Optimisation Trigonométrique avec Symétrie

Trigonometric Optimization with Symmetry

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JNCF 2023

The goal of trigonometric optimization is to find the global minimum of a function $\mathbb{R}^n \to \mathbb{R}$ such as

$$1/3 + (736 \sin(\pi y) \cos(\pi y) \sin(x\pi) \cos(x\pi)^3)/3 + (512 \sin(x\pi)^2 \cos(x\pi)^2 \sin(\pi y)^2 \cos(\pi y)^6)/3 + (512 \cos(\pi y)^7 \sin(x\pi) \cos(x\pi) \sin(\pi y))/3 - (1024 \cos(x\pi)^3 \cos(\pi y)^7 \sin(x\pi) \sin(\pi y))/3 + (1024 \sin(\pi y)^2 \cos(\pi y)^4 \sin(x\pi)^2 \cos(x\pi)^4)/3 + (736 \sin(x\pi) \cos(x\pi) \sin(\pi y) \cos(\pi y)^3)/3 + (424 \cos(\pi y)^2 \cos(x\pi)^2)/3 - (2048 \cos(\pi y)^5 \cos(x\pi)^5 \sin(x\pi) \sin(\pi y))/3 - (1024 \cos(\pi y)^3 \cos(x\pi)^7 \sin(\pi y) \sin(x\pi))/3 - (1280 \sin(\pi y)^2 \cos(\pi y)^4 \sin(x\pi)^2 \cos(x\pi)^2)/3 - (1280 \sin(\pi y)^2 \cos(\pi y)^2 \sin(x\pi)^2 \cos(x\pi)^4)/3 + (800 \sin(\pi y)^2 \cos(\pi y)^2 \sin(x\pi)^2 \cos(x\pi)^2)/3 - (512 \cos(x\pi)^2 \cos(\pi y)^8)/3 + (512 \cos(x\pi)^4 \cos(\pi y)^8)/3 - (512 \cos(\pi y)^2 \cos(x\pi)^8)/3 + (1024 \cos(\pi y)^6 \cos(x\pi)^6)/3 + (1408 \cos(\pi y)^6 \cos(x\pi)^2)/3 + (512 \cos(\pi y)^4 \cos(x\pi)^8)/3 + (1408 \cos(\pi y)^2 \cos(x\pi)^6)/3 + (2848 \cos(\pi y)^4 \cos(x\pi)^4)/3 + (176 \cos(\pi y)^4)/3 + (176 \cos(x\pi)^4)/3 - (256 \cos(\pi y)^6)/3 - (256 \cos(x\pi)^6)/3 + (128 \cos(x\pi)^8)/3 + (128 \cos(\pi y)^8)/3 - 416 \cos(x\pi)^2 \cos(\pi y)^4 - 768 \cos(\pi y)^4 \cos(x\pi)^6 - 768 \cos(\pi y)^6 \cos(x\pi)^4 - 416 \cos(\pi y)^2 \cos(x\pi)^4 - 16 \cos(\pi y)^2 - 16 \cos(x\pi)^2 + (3584 \cos(\pi y)^5 \cos(x\pi)^3 \sin(x\pi) \sin(\pi y))/3 + (512 \sin(\pi y)^2 \cos(\pi y)^2 \sin(x\pi)^2 \cos(x\pi)^6)/3 + (512 \cos(x\pi)^7 \sin(\pi y) \cos(\pi y) \sin(x\pi))/3 - 1088 \cos(\pi y)^3 \cos(x\pi)^3 \sin(\pi y) \sin(x\pi) - 384 \cos(\pi y)^5 \sin(x\pi) \cos(x\pi) \sin(\pi y) - 384 \cos(x\pi)^5 \sin(\pi y) \cos(\pi y) \sin(x\pi) - 40 \sin(\pi y) \cos(\pi y) \sin(x\pi) \cos(x\pi) + (3584 \cos(\pi y)^3 \cos(x\pi)^5 \sin(\pi y) \sin(x\pi))/3.$$

By exploiting "algebraic structures", one can simplify the problem. Here, we can instead minimize the polynomial $6z^2 - 2z - 1$!

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The presented results are based on joint work with
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Cordian Riener (UiT The Arctic University).

This work has been supported by European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions, grant agreement 813211 (POEMA).

