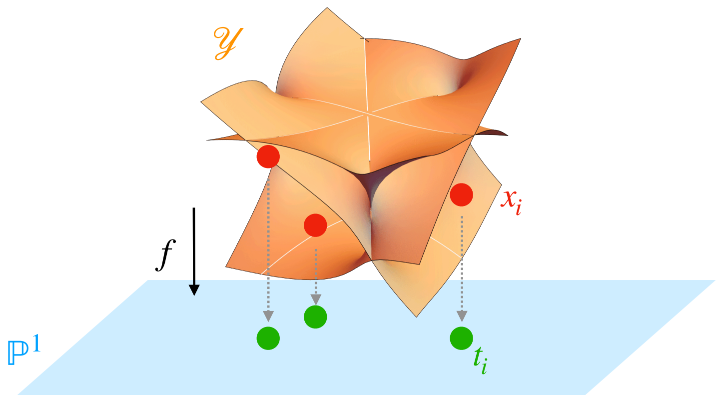
Eric Pichon-Pharabod

Computing periods of hypersurfaces

Joint work with Pierre Lairez and Pierre Vanhove



Periods are rational integrals

A is homogeneous of degree $k \deg P - \deg \Omega$

Ω is the volume form of $\mathbb{P}^n$

$$\int_\gamma \frac{A}{P^k}\Omega$$

Some integration domain without boundary

P defines a smooth complex projective hypersurface $\mathscr{X} = V(P) = \{P = 0\}$

The period matrix

We chose generating families $\gamma_1, \ldots, \gamma_r \in H_n(\mathscr{X})$ and $\omega_1, \ldots, \omega_r \in H^n_{DR}(\mathscr{X})$.

Define the period matrix

$$\Pi = \left(\int_{\gamma_j} \omega_i \right)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq r}}$$

It is an **invertible** matrix that describes the isomorphism between DeRham cohomology and homology.

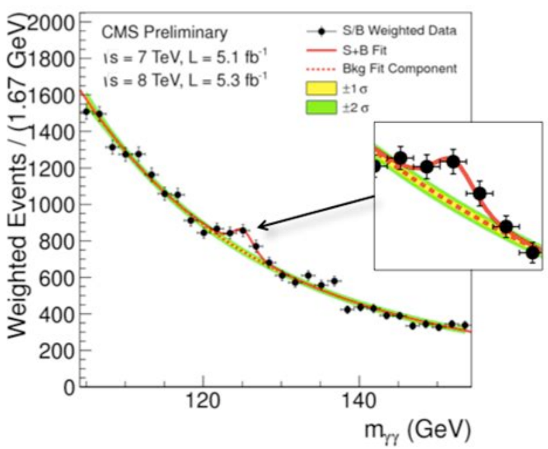
Our goal is to find a way, given P, to compute the period matrix of $\mathscr{X} = V(P)$.

Why are periods interesting?

The period matrix of $\mathscr{X}$ contains information about fine **algebraic invariants** $\mathscr{X}$.

Torelli-type theorems : the period matrix of $\mathscr{X}$ determines its isomorphism class (in certain cases).

Feynman integrals are relative periods that give scattering amplitudes of particle interactions in quantum field theory.



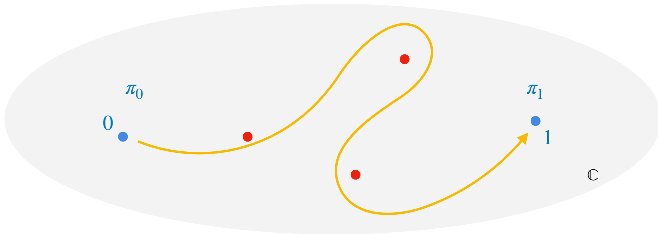
Previous works

[Deconinck,van Hoeij 2001], [Bruin, Sijsling, Zotine 2018], [Molin, Neurohr 2017]:
algebraic curves (Riemann surfaces)

Sertöz 2019: compute the period matrix by deformation:

We wish to compute $\displaystyle\int_\gamma \frac{\Omega}{X^3+Y^3+Z^3+XYZ}$.

