Further results on the computation of the annihilator of integro-differential operators

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March 7th, 2023

Plan

Introduction

Algebraic analysis is a mathematical theory which studies linear systems of PDEs using module theory, homological algebra...

It was developed by Malgrange, Bernstein, Kashiwara... in the 70's.

It nowadays plays a fundamental role in modern mathematics (algebraic geometry, representation theory, singularity theory...).

Question: What does algebraic analysis yield if we consider *rings of integro-differential operators* instead of rings of differential operators?

$$\dot{y}(t) + t^2\, y(t) + t \int_0^t y(\tau)\, d\tau - t \int_0^t \tau\, y(\tau)\, d\tau + (t-1)\, y(0) = 0$$

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Integro-differential operators

$\mathbb{k}$ = a field of characteristic 0.

Let us consider the following $\mathbb{k}$-endomorphisms of $\mathbb{k}[t]$:

$$t : \begin{array}{ccc} \mathbb{k}[t] & \longrightarrow & \mathbb{k}[t] \\ p & \longmapsto & t\,p, \end{array} \quad \partial : \begin{array}{ccc} \mathbb{k}[t] & \longrightarrow & \mathbb{k}[t] \\ p & \longmapsto & p', \end{array} \quad I : \begin{array}{ccc} \mathbb{k}[t] & \longrightarrow & \mathbb{k}[t] \\ p & \longmapsto & \int_{t_0}^{t} p(\tau)d\tau. \end{array}$$

The *fundamental theorem of calculus* can be written as

$$\partial \circ I = 1,$$

where 1 denotes the identity endomorphism.

We can also see that

$$\forall\, p \in \mathbb{k}[t], \quad (1 - I \circ \partial)(p) = p - \int_{t_0}^{t} \dot{p}(\tau)\, d\tau = p(t_0).$$

Fix $t_0 \in \mathbb{k}$ and consider the following endomorphism of $\mathbb{k}[t]$:

$$e = 1 - I \circ \partial : \begin{array}{ccc} \mathbb{k}[t] & \longrightarrow & \mathbb{k}[t] \\ p & \longmapsto & p(t_0). \end{array}$$

Definitions of $\mathbb{A}_1(\Bbbk)$ and $\mathbb{I}_1(\Bbbk)$

Definition

$\mathbb{A}_1(\Bbbk)$ is the sub-$\Bbbk$-algebra of $\mathrm{end}_{\Bbbk}(\Bbbk[\mathrm{t}])$ generated by t and ∂.

Definition

$\mathbb{I}_1(\Bbbk)$ is the sub-$\Bbbk$-algebra of $\mathrm{end}_{\Bbbk}(\Bbbk[\mathrm{t}])$ generated by t, ∂, I and e.

Identities of $\mathbb{I}_1(\Bbbk)$: ($\circ$ is ommited)

$$\partial\, I = 1 \quad \text{: 1st fundamental thm}$$

$$I\,\partial = 1 - e \quad \text{: 2nd fundamental thm}$$

$$\partial\, p = p\,\partial + \dot{p} \quad \text{: Leibniz rule}$$

$$I\, p\,\partial = -I\,\partial\, p + p - e(p)\, e \quad \text{: integration by parts}$$

$$I\, p\, I = I(p)\, I - I\, I(p) \quad \text{: double integration}$$

$$e^2 = e,\ \partial\, e = 0,\ e\, p = e(p)\, e = p(t_0)\, e \quad \text{: relations with the evaluation}$$

Normal forms

A consequence of these identities is that every operator of $\mathbb{I}_1(\mathbb{k})$ can be written in a canonical way (*normal form*).

Any operator of $\mathbb{I}_1(\mathbb{k})$ can uniquely be written as

$$d = \underbrace{\sum_{i=0}^{m} a_i(t)\,\partial^i}_{\in \mathbb{A}_1(\mathbb{k})} + \sum_{j=0}^{p} b_j(t)\, I\, c_j(t) + \underbrace{\sum_{k=0}^{q} f_k(t)\, e\,\partial^k}_{\in \langle e \rangle},$$

where $a_i, b_j, c_j, f_k \in \mathbb{k}[t]$, $m, p, q \in \mathbb{N}$ and $\langle e \rangle$ is the only two-sided ideal of $\mathbb{I}_1(\mathbb{k})$ generated by e, i.e., $\langle e \rangle = \mathbb{I}_1(\mathbb{k})\, e\, \mathbb{I}_1(\mathbb{k})$.