From trigonometric to generalized Chebyshev polynomials

Trigonometric optimization

Let $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n \leqslant \mathbb{R}^n$ be a lattice and $\langle \cdot, \cdot \rangle$ be the Euclidean scalar product.

$$\mathfrak{e}^{\mu}\,\mathfrak{e}^{\nu} = \mathfrak{e}^{\mu+\nu}$$
$$\mathfrak{e}^{\mu}\,\mathfrak{e}^{-\mu} = \mathfrak{e}^{0} = 1$$

The algebra of trigonometric polynomials

For $\mu \in \Omega$, define $\mathfrak{e}^{\mu} : \mathbb{R}^n \to \mathbb{C}$ with

$$\mathfrak{e}^{\mu}(u) := \exp(-2\pi\mathrm{i}\,\langle \mu, u \rangle)$$

and write $\mathbb{R}[\Omega] = \mathbb{R}[\mathfrak{e}^{\pm\omega_1}, \ldots, \mathfrak{e}^{\pm\omega_n}]$.

$$f = \sum_{\mu} f_{\mu}\,\mathfrak{e}^{\mu} \in \mathbb{R}[\Omega]$$

$$\mu = \textstyle\sum_i \alpha_i\,\omega_i \in \Omega$$
$$\Rightarrow \mathfrak{e}^{\mu} = \textstyle\prod_i (\mathfrak{e}^{\omega_i})^{\alpha_i}$$

Periodicity

Let $\Lambda := \{\lambda \in \mathbb{R}^n \,|\, \forall\, \mu \in \Omega : \langle \mu, \lambda \rangle \in \mathbb{Z}\}$ be the **dual lattice**.
Then, for $f \in \mathbb{R}[\Omega]$, $\lambda \in \Lambda$, $u \in \mathbb{R}^n$, we have $f(u+\lambda) = f(u)$.

The trigonometric optimization problem

For $f = \sum_{\mu} f_{\mu}\,\mathfrak{e}^{\mu} \in \mathbb{R}[\Omega]$ with $f_{\mu} = f_{-\mu} \in \mathbb{R}$, find $f^* := \min_{u \in \mathbb{R}^n} f(u)$.

Symmetry in trigonometric optimization

Let $\mathcal{W} \leqslant \mathrm{O}_n(\mathbb{R})$ be a finite orthogonal group and
Ω be a **$\mathcal{W}$–lattice**, that is, for $A \in \mathcal{W}$, $\mu \in \Omega$, we have $A\,\mu \in \Omega$.

The linear action of $\mathcal{W}$ on $\mathbb{R}[\Omega]$

$$\begin{array}{rccc} \cdot: & \mathcal{W} \times \mathbb{R}[\Omega] & \to & \mathbb{R}[\Omega], \\ & (A, \mathfrak{e}^{\mu}) & \mapsto & \mathfrak{e}^{A\,\mu} \end{array}$$

- Say f is **$\mathcal{W}$–invariant**, if $\mathcal{W} \cdot f = \{f\}$
- $\mathbb{R}[\Omega]^{\mathcal{W}}$ the **algebra of $\mathcal{W}$–invariants**

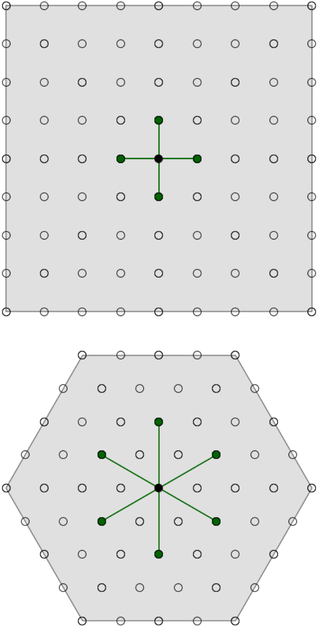
$$A \cdot \sum_{\mu} f_{\mu}\, \mathfrak{e}^{\mu} = \sum_{\mu} f_{\mu}\, \mathfrak{e}^{A\,\mu}$$

$$A \cdot (f\,g) = (A \cdot f)(A \cdot g)$$
$$A \cdot (f + g) = A \cdot f + A \cdot g$$

Generators (Lorenz'05: Multiplicative Invariant Theory)

- As a space, $\mathbb{R}[\Omega]^{\mathcal{W}}$ is generated by the $\frac{1}{|\mathcal{W}|} \sum_{A \in \mathcal{W}} \mathfrak{e}^{A\,\mu}$, $\mu \in \Omega$.
- As an algebra, $\mathbb{R}[\Omega]^{\mathcal{W}}$ is finitely generated.