$\rightarrow$ introduce a parameter t, look at $\displaystyle\pi_t = \int_\gamma \frac{\Omega}{X^3+Y^3+Z^3+tXYZ}$,



π_t is a solution of $(t^3+27)\partial_t^2 + 3t^2\partial_t + t$ and we have analytic formulae for π_0 **[Pham]**

Using numerical analytic continuation **[Mezzarobba]** we can recover π_1.

- **Computationally expensive:** the differential equations we need to integrate quickly get out of hand
- **Not easily generalisable:** need to know periods of some variety ($V(X^3+Y^3+Z^3)$ in example)

Our goal: get a description of the cycles that is well adapted to integration

Contributions

100s of digits

New method for computing periods with very high numerical precision

coming soon!

→ implementation in Sagemath (using OreAlgebra)

→ efficient enough to tackle new varieties (generic quartic surfaces)

→ byproduct: homology of complex projective varieties

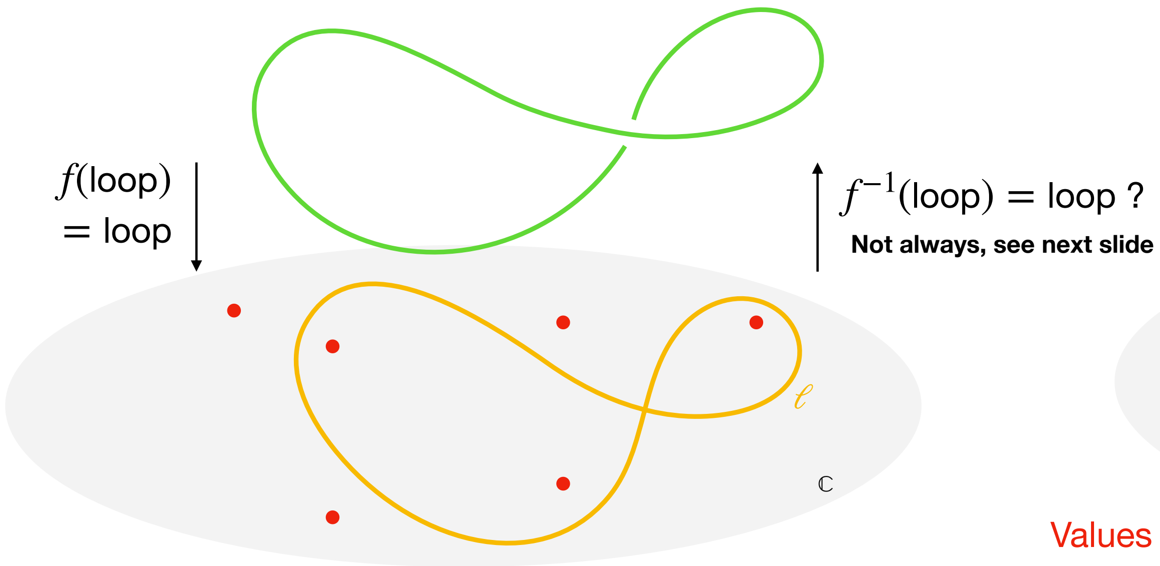
→ generalisable to other types of varieties (e.g. complete intersections, singular varieties)