Algebraic analysis

$\mathcal{R}$ = a ring such as $\mathbb{A}_1(\mathbb{k})$ or $\mathbb{I}_1(\mathbb{k})$.

$\mathcal{F}$ = left-$\mathcal{R}$-module such as $\mathbb{k}[t]$ or $C^\infty(\mathbb{R})$.

$$\underbrace{\begin{pmatrix} T_{11} & \dots & T_{1p} \\ \vdots & & \vdots \\ T_{q1} & \dots & T_{qp} \end{pmatrix}}_{T} \begin{pmatrix} h_1 \\ \vdots \\ h_p \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$T \in \mathcal{R}^{q \times p}, \quad h = (h_1, \dots, h_p)^T \in \mathcal{F}^{p \times 1}.$

The *generators* $g = (g_1, \dots, g_p)^T$ of the left-$\mathcal{R}$-module $\mathcal{M} := \mathcal{R}^{1 \times p} / (\mathcal{R}^{1 \times q}\, T)$, verify the *relations*

$$T\, g = 0.$$

Theorem (Malgrange)

$$\ker_{\mathcal{F}}(T.) = \{h \in \mathcal{F}^{p \times 1} \mid T\, h = 0\} \cong \mathrm{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{F}).$$

Compatibility conditions

An important problem in linear system theory is to solve inhomogeneous systems over a left $\mathbb{I}_1(\mathbb{k})$-module $\mathcal{F}$:

$$T\,h = g, \quad T \in \mathbb{I}_1(\mathbb{k})^{q\times p}, \quad g \in \mathcal{F}^{q\times 1} \text{ fixed}, \quad h \in \mathcal{F}^{p\times 1} \text{ sought}$$

Note that

$$\forall\, \lambda \in \mathbb{I}_1^{1\times q}: \; \lambda\, T = 0 \;\Rightarrow\; \lambda\, g = \lambda\, T\, h = 0,$$

i.e., g must satisfy $\lambda\, g = 0$ (*compatibility condition*, *elimination*).

Thus, we want to characterize

$$\ker_{\mathbb{I}_1(\mathbb{k})}(.T) = \left\{\lambda \in \mathbb{I}_1^{1\times q} \mid \lambda\, T = 0\right\}.$$

Question: Is $\ker_{\mathbb{I}_1}(.T)$ *finitely generated* and if so, *calculate* a finite set of generators?

Solving this question yields an *effective* $\mathbb{I}_1(\mathbb{k})$-*module theory*.

Plan

$\mathbb{I}_1(\Bbbk)$ is not a noetherian ring

Theorem

$\mathbb{I}_1(\Bbbk)$ is neither a left nor right noetherian ring.

For $N \in \mathbb{N}$, let us introduce

$$T_N = \sum_{k=0}^{N} \frac{t^k}{k!}\, e\, \partial^k \quad (\textit{Taylor operators for } t_0 = 0)$$

For instance, $T_0 = e$, $T_1 = e + t\, e\, \partial$. Notice that

$$e\,(e + t\, e\, \partial) = e^2 + e\, t\, e\, \partial = e^2 + 0 = e \;\Rightarrow\; \mathbb{I}_1\, T_0 \subset \mathbb{I}_1\, T_1.$$

More generally:

$$T_N = T_N\, T_{N+1} \;\Rightarrow\; \mathbb{I}_1\, T_N \subseteq \mathbb{I}_1\, T_{N+1}, \quad \mathbb{I}_1\, T_N \neq \mathbb{I}_1\, T_{N+1}.$$

Coherence definition

Finitely presented module

Let $\mathcal{R}$ be a ring and $\mathcal{M}$ a left $\mathcal{R}$-module finitely generated by $g_1, \ldots, g_p$. Then, we have the following surjective homomorphism:

$$\begin{array}{rcl} \pi : \mathcal{R}^{1\times p} & \longrightarrow & \mathcal{M} \\ e_i = (0 \ \ldots \ 1 \ \ldots 0) & \longmapsto & g_i, \quad i = 1, \ldots, p. \end{array}$$

$\mathcal{M}$ is said to be *left finitely presented* if the left $\mathcal{R}$-module

$$\ker \pi = \left\{ (\lambda_1, \ldots, \lambda_p) \in \mathcal{R}^{1\times p} \mid \pi(\lambda) = \sum_{i=1}^{p} \lambda_i \, g_i = 0 \right\}$$

is finitely generated. This is equivalent to the existence of a matrix $S \in \mathcal{R}^{q\times p}$ such that $\ker \pi = \mathrm{im}_{\mathcal{R}}(\cdot S)$, i.e., we have the following exact sequence:

$$\mathcal{R}^{1\times q} \xrightarrow{\cdot S} \mathcal{R}^{1\times p} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0.$$

Coherent module

A left $\mathcal{R}$-module $\mathcal{M}$ is *coherent* if all of its finitely generated left $\mathcal{R}$-modules are left finitely presented.

