# How to certify transcendence of D-finite functions?

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Der Wissenschaftsfonds.

**What is a function?**

- For us here: "something" which can be added, multiplied, differentiated...
- Elements of a differential algebra
- Examples: polynomials, rational functions, power/Laurent/Puiseux series...

**Algebraic function**

$$\exists \alpha_0, \dots, \alpha_r \in C(x) \text{ with } 0 = \alpha_r f^r + \cdots + \alpha_1 f + \alpha_0$$
$$\Updownarrow$$
$$\exists\, P = \alpha_r(x) y^r + \cdots + \alpha_1(x) y + \alpha_0(x) \in C(x)[y] \text{ with } 0 = P(x, f)$$

**D-finite function**

$$\exists \alpha_0, \dots, \alpha_r \in C(x) \text{ with } 0 = \alpha_d f^{(d)} + \cdots + \alpha_1 f' + \alpha_0 f$$
$$\Updownarrow$$
$$\exists\, L = \alpha_d D^r + \cdots + \alpha_1 D + \alpha_0 \in C(x)[D] \text{ with } 0 = L(f)$$

**Important examples**: functions with a finite representation and effective arithmetic

# A very general problem

**Question** *"Given a function, is it algebraic ?"*

This question is way too general: how is the function given?

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The function is given by the D-finite equation and initial conditions.

**Reduces to** *"Given a D-finite equation, does it have only algebraic solutions?"*

**Previous work**

- Schwarz, 1873 : classification of algebraic ${}_2F_1$ hypergeometric functions
- Singer, 1979 : algorithm finding all algebraic solutions of a D-finite equation
- Kovacic, 1986: same but practical, for equations of order 2
- Beukers, Heckman, 1989 : same, for all hypergeometric equations

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**Alternatives: sufficient conditions** and verifiable certificates

- For algebraicity: polynomial equation: ✓
- For transcendence:
  - Transcendental singularities: ✓
  - Asymptotics, monodromy: numerics, closely related to transcendental singularities
  - $p$-curvature: Grothendieck's conjecture, "almost all" primes
  - Pseudoconstants

# A few examples

$$L_1 = \left(x^2 - x\right) D^2 + \left(\frac{1}{3}x - \frac{2}{3}\right) D + \frac{1}{12}$$

$$L_2 = \left(x^2 - x\right) D^2 + (x - 2) D - \frac{1}{9}$$

$$L_3 = \left(x^2 - x\right) D^2 + \left(\frac{31}{24}x - \frac{5}{6}\right) D + \frac{1}{48}$$

$$L_4 = \left(x^2 - x\right) D^2 + \left(-\frac{49}{6}x + 2\right) D - 12$$

$$L_5 = \left(x^2 - x\right) D^2 + \left(\frac{49}{6}x - \frac{7}{3}\right) D + 12$$

$$L_6 = \left(x^2 - x\right) D^2 + \left(\frac{164}{15}x - \frac{16}{3}\right) D + \frac{1403}{60}$$

**Guess! (and prove)** (Polya, 1945; Bostan, Kauers, 2009)

1. Pick a power series solution $f$ of the differential equation
2. Compute a large number of coefficients of $f$

**Example** $L_1 = \left(x^2 - x\right) D^2 + \left(\frac{1}{3}x - \frac{2}{3}\right) D + \frac{1}{12}$

- Solution $f = 1 - \frac{1}{48}(x+1) - \frac{1}{72}(x+1)^2 - \frac{5}{512}(x+1)^3 + \cdots + (\ldots)(x+1)^{500} + O((x+1)^{500})$

## How to find algebraic equations?

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1. Pick a power series solution $f$ of the differential equation
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3. Form an ansatz $P(x, y)$: polynomial in $x$ and $y$ with fixed degree and undeterminate coefficients $c_\alpha$

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# How to find algebraic equations?

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2. Compute a large number of coefficients of $f$
3. Form an ansatz $P(x, y)$: polynomial in $x$ and $y$ with fixed degree and undeterminate coefficients $c_\alpha$
4. Solve $P(x, f(x)) = 0$ for the $c_\alpha$'s
5. If no solution is found: $f$ is not annihilated by any polynomial with the chosen degree bounds
6. If a solution is found, congratulations for the successful guess! (Now prove that this algebraic equation actually annihilates $f$)

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- Ansatz $P = \sum_{i=0}^{12} \sum_{j=0}^{24} c_{i,j} x^i y^j$
- Found! $P = y^{24} - (x^2 + 14x + 1)y^{20} + (-\frac{1}{3}x^3 + 11x^2 + 11x - \frac{1}{3})y^{18} + \cdots$
- From there one proves that the solutions of $L_1$ are all algebraic.

# How to find transcendental solutions? (1)

**Example** $L_2 = \left(x^2 - x\right) D^2 + (x - 2)\, D - \frac{1}{9}$

- Basis of solutions of $L_2$ at $-1$: $1 + \frac{1}{36}(x+1)^2 + \frac{1}{36}(x+1)^3 + \frac{13}{486}(x+1)^4 + O((x+1)^5)$
  $(x+1) + \frac{3}{4}(x+1)^2 + \frac{73}{108}(x+1)^3 + \frac{23}{36}(x+1)^4 + O((x+1)^5)$

**Theorem**
Let $f \in V(L)$, then for almost all $v \in C \cup \{\infty\}$, $f$ has a power series expansion at $v$:
$f(x) = \sum_{n=0}^{\infty} a_n (x - v)^n$.