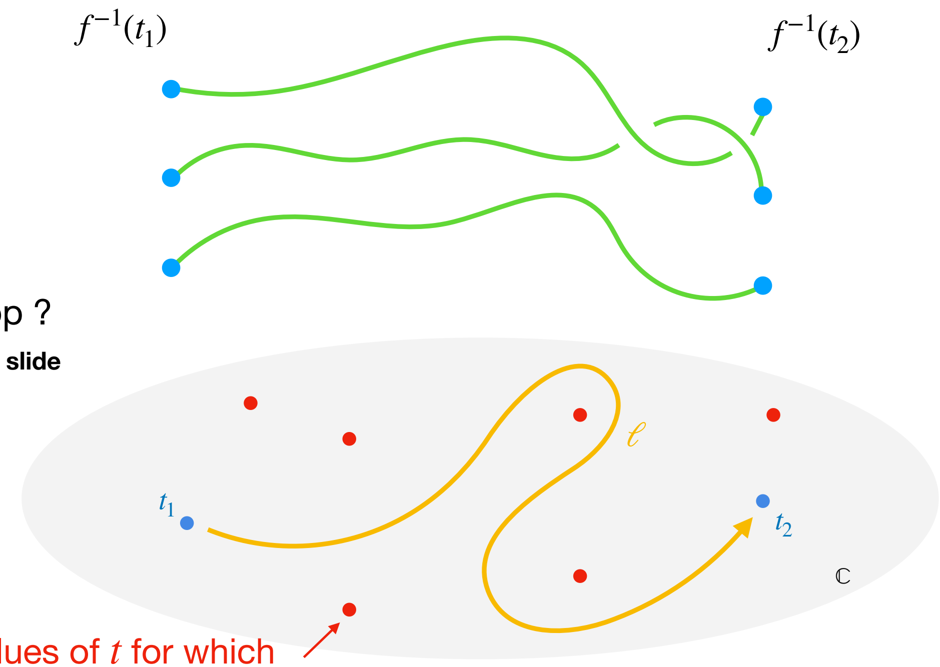
First example: algebraic curves

Let $\mathscr{X}$ be the elliptic curve defined by $P = y^3 + x^3 + 1 = 0$ and let $f : (x, y) \mapsto y/(2x+1)$.

In dimension 1, we are looking for closed paths in $\mathscr{X}$, up to deformation (1-cycles).

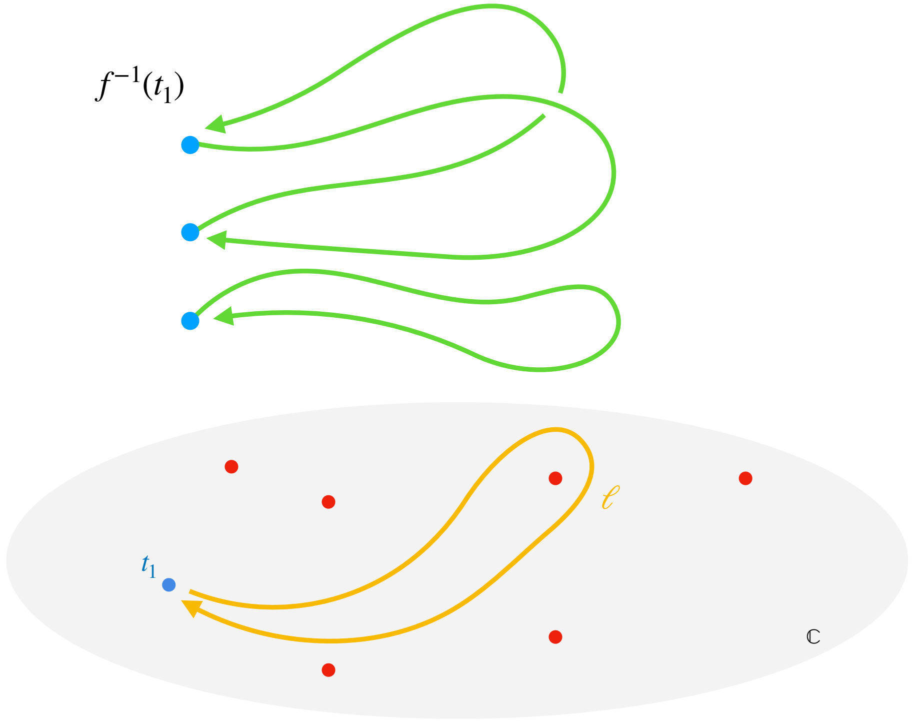
The fiber above $t \in \mathbb{C}$ is $\mathscr{X}_t = f^{-1}(t)$
$= \{(x, t(2x+1)) \mid P\left(x, t(2x+1)\right) = 0\}$.
It deforms continuously with respect to t.