$\mathbb{I}_1(\mathbb{k})$ is coherent

Coherence characterization

Let $\mathcal{R}$ be a ring. The following assertions are equivalent:

1. $\mathcal{R}$ is a left coherent ring.
2. i) For all $a \in \mathcal{R}$, $\mathrm{ann}_{\mathcal{R}}(.a) = \{r \in \mathcal{R} \mid r\,a = 0\}$ is a finitely generated left ideal.

   ii) For all pairs of ideals $\mathcal{I}$ and $\mathcal{J}$ finitely generated, the left ideal $\mathcal{I} \cap \mathcal{J}$ is finitely generated.

Theorem (*Bavula 2013*)

$\mathbb{I}_1(\mathbb{k})$ is a coherent ring, i.e., left coherent and right coherent.

END GOAL: Give an effective proof of this theorem.

$\Rightarrow$ *Effective development of module theory over* $\mathbb{I}_1(\mathbb{k})$.

Plan

Strategy

Goal

Give an effective proof of the condition 2.i in the characterization of the coherence property.

Given $a \in \mathbb{I}_1(\mathbb{k})$, we want to calculate a set of generators of

$$\mathrm{ann}_{\mathcal{R}}(.a) = \{r \in \mathcal{R} \mid r\, a = 0\}.$$

Two possible cases: $a \in \langle e \rangle$ or $a \notin \langle e \rangle$.

The effective study of $a \notin \langle e \rangle$ was done in *Quadrat-Regensburger*'20.

Thus, we concentrate on $a \in \langle e \rangle$, i.e., a is an element of the form

$$a = \sum_{i=0}^{r} a_i(t)\, e\, \partial^i \in \langle e \rangle, \quad a_i \in \mathbb{k}[t], \quad i = 0, \ldots, r.$$

Link with $\mathbb{A}_1(\mathbb{k})$

Proposition (*Chartouny, Cluzeau, Quadrat, 2021*)

Let $p \in \mathbb{k}[t]$ of degree r, $Q_1 = \partial^{r+1}$ and $Q_2 = p\,\partial^r - p^{(r)}(0)$. Then:
$\mathrm{Ann}_{\mathbb{A}_1(\mathbb{k})}(.p) = \{f \in \mathbb{A}_1(\mathbb{k}) \mid f(p) = 0\} = \mathbb{A}_1(\mathbb{k})\, Q_1 + \mathbb{A}_1(\mathbb{k})\, Q_2$.

Example

If $p = t^3 + 1$, then $Q_1 = \partial^4$ and $Q_2 = (t^3+1)\,\partial^3 - 6$.

$\partial^4(p) = p^{(4)} = 0$,

$((t^3+1)\,\partial^3 - 6)\,(p) = (t^3+1)\,p^{(3)} - 6\,p = (t^3+1)\,6 - 6\,(t^3+1) = 0$.

Lemma

Let $P \in \mathbb{I}_1(\mathbb{k})$. Then, there exists $N \in \mathbb{N}$ such that $\partial^N P \in \mathbb{A}_1(\mathbb{k})$.

Example

If $P = t\,I - e$. Applying ∂ to P, we get

$$\partial\, t\, I - \partial\, e = (t\,\partial + 1)\, I - 0 = t\,\partial\, I + I = t + I.$$

If we apply ∂^2 to P, we then have

$$\partial\, t + \partial I = t\,\partial + 1 - 1 = t\,p \in \mathbb{A}_1(\mathbb{k}) \;\Rightarrow\; N = 2.$$

Important preliminary results

Lemma

Let $P \in \mathbb{I}_1(\mathbb{k})$ and $a = \sum_{j=0}^{q} a_j(t)\, e\, \partial^j \in \langle e \rangle$, where $a_j \in \mathbb{k}[t]$ for $j = 0, \ldots, q$. Then, we have

$$P\, a = \sum_{j=0}^{q} P(a_j(t))\, e\, \partial^j.$$

Lemma

For all $N \in \mathbb{N}$, we have the identity $I^N \partial^N + T_{N-1} = 1$ in $\mathbb{I}_1$, which corresponds to the *Taylor's theorem with an integral form of the remainder*, i.e.,

$$f(t) = f(0) + \dot{f}(0)\, t + \cdots + f^{(n-1)}(0)\, \frac{t^{n-1}}{(n-1)!} + \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!}\, f^{(n)}(\tau)\, d\tau.$$

Hence, for all $P \in \mathbb{I}_1$, there exists $N \in \mathbb{N}$ such that

$$P = I^N \partial^N P + T_{N-1}\, P, \quad \partial^N P \in \mathbb{A}_1(\mathbb{k}), \quad T_{N-1}\, P \in \langle e \rangle.$$

First result

Let us consider the simple case $a = p(t)\, e\, \partial^s$, where $s \in \mathbb{N}$ and $p \in \mathbb{k}[t]$.

