathbf{u}, F(t, 0), \dots, \partial_u^{k-1} F(t, 0)) = 0, \quad \mathbf{u} \neq 0, \\ \partial_u P(x, \mathbf{u}, F(t, 0), \dots, \partial_u^{k-1} F(t, 0)) = 0. \end{cases}$$

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Define

$$\pi_x : (x, \mathbf{u}, \underline{z}) \in \overline{\mathbb{K}(t)}^{k+2} \mapsto (\mathbf{u}, \underline{z}) \in \overline{\mathbb{K}(t)}^{k+1},$$

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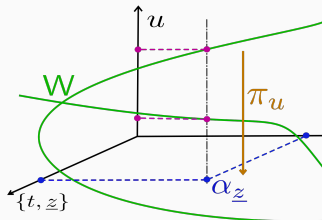
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$$\# \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} = 2$$

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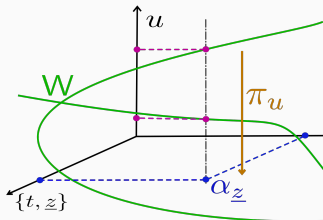
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$$\# \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} = 2$$

Objective:

Characterize with polynomial constraints

$$\mathcal{F}_k := \{\alpha_{\underline{z}} \in \overline{\mathbb{K}(t)}^k \mid \# \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} \geq k\}$$

Example: (Walks in $\mathbb{N}$ with steps in $\{+1, -2\}$)

$$P := (1 - x)u^2 + tu^3x + t(x - z_0 - uz_1) \in \mathbb{K}(t)[x, u, z_0, z_1], \quad k = 2.$$

G_u Gröbner basis of $\langle P, \partial_1 P, \partial_u P, mu - 1 \rangle \cap \mathbb{K}[u, t, z_0, z_1]$ for $\{u\} \succ_{lex} \{t, z_0, z_1\}$:

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$$\begin{aligned} B_0 : & \quad \gamma_0 \\ B_1 : & \left\{ \begin{array}{l} \beta_1 \cdot u + \gamma_1 \\ \vdots \\ \beta_r \cdot u + \gamma_r \end{array} \right. , \gamma_i, \beta_j \in \mathbb{K}[t, z_0, z_1] \\ B_2 : & \quad g_2 := u^2 + \beta_{r+1} \cdot u + \gamma_{r+1} \end{aligned}$$

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Necessary condition:

At $\alpha \in V(G_u \cap \mathbb{K}[t, z_0, z_1])$ fixed,
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The **pre-image** of α by π_u is well-defined

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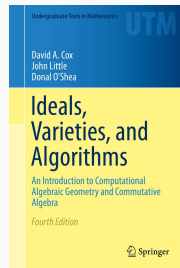
After adding these constraints to G_u and eliminating u and z_1 :

$$R(t, z_0) = t^3 z_0^3 - z_0 + 1 \text{ satisfies } R(t, F(t, 0)) = 0$$

- **Projecting:** **Elimination theorem**
- **Lifting points of the projections:** **Extension theorems**

Cardinality conditions on the fibers:

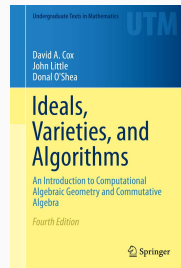
- **Extension theorem** (Gröbner bases version)
- $g(u, \alpha_z) \in \overline{\mathbb{K}(t)}[u]$ of degree $k + j$ has at least k distinct roots
 $\iff \neg (g = 0 \wedge \partial_u g = 0 \wedge \dots \wedge \partial_u^j g = 0)$



- **Projecting:** **Elimination theorem**
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Cardinality conditions on the fibers:

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[Bostan, N., Safey El Din '23] :

Conjunctions of **polynomial equations** and **inequations** whose zero set is $\mathcal{F}_k$

3-Tamari, $k = 3$

A	T	d_t	d_{z_0}
D	2d2h	5	16
DE	2m	5	16
HGP + DE	1h40m	5	16

5-constellations, $k = 4$

A	T	d_t	d_{z_0}
D	∞	—	—
DE	∞	26 ●	53 ●
HGP + DE	6h7m	2	5

●: data obtained after a computation mod $p = 65521$.