Root systems, Weyl groups and lattices (Example)



Root systems, Weyl groups and lattices (Definition)

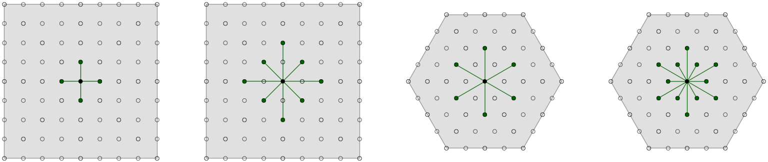
$\mathrm{R} \subseteq \mathbb{R}^n$ root system (Bourbaki'68 Ch. VI: Systèmes de Racines)

R1 R is finite, spans $\mathbb{R}^n$ and does not contain 0.

R2 If $\rho, \tilde{\rho} \in \mathrm{R}$, then $\langle \tilde{\rho}, \rho^\vee \rangle \in \mathbb{Z}$, where $\rho^\vee := 2\,\rho / \langle \rho, \rho \rangle$.

R3 If $\rho, \tilde{\rho} \in \mathrm{R}$, then $A_\rho(\tilde{\rho}) \in \mathrm{R}$, where $A_\rho(u) := u - \langle u, \rho^\vee \rangle \rho$.

- The **Weyl group** $\mathcal{W}$ is the group generated by the A_ρ.
- The **coroot lattice** Λ is the lattice spanned by the $\rho^\vee$.
- The **weight lattice** Ω is the dual lattice of Λ.



What are the generators of $\mathbb{R}[\Omega]^{\mathcal{W}}$ (as an algebra)?

Generalized Chebyshev polynomials

The generalized cosine functions

For $\mu \in \Omega$, define $\mathfrak{c}_\mu \in \mathbb{R}[\Omega]^{\mathcal{W}}$ with

$$\mathfrak{c}_\mu(u) := \frac{1}{|\mathcal{W}|} \sum_{A \in \mathcal{W}} \mathfrak{e}^{A\mu}(u).$$

$$\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$$

$$\mathbb{R}[\Omega] = \mathbb{R}[\mathfrak{e}^{\pm\omega_1}, \ldots, \mathfrak{e}^{\pm\omega_n}]$$

The algebra of $\mathcal{W}$–invariants (Bourbaki'68 Ch. VI)

- The $\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}$ are algebraically independent.
- $\mathbb{R}[\Omega]^{\mathcal{W}} = \mathbb{R}[\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}]$ is a polynomial algebra.

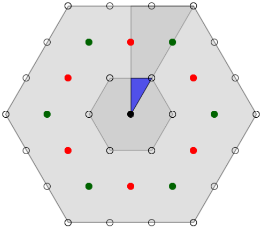
The generalized Chebyshev polynomial associated to $\mu \in \Omega$

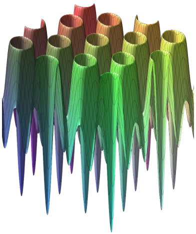
$T_\mu \in \mathbb{R}[z] = \mathbb{R}[z_1, \ldots, z_n]$, so that $T_\mu(\mathfrak{c}_{\omega_1}(u), \ldots, \mathfrak{c}_{\omega_n}(u)) = \mathfrak{c}_\mu(u)$.

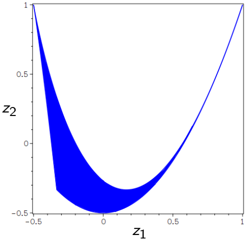
Example ($n = 1$, $\Omega = \mathbb{Z}$)

$\mathbb{R}[\mathfrak{e}^{\pm 1}(u)]^{\{\pm 1\}} = \mathbb{R}[\cos(2\pi u)]$ and $T_\mu(\cos(2\pi u)) = \cos(2\pi \mu u)$.

Rewriting the trigonometric optimization problem







Example ($\mathcal{W} = \mathfrak{S}_3 \ltimes \{\pm 1\}$, Ω hexagonal lattice)

For $S := \mathcal{W}\,\{2\omega_1, \omega_2\}$ and $f_{2\omega_1} := 1$, $f_{\omega_2} := 2$, we have

$$\inf_{u\in\mathbb{R}^2} \sum_{\mu\in S} f_\mu\, \mathfrak{c}_\mu(u) = \inf_{z\in\mathcal{T}} T_{2\omega_1}(z) + 2\,T_{\omega_2}(z) = \inf_{z\in\mathcal{T}} 6\,z_1^2 - 2\,z_1 - 1 = -\frac{7}{6}$$

New feasible region: The image of the generalized cosines

$$\mathcal{T} := \{\mathfrak{c}(u) := (\mathfrak{c}_{\omega_1}(u), \ldots, \mathfrak{c}_{\omega_n}(u)) \mid u \in \mathbb{R}^n\}$$

The image of the generalized cosines as a semi–algebraic set

$\rightarrow$ (Hubert, M, Riener'22)