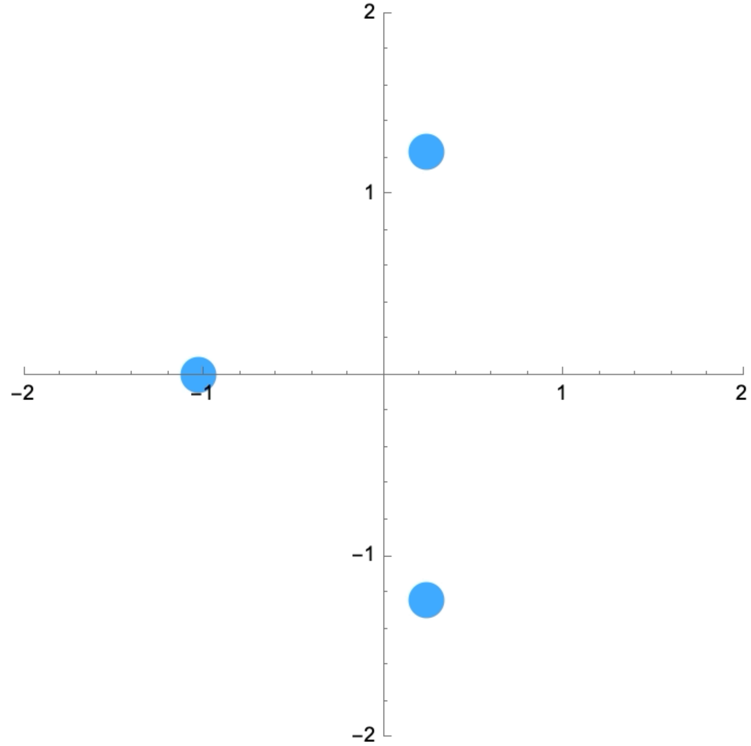




What happens when you loop around a critical point?

A loop ℓ in $\mathbb{C}$ pointed at t_1 induces a permutation of $\mathscr{X}_{t_1}$.



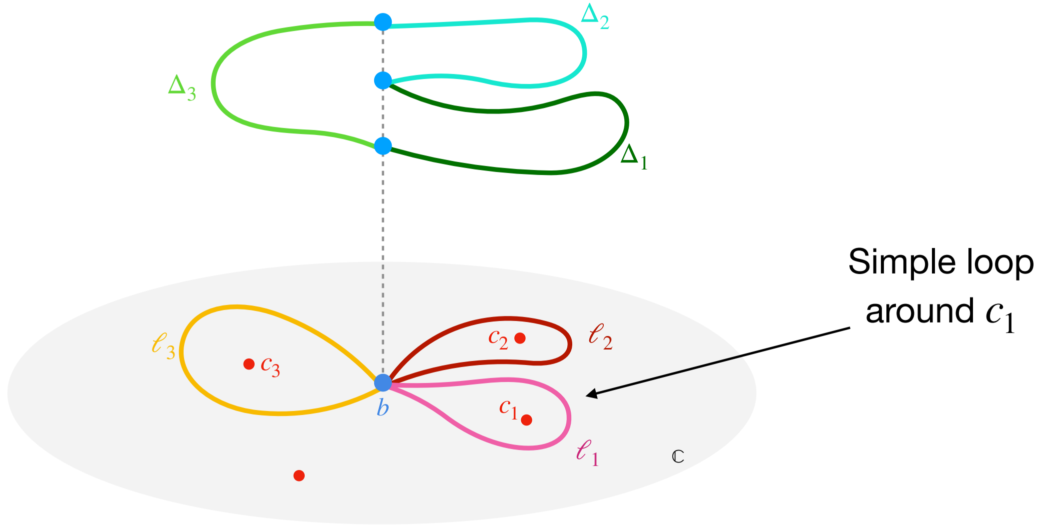


This permutation is called the **action of monodromy along ℓ on $\mathscr{X}_{t_1}$**. It is denoted ℓ_*

If ℓ is a simple loop around a critical value, ℓ_* is a transposition.

Computing periods of algebraic curves

The lift of the simple loop ℓ around a critical value c that has boundary in $\mathscr{X}_b$ is called the **thimble** of c.