- **A**: Algorithm used (D: duplication, DE: direct elimination, HGP: Hybrid Guess-and-Prove [Bostan, Chyzak, N., Safey El Din '22]),
- **T**: total timing needed to obtain an output in $\mathbb{Q}[t, z_0]$,
- **d_z** : degree in $Z \in \{t, z_0\}$ of output $R \in \mathbb{Q}[t, z_0]$ s.t. $R(t, F(t, a)) = 0$,

Intel® Xeon® Gold CPU 6246R v4 @ 3.40GHz and 1.5TB of RAM with a single thread.

Gröbner bases computations are performed using the C library **msolve**, and all guessing computations are performed using the **gfun** Maple package.

Conclusion

- New geometric interpretations of the problem “solving a DDE”,
- New algorithm based on algebraic elimination and Gröbner bases,
- Some promising practical results,
- (In the preprint) New geometric algorithm based on Stickelberger’s theorem.

Future works

- Study the *minimality* and the *genericity* of the introduced assumptions,
- Provide a maple package for solving DDEs, together with a tutorial paper.

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Input: $P(F(t, u), F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, u) = 0$, $\delta := \deg(P)$.

Output: $R \in \mathbb{K}[t, z_0] \setminus \{0\}$ annihilating $F_0 = F(t, 0)$, i.e. $R(t, F_0) = 0$.

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geometry

(1) Functional
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(2) Polynomial
system



(3) Bounds

- $\deg_t(R) \leq b_t$,
- $\deg_{z_0}(R) \leq b_z$.

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guess-and-prove

(4) Expand F_0



(5) Compute $R \in \mathbb{K}[t, z_0]$
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 $R(t, F_0) = O(t^{\sim b_t} b_z)$



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tools

- Newton iteration



- Algebraic approximants
"seriestoalgeq"



- Multiplicity lemma:
 $R(t, F_0) = O(t^{\sim 2b_t b_z})$
implies $R(t, F_0) = 0$

Solving our toy example using the Hybrid Guess-and-Prove strategy

Counting walks in $\mathbb{N}$ with steps in $\{+1, -2\}$

$$F(t, u) = 1 + t \cdot u \cdot F(t, u) + t \cdot \frac{F(t, u) - F(t, 0) - u \cdot \partial_u F(t, 0)}{u^2}$$

- Draw a random $c = 1341$, and a prime number $p = 19541$,
- Using the new algorithm based on elimination theory, we obtain:
 - $R(t, c) \bmod p = t^3 + 15794$,
 - $R(c, z_0) \bmod p = z_0^3 + 18182z_0 + 1319$.
- Set $b_t = 3$, $b_{z_0} = 3$,
- Compute
$$F(t, 0) = 1 + t^3 + 3t^6 + 12t^9 + 55t^{12} + 273t^{15} + 1428t^{18} + O(t^{2 \cdot b_t \cdot b_{z_0} + 1})$$
- Guess $A := t^3 z_0^3 - z_0 + 1$ such that $A(t, F(t, 0)) = O(t^{(b_t+1) \cdot (b_{z_0}+1)-1})$,
- Check that $A(t, F(t, 0)) = O(t^{2 \cdot b_t \cdot b_{z_0} + 1})$

The output A is certified.

Let $P \in \mathbb{K}[x, \underline{z}, t, u]$ be the numerator of a DDE, $\underline{z}_1 \equiv z_0, z_2, \dots, z_{k-1}$,

$\mathcal{I} := \langle P, \partial_1 P, \partial_u P \rangle \subset \mathbb{K}(\underline{t}, \underline{z}_1)[x, u, z_1]$ of dimension 0 $\leftrightarrow \mathcal{J} := \langle P, \partial_1 P, \partial_u P \rangle \subset \mathbb{K}(t)[x, \underline{z}, t, u]$,

Define $\pi : (x, \underline{z}, u) \in \overline{\mathbb{K}(t)}^{k+2} \mapsto (\underline{z}) \in \overline{\mathbb{K}(t)}^k$ and $\pi_{\underline{z}_1} : (x, \underline{z}, u) \in \overline{\mathbb{K}(t)}^{k+2} \mapsto (\underline{z}) \in \overline{\mathbb{K}(t)}^{k-1}$:

$\pi|_{V(\mathcal{J})}$ and $\pi_{\underline{z}_1}|_{V(\mathcal{J})}$ have finite fibers and $\pi_{\underline{z}}|_{V(\mathcal{J})}$ is dominant,

Define $S_k(\underline{z}_1, V(\mathcal{J})) := \{\alpha \in \pi(V(\mathcal{J})) \text{ s.t. } \pi_{\underline{z}_1}^{-1}(\alpha_0, \alpha_2, \dots, \alpha_{k-1}) \geq k\}$.

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$(x, u, z_1) = (F(t, U_1), U_1, \partial_u F(t, 0)), \dots, (F(t, U_k), U_k, \partial_u F(t, 0)) \in \overline{\mathbb{K}}[[t^{\frac{1}{k}}]]^2 \times \mathbb{K}[[t]]$
are k distinct solutions with same z_1 -coordinate of the system

$$P(x, F(t, 0), z_1, \partial_u^2 F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, u) = 0,$$

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$\pi|_{V(\mathcal{J})}$ and $\pi_{\underline{z}_1}|_{V(\mathcal{J})}$ have **finite fibers** and $\pi_z|_{V(\mathcal{J})}$ is **dominant**,

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$(x, u, z_1) = (F(t, U_1), U_1, \partial_u F(t, 0)), \dots, (F(t, U_k), U_k, \partial_u F(t, 0)) \in \overline{\mathbb{K}}[[t^{\frac{1}{k}}]]^2 \times \mathbb{K}[[t]]$
are k distinct solutions with same z_1 -coordinate of the system

$$P(x, F(t, 0), z_1, \partial_u^2 F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, u) = 0,$$

$$\partial_1 P(x, F(t, 0), z_1, \partial_u^2 F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, u) = 0,$$

$$\partial_u P(x, F(t, 0), z_1, \partial_u^2 F(t, 0), \dots, \partial_u^{k-1} F(t, 0), t, u) = 0.$$

multiplication map

$m_{z_1} : f \mapsto f \cdot z_1$
in $\mathbb{K}(t, \underline{z}_1)[x, u, z_1]/\mathcal{I}$

Stickelberger's theorem: The roots of $\chi := \text{CharPoly}(m_{z_1})$
are the z_1 -coordinates of the points in $V(\mathcal{I})$.

Let $P \in \mathbb{K}[x, \underline{z}, t, u]$ be the numerator of a DDE, $\underline{z}_1 \equiv z_0, z_2, \dots, z_{k-1}$,
 $\mathcal{I} := \langle P, \partial_1 P, \partial_u P \rangle \subset \mathbb{K}(t, \underline{z}_1)[x, u, z_1]$ of dimension 0 $\leftrightarrow \mathcal{J} := \langle P, \partial_1 P, \partial_u P \rangle \subset \mathbb{K}(t)[x, \underline{z}, t, u]$,
 Define $\pi : (x, \underline{z}, u) \in \overline{\mathbb{K}(t)}^{k+2} \mapsto (\underline{z}) \in \overline{\mathbb{K}(t)}^k$ and $\pi_{\underline{z}_1} : (x, \underline{z}, u) \in \overline{\mathbb{K}(t)}^{k+2} \mapsto (\underline{z}) \in \overline{\mathbb{K}(t)}^{k-1}$:
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Proposition: [Bostan, N., Safey El Din '23] There exists a subset $\mathcal{Z} \subset \pi(V(\mathcal{J}))$ Zariski open dense such that

$$\pi(V(\mathcal{I} \cap \mathbb{K}[t, \underline{z}] + \langle \chi, \dots, \partial_{z_1}^{k-1} \chi \rangle)) \cap \pi(\mathcal{Z}) = \mathcal{S}_k(\underline{z}_1, V(\mathcal{J}) \cap \mathcal{Z}).$$

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If $S_k(\underline{z}_1, V(\mathcal{J}) \cap \mathcal{Z}) = S_k(\underline{z}_1, V(\mathcal{J}))$ and has dimension 0 $\rightarrow$ **Compute** χ