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**Theorem**

Let $f \in V(L)$, then for all regular point $v$ of $L$, $f$ has a power series expansion at $v$: $f(x) = \sum_{n=0}^{\infty} a_n (x - v)^n$.

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- Basis of solutions of $L_2$ at 0: $f_1 = 1 - \frac{1}{18}x - \frac{2}{243}x^2 - \frac{35}{13122}x^3 - \frac{70}{59049}x^4 + O(x^5)$
  $f_2 = \frac{9}{8}x^{-1} - \frac{9}{4} + \frac{5}{24}x + O(x^2) + \log(x)\left(1 - \frac{1}{18}x + O(x^2)\right)$

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- So $L_2$ has a transcendental solution!

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**Theorem**
An algebraic function has a Puiseux series expansion at all points: $f(x) = \sum_{n=-v}^{\infty} a_n (x-v)^{\frac{n}{u}}$.

# How to find transcendental solutions? (2)

**Example** $L_3 = \left(x^2 - x\right) D^2 + \left(\frac{31}{24}x - \frac{5}{6}\right) D + \frac{1}{48}$

- Singular points: $0, 1, \infty$
- Basis of solutions at 0: $x^{\frac{1}{6}} \cdot \left(1 + \frac{1}{12}x + \frac{31}{936}x^2 + O(x^3)\right)$
  $1 + \frac{1}{40}x + \frac{63}{7040}x^2 + O(x^3)$
- Basis of solutions at 1: $(x-1)^{\frac{13}{24}} \cdot \left(1 - \frac{34}{111}(x-1) + \frac{3485}{20313}(x-1)^2 + O((x-1)^3)\right)$
  $1 - \frac{1}{22}(x-1) + \frac{9}{440}(x-1)^2 + O((x-1)^3)$
- Basis of solutions at $\infty$: $(1/x)^{\frac{1}{6}} \cdot \left(1 + \frac{4}{75}(1/x) + \frac{32}{1575}(1/x)^2 + O((1/x)^3)\right)$
  $(1/x)^{\frac{1}{8}} \cdot \left(1 + \frac{7}{184}(1/x) + \frac{1953}{138368}(1/x)^2 + O((1/x)^3)\right)$

All solutions are Puiseux series, but...

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All solutions are Puiseux series, but...

**Theorem**
Any non-constant algebraic function must have at least one pole.

So $L_3$ has only transcendental solutions.

## Generalization: pseudoconstants

**Closure properties:** the sum, product, and differential of algebraic functions is algebraic

In particular, if $P \in C(x)[D]$ and $f$ is algebraic, then $P(f)$ is algebraic.

**Theorem**

Let $f \in V(L)$ such that there exists $P \in C(x)[D]$ such that:

- at all points of $C \cup \{\infty\}$, $P(f)$ does not have a pole
- $P(f)$ is not constant

Then $f$ is transcendental.

Finding such a $P$ is hard!

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Finding such a $P$ is hard! This is easier:

> **Theorem**
> Assume that there exists $P \in C(x)[D]$ such that:
> - at all points of $C \cup \{\infty\}$, and for all $f \in V(L)$, $P(f)$ does not have a pole
> - there exists $f \in V(L)$ such that $P(f)$ is not constant ( $\iff D \cdot P(f) \neq 0$)
>
> Then $L$ admits a transcendental solution.

**Definition**: $P$ is called a pseudoconstant of $L$.

**Example**: 1 is a pseudoconstant of $L_3$.

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**Closure properties:** the sum, product, and differential of algebraic functions is algebraic

In particular, if $P \in C(x)[D]$ and $f$ is algebraic, then $P(f)$ is algebraic.

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Let $f \in V(L)$ such that there exists $P \in C(x)[D]$ such that:

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Then $f$ is transcendental.

Finding such a $P$ is hard! This is easier:

**Theorem**

Assume that there exists $P \in C(x)[D]$ such that:

- at all singularities of $L$, and for all $f$ in a basis of $V(L)$, $P(f)$ does not have a pole
- $D \cdot P$ is not divisible by $L$

Then $L$ admits a transcendental solution.

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# How to find pseudoconstants? (1)

**Method 1** make an ansatz and solve (requires guessing bounds)

**Method 2** use integral bases

**Definition**

Let $L \in C(x)[D]$ be an operator with Puiseux series solutions at all places.