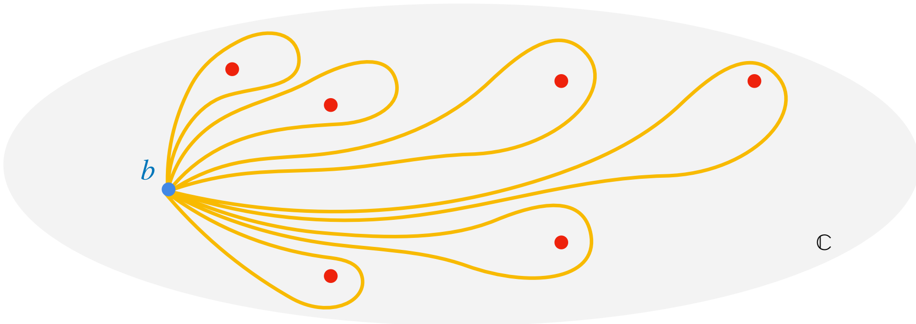
Thimbles serve as building blocks for the closed paths of $\mathscr{X}$.

Indeed, to find a loop that lifts to $\mathscr{X}$, we just need to take glue thimbles so that the boundaries cancel.

Fact: all (homology classes of) loops of $\mathscr{X}$ can be obtained this way.

Computing periods of algebraic curves

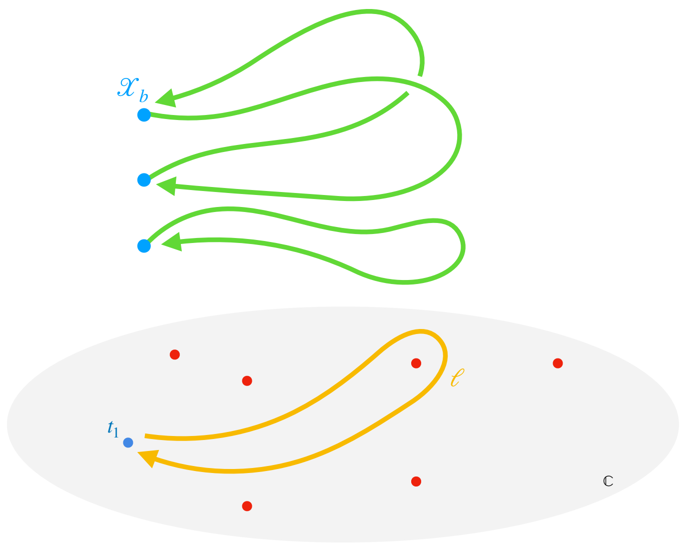
1. Pick simple loops $\ell_1, \ldots, \ell_{\#\text{crit.}}$ around the critical values — basis of $\pi_1(\mathbb{C} \setminus \{\text{crit. values}\})$



Computing periods of algebraic curves

1. Pick simple loops $\ell_1, \ldots, \ell_{\#\text{crit.}}$ around the critical values — basis of $\pi_1(\mathbb{C} \setminus \{\text{crit. values}\})$

2. For every i compute the action of monodromy along ℓ_i on $\mathscr{X}_b$ (transposition)



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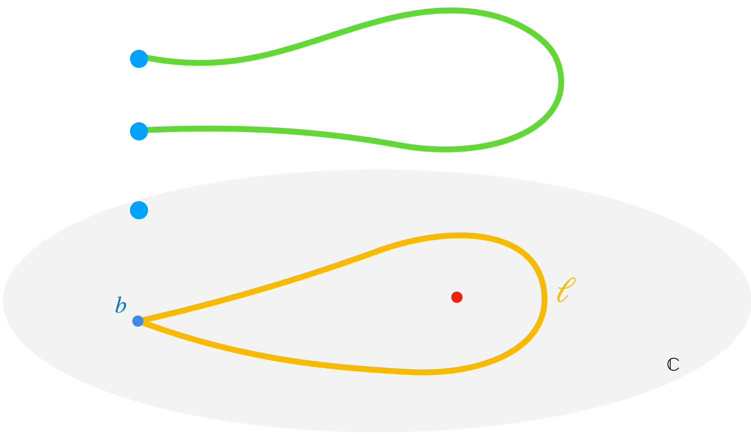
3. This gives the thimble Δ_i. Its border is the two points of $\mathscr{X}_b$ that get permuted.