Describing $\mathcal{T}$ for irreducible root systems

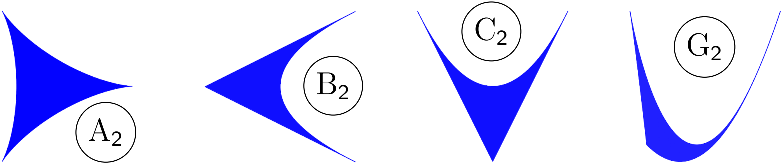
Semi–algebraic description

If R is of type A_{n-1}, B_n, C_n, D_n or G_2, then there exists a symmetric matrix polynomial $H \in \mathbb{R}[z]^{n\times n}$, such that

$$\mathcal{T} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}.$$

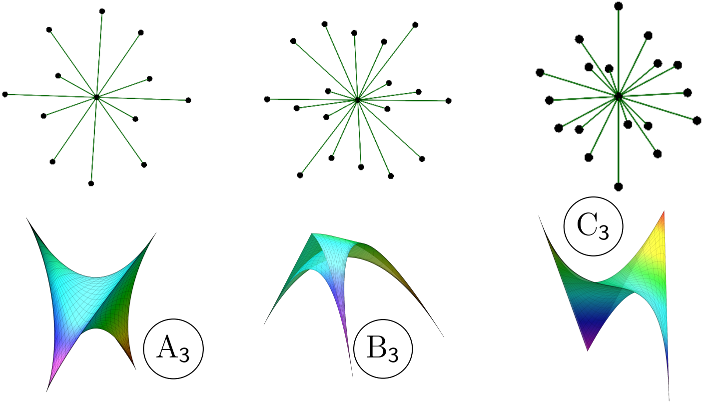
The closed formula in the Chebyshev basis is

$$H = \begin{pmatrix} (T_0 - T_{2\omega_1})/2 & (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & \cdots \\ (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & \cdots \\ (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & (2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1})/32 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$



Describing $\mathcal{T}$ for irreducible root systems

$$H = \begin{pmatrix} (T_0 - T_{2\omega_1})/2 & (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & \cdots \\ (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & \cdots \\ (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & (2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1})/32 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



Optimization with Chebyshev polynomials in practice

$\rightarrow$ (Hubert, M, Moustrou, Riener'22)

Matrix SOS reinforcement

$$\begin{aligned} f^* &= \min \quad \textstyle\sum_\mu f_\mu\, T_\mu(z) \\ &\quad\ \text{s.t.} \quad z \in \mathbb{R}^n,\ H(z) \succeq 0 \\ &= \max \quad r \\ &\quad\ \text{s.t.} \quad r \in \mathbb{R},\ \forall\, H(z) \succeq 0 : \\ &\qquad\qquad \textstyle\sum_\mu f_\mu\, T_\mu(z) - r \geq 0. \\ &\geq \sup \quad r \\ &\quad\ \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\ &\qquad\qquad \textstyle\sum_\mu f_\mu\, T_\mu - r = q + \mathrm{tr}(H\, Q) \end{aligned}$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n})$, if

$\exists\, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^{k} Q_i(z)\, Q_i(z)^t$$

For computations, restrict q, Q to finite space ($d \in \mathbb{N}$)

$$\mathcal{F}_d := \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$$

$$T_\mu\, T_\nu = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \mu+\nu, \rho_0^\vee \rangle} t_\omega\, T_\omega$$

If $T_\mu \in \mathcal{F}_{d_1}$ and $T_\nu \in \mathcal{F}_{d_2}$,
then $T_\mu\, T_\nu \in \mathcal{F}_{d_1+d_2}$.

Semi–definite lower bounds

SOS hierarchy for trigonometric polynomials with $\mathcal{W}$–symmetry

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

$$f^* \geq f_{\mathrm{sym}}^d := \begin{array}{rl} \sup & r \\ \text{s.t.} & r \in \mathbb{R},\, q \in \mathrm{SOS}(\mathcal{F}_d),\, Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n \times n}), \\ & \sum\limits_{\mu} f_\mu\, T_\mu - r = q + \mathrm{tr}(H\, Q). \end{array}$$

Then $f_{\mathrm{sym}}^d \leq f_{\mathrm{sym}}^{d+1}$ and $\lim\limits_{d \to \infty} f_{\mathrm{sym}}^d = f^*$.