- Integral element at $v \in C \cup \{\infty\}$: $P \in C(x)[D]$ such that for all $f \in V(L)$, $P(f)$ does not have a pole at $v$
- Globally integral element: integral at all $v \in C$
- Completely integral element: integral at all $v \in C \cup \{\infty\}$
- Integral basis (at $v$ or global): basis of $C(x)[D]/\langle L\rangle$ made of integral elements
- Normal integral basis: global integral basis $(b_1, \dots, b_r)$ such that there exists $\tau_1, \dots, \tau_r \in \mathbb{N}$ such that $x^{\tau_1} b_1, \dots, x^{\tau_r} b_r$ is an integral basis at $\infty$

**Algorithm**

1. Compute a normal integral basis $b_1, \dots, b_r$ with the corresponding $\tau_1, \dots, \tau_r$
2. Return all $b_i$ such that $\tau_i = 0$ and $D \cdot b_i \neq 0 \mod L$

# How to find pseudoconstants? (2)

**Example** $L_4 = \left(x^2 - x\right) D^2 + \left(-\frac{49}{6} x + 2\right) D - 12$

- $L_4$ has a basis of Puiseux series solutions at all places
- Finite singularities: $0, 1$
- Integral basis at 0 and 1 (global): $x(x-1)^6$

$$x^2(x-1)D + \frac{143}{30} x^2 + \frac{2}{5} x$$

- Normal integral basis: $b_1 = x^2(x-1)D + x\left(\frac{143}{30} x + \frac{2}{5}\right)$ $\qquad \tau_1 = 0$

$$b_2 = x^6(x-1)(x-10)D + \tfrac{8}{3} x^7 + \cdots - \tfrac{21}{10} x \qquad \tau_2 = -3$$

- $D \cdot b_1 = \left(-\frac{2}{5} x^2 + \frac{2}{5} x\right) D - \frac{37}{15} x + \frac{2}{5} \mod L \neq 0 \mod L$
- So $b_1$ is a pseudoconstant and $L$ has a transcendental solution.

# What if we cannot find pseudo-constants?

**Closure properties (again)** $f$ is algebraic if and only if $f^2$ is algebraic

**Symmetric product**

$L_1 \otimes L_2$ is a differential operator such that $V(L_1 \otimes L_2)$ is spanned by $V(L_1)V(L_2)$ (products)

$L^{\otimes s} = L \otimes \cdots \otimes L$ is such that $V(L^{\otimes s})$ is spanned by the products of $s$ solutions of $L$

**Theorem**

$L$ has a transcendental solution if and only if $L^{\otimes s}$ has a transcendental solution

**Example** $L_5 = \left(x^2 - x\right) D^2 + \left(\frac{49}{6} x - \frac{7}{3}\right) D + 12$

- $L_5$ has a basis of Puiseux series solutions at all places
- $L_5$ does not have pseudoconstants
- $L_5^{\otimes 2}$ does have a pseudoconstant
- So $L_5$ has a transcendental solution

# Question

**Remark** If $P(x, y) = y^d + \alpha_{d-1}(x)y^{d-1} + \cdots + \alpha_0(x) \in C(x)[y]$ is a polynomial annihilating $f \in V(L)$, then for all $i$, $\alpha_i$ is a rational solution of $L^{\otimes(d-i)}$.

$\Rightarrow$ possible to guess an algebraic equation without relying on a bound on the degree in $x$

$\Rightarrow$ semi-algorithm for finding algebraic equations

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$\Rightarrow$ semi-algorithm for finding algebraic equations

**Question:** can we also do it for transcendence?

What we have now is a way to search for pseudoconstants in an increasing search space. Does this terminate?

*"If $L$ has a transcendental solution, does there exist $s$ such that $L^{\otimes s}$ has a pseudoconstant?"*

**Example** $L_6 = \left(x^2 - x\right) D^2 + \left(\frac{164}{15}x - \frac{16}{3}\right) D + \frac{1403}{60}$

- $L_6$ has a basis of Puiseux series solutions at all places
- $L_6$ does not have a pseudoconstant
- For $s = 2, \dots, 6$, $L_6^{\otimes s}$ does not have a pseudoconstant (using integral bases)
- For $s = 7, \dots, 30$, $L_6^{\otimes s}$ probably does not have a pseudoconstant (using an ansatz)
- Does there exist $s$ such that $L_6^{\otimes s}$ has a pseudoconstant?

# A few examples with certificates

$$L_1 = \left(x^2 - x\right) D^2 + \left(\frac{1}{3} x - \frac{2}{3}\right) D + \frac{1}{12}$$

✓ algebraic

$$L_2 = \left(x^2 - x\right) D^2 + (x - 2) D - \frac{1}{9}$$

✗ log singularities

$$L_3 = \left(x^2 - x\right) D^2 + \left(\frac{31}{24} x - \frac{5}{6}\right) D + \frac{1}{48}$$

✗ 1 is a pseudoconstant

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✗ $L_4$ has a pseudoconstant

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✗ $L_5^{\otimes 2}$ has a pseudoconstant

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(✗ Schwarz's list)

# Conclusion

**Summary**

- Pseudoconstants: new certificate for the transcendence of D-finite functions
- Can be computed in practice using integral bases
- Available in the `ore_algebra` package for SageMath
- Can be used with an expanding search space

**Open questions and future work**

- Are all transcendental solutions certified by a pseudoconstant?
- If not: can we prove that an equation does not have a completely integral element?

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  - Multiplication by algebraic function?
  - Right-composition with an algebraic function?

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Thank you for your attention!