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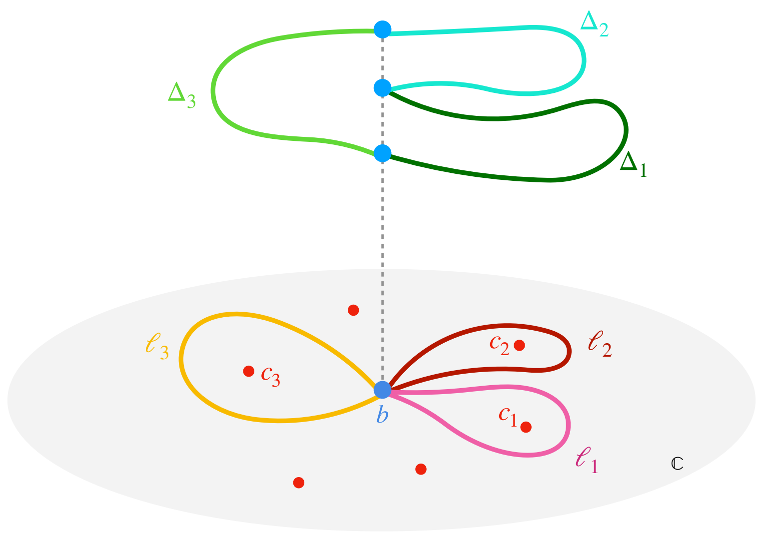
4. Compute sums of thimbles without boundary



Computing periods of algebraic curves

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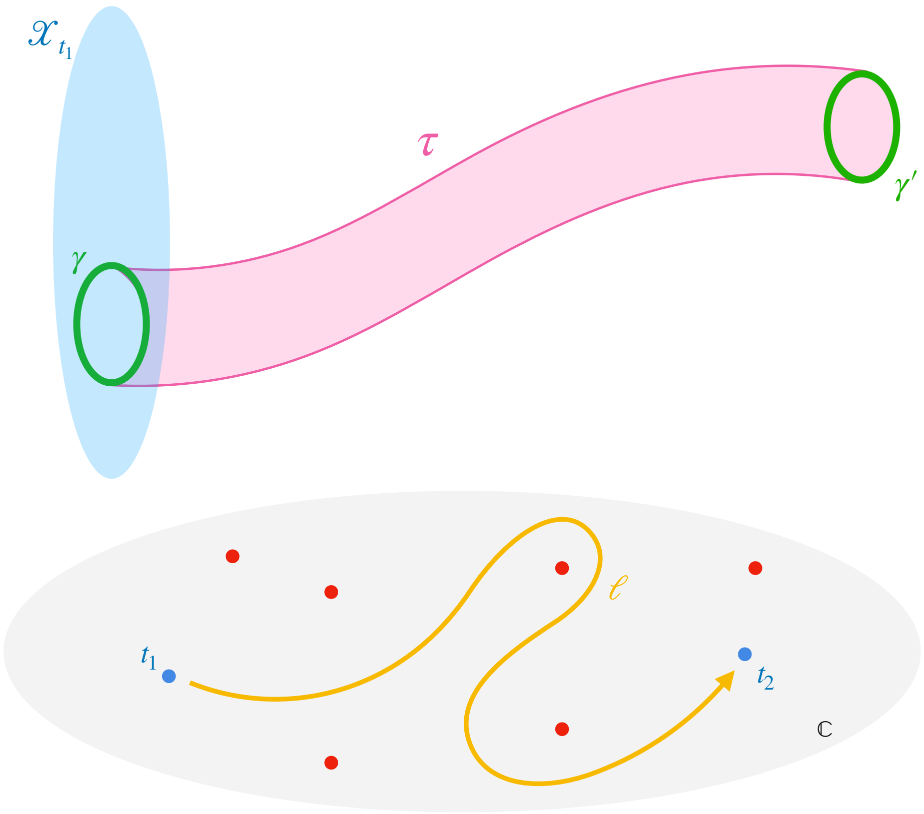
4. Compute sums of thimbles without boundary → loops in $\mathcal{X}$

5. Periods are integrals along these loops
→ we have an explicit parametrisation of the path → numerical integration.

$$\int_\gamma \omega = \int_\ell \omega_t$$

Insight into higher dimensions: surfaces

The fibre $\mathscr{X}_t$ is a variety of dimension 1.
It deforms continuously with respect to t.