Proposition

Let $p \in \mathbb{k}[t]$ of degree r. Then, we have

$$\mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.p\, e\, \partial^s) = \mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.p\, e) = \mathbb{I}_1(\mathbb{k})\, Q_1 + \mathbb{I}_1(\mathbb{k})\, Q_2,$$

where $Q_1 = \partial^{r+1}$ and $Q_2 = p\, \partial^r - p^{(r)}(0)$.

Proof scheme

1. $P\, p\, e\, \partial^s = P(p)\, e\, \partial^s = 0$, i.e., $P(p) = 0$

2. We "push" in $\mathbb{A}_1(\Bbbk)$, i.e., there exists N such that $\partial^N P \in \mathbb{A}_1(\Bbbk)$

3. $\partial^N P \in \mathrm{ann}_{\mathbb{A}_1(\Bbbk)}(.a) = \mathbb{A}_1(\Bbbk)\, Q_1 + \mathbb{A}_1(\Bbbk)\, Q_2$

$$\Rightarrow \; \partial^N P = \alpha\, Q_1 + \beta\, Q_2, \quad \alpha, \beta \in \mathbb{A}_1(\Bbbk)$$

4. We apply I^N to $\partial^N P$ and we use the identity $I^N \partial^N + T_{N-1} = 1$ to get

$$P = T_{N-1}\, P + (I^N \alpha)\, Q_1 + (I^N \beta)\, Q_2,$$

which shows that $T_{N-1}\, P \in \mathrm{ann}_{\mathbb{I}_1(\Bbbk)}(.a)$.

5. We can then study

$$\mathrm{ann}_{\mathbb{I}_1(\Bbbk)}(.a) \cap \langle e \rangle$$

General case

We now consider the general case $a = \sum_{i=0}^{r} a_i(t)\, e\, \partial^i$.

Lemma

Let $a_i \in \mathbb{k}[t]$, $i = 0, \ldots, r$ $\mathbb{k}$-linearly independent, $m = \max_{i\ldots r}\{\deg_t a_i\}$, and $J = (1, \ldots, \partial^{m+1})^T$. Then,

$$\bigcap_{i=0}^{r} \mathrm{Ann}_{\mathbb{A}_1(\mathbb{k})}(.a_i) = \sum_{j=1}^{l} \mathbb{A}_1(\mathbb{k})\, f_j = \left\{ \sum_{j=1}^{l} \alpha_j\, f_j \mid \alpha_j \in \mathbb{A}_1(\mathbb{k}),\ j = 1, \ldots, l \right\}$$

where the f_j's are defined by $(f_1, \ldots, f_l)^T = D\, J$ and $D \in \mathbb{k}[t]^{l \times (m+2)}$ is a full row rank matrix such that

$$\ker_{\mathbb{k}[t]}\left(. \underbrace{\begin{pmatrix} a_0 & \ldots & a_r \\ \vdots & & \vdots \\ a_0^{(m+1)} & \ldots & a_r^{(m+1)} \end{pmatrix}}_{C} \right) = \mathrm{im}_{\mathbb{k}[t]}(.D).$$

Proof scheme

- $\forall\, i = 0, \ldots, r,\; P(a_i) = \sum_{j=0}^{m+1} c_j(t)\, a_i^{(j)}(t) = 0$ is equivalent to

$$(c_0(t)\ \ldots\ c_{m+1}(t))\ \underbrace{\begin{pmatrix} a_0 & \ldots & a_r \\ \vdots & & \vdots \\ a_0^{(m+1)} & \ldots & a_r^{(m+1)} \end{pmatrix}}_{C} = (0\ \ldots\ 0)$$

- If $\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D)$, then $(c_0\ \ldots\ c_{m+1}) = \nu\, D$ for a certain $\nu \in \mathbb{k}[t]^{1 \times l}$

- If we write $P = (c_0\ \ldots\ c_{m+1})\, J$, then $P = \nu\, D\, J$.