Translation to an SDP $\to$ Maple

Compute $A_0,\, A_\mu \in \mathrm{Sym}^{N(d)}$, such that

$$f_{\mathrm{sym}}^d = \begin{array}{rl} \sup & f_0 - \mathrm{tr}(A_0\, X) \\ \text{s.t.} & X \in \mathrm{Sym}_{\succeq 0}^{N(d)},\, \forall\, 0 \neq \mu : \\ & \mathrm{tr}(A_\mu\, X) = f_\mu. \end{array}$$

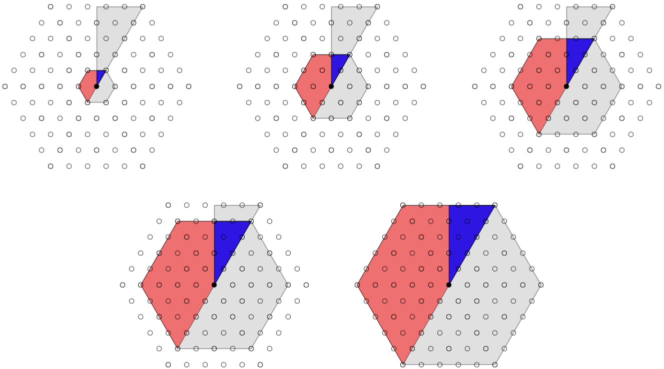
Matrix size:
$N(d) := \dim(\mathcal{F}_d)$
$\qquad + n \dim(\mathcal{F}_{d-n})$

Comparison with the dense approach

SOHS hierarchy for trigonometric polynomials without symmetry

For $f = \sum_\mu f_\mu \mathfrak{e}^\mu \in \mathbb{R}[\Omega]$ with $f_\mu = f_{-\mu} \in \mathbb{R}$, find $f^* := \min_{u \in \mathbb{R}^n} f(u)$.

(Dumitrescu'07) $f^d_{\mathrm{dense}} := \sup\{r \in \mathbb{R} \mid f - r \in \mathrm{SOHS}(d)\} \to$ SDP.

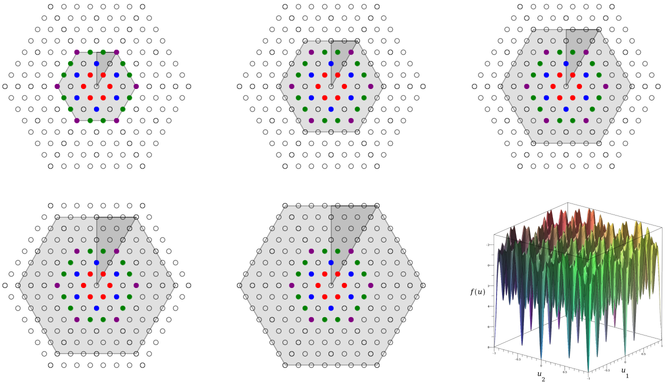


What is better, f^d_{dense} or f^d_{sym}? We solve with MOSEK and compare.

Comparison with the dense approach

$$f = 2\,T_{\omega_1} + T_{\omega_2} + T_{\omega_1+\omega_2} + 4\,T_{3\omega_1}\ (\mathrm{G}_2\text{–symmetry})$$

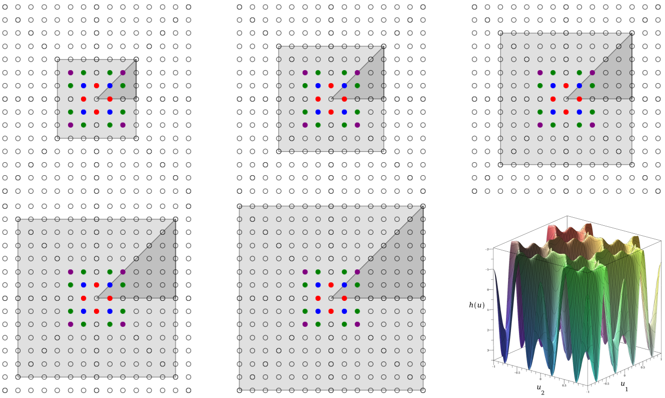
d	3	4	5	6	7
f^d_{dense}	-3.50118	-3.40372	-3.31195	-3.25383	-3.22049
f^d_{sym}	-3.20499	-3.10220	-2.98718	-2.98718	-2.98718



Comparison with the dense approach

$$h = 2\,T_{\omega_1} + T_{\omega_2} - 3\,T_{\omega_1+\omega_2} - T_{2\omega_1}\ (\mathrm{C}_2\text{–symmetry})$$

d	3	4	5	6	7
h^d_{dense}	-2.12159	-2.10672	-2.1012	-2.09959	-2.09073
h^d_{sym}	-2.27496	-2.06250	-2.06250	-2.06250	-2.06250



Merci.

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