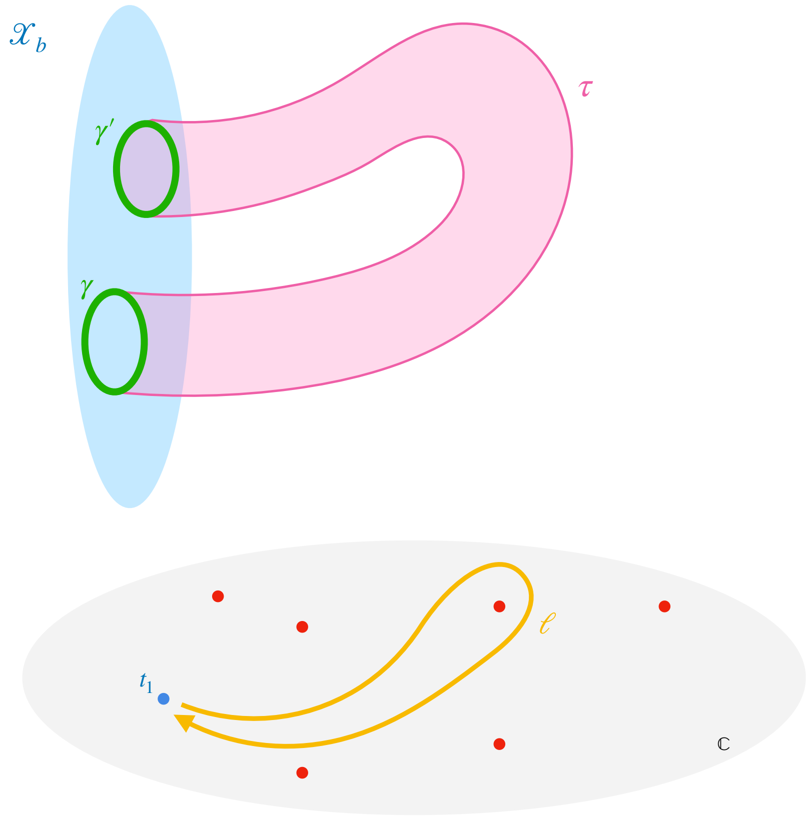


Period of algebraic surface

$$\int_\tau f(x,y)\mathrm{d}x\mathrm{d}y = \int_\ell \left(\int_{\gamma_y} f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

Period of algebraic curve

Comparison with dimension 1



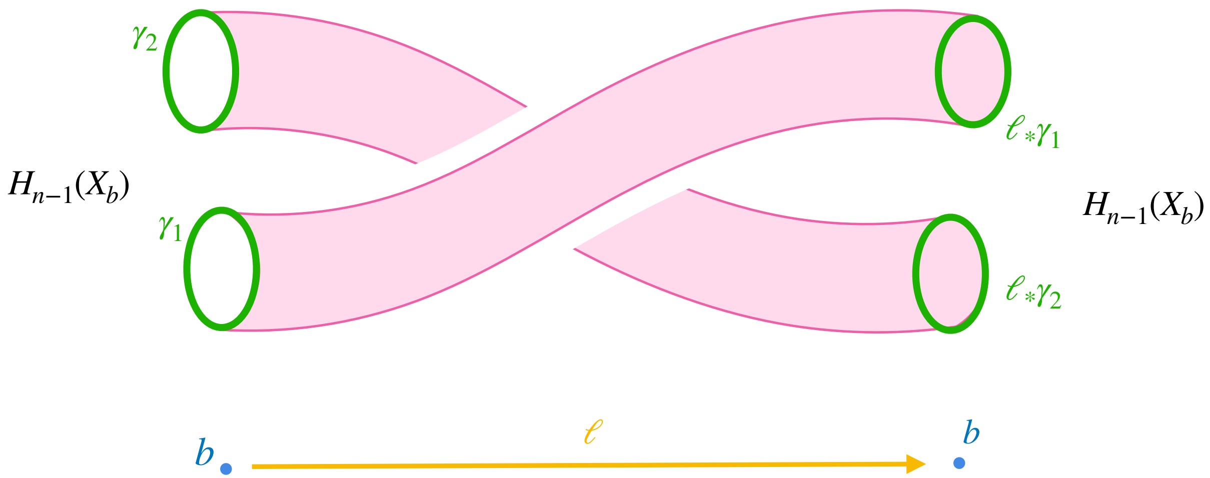
Dimension 1	**Dimension 2**
looking for loops in $\mathscr{X}$	looking for closed 2-manifolds in $\mathscr{X}$
Thimbles are paths obtained by extending points along loops.	Thimbles are tubes (pink) obtained by extending 1-cycles (green) along loops.
The monodromy along a loop ℓ is an isomorphism of $H_0(\mathscr{X}_b)$ (induced by a permutation of $\mathscr{X}_b$)	The monodromy along a loop ℓ is an isomorphism on $H_1(\mathscr{X}_b)$
Complex analysis	Picard-Lefschetz theory
We get every possible loop up to deformation by gluing thimbles.	We get almost every possible 2-cycle by gluing thimble tubes.

There is one thimble per critical point.

We obtain the periods as integrals parametrised on a path.

Computing monodromy

$$\pi_1(\mathbb{C}\backslash\{\text{critical values}\}) \to GL(H_{n-1}(X_b))$$



Given by periods!

Tools we use:

- induction on dimension — we know cycles of $H_{n-1}(X_b)$
- Isomorphism between homology and DeRham cohomology → we gain analytical structure
- Monodromy of a differential operator (Picard-Fuchs equation) **[Mezzarobba]**

Results and perspectives

holomorphic periods of quartic surfaces in an hour (previously unfeasible in most cases).

A singular example: Tardigrade family (a very generic family of quartic K3 surfaces).
[Doran, Harder, Vanhove 2023, Appendix by EPP]
→ able to embed Néron-Severi lattice in standard K3 lattice

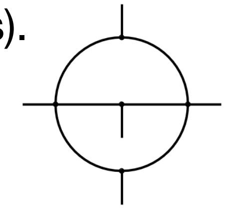


FIGURE 13. The tardigrade graph

Found smooth quartic surface in $\mathbb{P}^3$ with Picard rank 2

$$\mathscr{X} = V\left(\begin{array}{c} X^4 - X^2Y^2 - XY^3 - Y^4 + X^2YZ + XY^2Z + X^2Z^2 - XYZ^2 + XZ^3 \\ -X^3W - X^2YW + XY^2W - Y^3W + Y^2ZW - XZ^2W + YZ^2W - Z^3W + XYW^2 \\ +Y^2W^2 - XZW^2 - XW^3 + YW^3 + ZW^3 + W^4 \end{array}\right)$$

This approach can be applied to more general types of varieties, e.g. complete intersections

Bottleneck for accessing higher dimensions is still the order/degree of the differential operators

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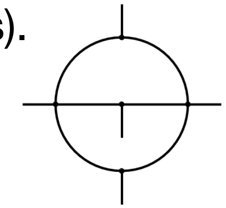

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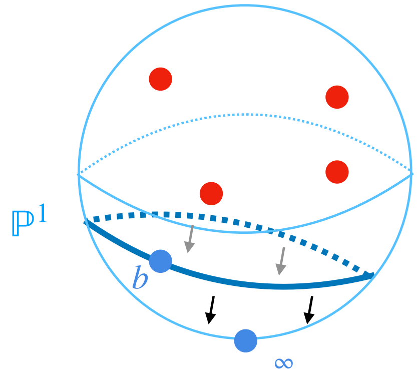
Bottleneck for accessing higher dimensions is still the order/degree of the differential operators

Thank you!

Diagram chasing to recover $H_n(X)$ from thimbles

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_{n+1}(Y) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_{n+1}(Y,Y^+) \cong H_{n-1}(Y_b) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & H_n(Y_b) & \longrightarrow & H_n(Y^+) & \longrightarrow & \boxed{H_n(Y^+,Y_b)} & \longrightarrow & H_{n-1}(Y_b) \longrightarrow 0 \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & H_{n-2}(X') & \longrightarrow & \boxed{H_n(Y)} & \longrightarrow & H_n(X) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & H_n(Y,Y^+) \cong H_{n-2}(Y_b) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$



$$
\boxed{H_n(Y) \simeq H_n(Y_b) \oplus \frac{\ker \delta}{\operatorname{im} \tau_\infty^n} \oplus H_{n-2}(Y_b)}
$$