Annihilator in the evaluation

Proposition

Let $a = \sum_{i=0}^{r} a_i \, e \, \partial^i \in \langle e \rangle$, $a_i \in \mathbb{k}[t]$, for $k = 0, \ldots, r$, $m = \max_{i \in [\![0,\, r]\!]} \{\deg_t(a_i)\}$, and $C \in \mathbb{k}[t]^{(m+2)\times(r+1)}$ and J be the matrices defined before. If $E \in \mathbb{k}^{v\times(m+2)}$ is a full row rank matrix such that $\ker_{\mathbb{k}}(.e(C)) = \mathrm{im}_{\mathbb{k}}(.E)$, then we have

$$\mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.a) \bigcap \langle e \rangle = \sum_{k=1}^{v} \mathbb{I}_1(\mathbb{k}) \, g_k,$$

where the g_k's are defined by

$$\begin{pmatrix} g_1 & \ldots & g_v \end{pmatrix}^T = E \, e \, J \in \langle e \rangle^{v\times 1}.$$

Proof scheme

$P = \sum_{j=0}^{m+1} \alpha_j \, e \, \partial^j \in \mathrm{ann}_{\mathbb{I}_1(\Bbbk)}(.a) \bigcap \langle e \rangle$. We then have:

$$\begin{aligned} P\, a = 0 &\iff P(a_i) = 0, \quad i = 0, \dots, r, \\ &\iff P(a_i) = \sum_{j=0}^{m+1} \alpha_j \, e(\partial^j a_i) = 0, \quad i = 0, \dots, r, \\ &\iff (\alpha_0 \ \dots \ \alpha_{m+1}) \, e(C) = (0 \ \dots \ 0), \\ &\iff (\alpha_0 \ \dots \ \alpha_{m+1}) \in \ker_{\Bbbk[t]}(.e(C)), \end{aligned}$$

Main theorem

Theorem

Let $a = \sum_{i=0}^{r} a_i \, e \, \partial^i \in \langle e \rangle$, $a_i \in \mathbb{k}[t]$ $\mathbb{k}$-linearly independent, $m = \max_{i=0,\dots,r} \deg a_i$,

$$C = \begin{pmatrix} a_0 & \dots & a_r \\ \vdots & & \vdots \\ a_0^{(m+1)} & \dots & a_r^{(m+1)} \end{pmatrix} \in \mathbb{k}[t]^{(m+2)\times(r+1)},$$

$D \in \mathbb{k}[t]^{l\times(m+2)}$ a matrix whose rows define a basis of the left kernel of the matrix C,

$E \in \mathbb{k}^{n\times(r+1)}$ a matrix whose rows define a basis of the left kernel of the matrix $e(C) = C(t_0) \in \mathbb{k}^{(m+2)\times(r+1)}$,

$$J = \begin{pmatrix} 1 \\ \vdots \\ \partial^{m+1} \end{pmatrix}, \quad \begin{pmatrix} f_1 \\ \vdots \\ f_l \end{pmatrix} = D\, J, \quad \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = E\, e\, J.$$

Then we have $\mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.a) = \sum_{j=1}^{l} \mathbb{I}_1(\mathbb{k})\, f_j + \sum_{k=1}^{n} \mathbb{I}_1(\mathbb{k})\, g_k$.

Consequence

Corollary

With the previous notations, let us define

$$\mathcal{N} = \mathrm{coker}_{\Bbbk[t]}(.C) = \Bbbk[t]^{1\times(r+1)} / \left(\Bbbk[t]^{1\times(m+2)}\, C\right).$$

Then, $\mathcal{N} = 0$ and we have

$$\mathrm{ann}_{\mathbb{I}_1(\Bbbk)}(.a) = \sum_{j=1}^{l} \mathbb{I}_1(\Bbbk)\, f_j,$$

with

$$l = m - r + 1.$$

Plan

Example

Let us consider $a_0(t) = t,\ a_1(t) = t^2 + 1,\ a_2(t) = t^3 - 2\,t$ and

$$a = a_0\, e + a_1\, e\, \partial + a_2\, e\, \partial^2 \in \langle e \rangle.$$

Then, we have

$$C = \begin{pmatrix} t & t^2+1 & t^3 - 2\,t \\ 1 & 2\,t & 3\,t^2 - 2 \\ 0 & 2 & 6\,t \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using, e.g., the command `SyzygyModule` of the `Maple` package `OreModules`, we obtain $\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D)$, where

$$D = \begin{pmatrix} -6 & 6\,t & -3\,t^2 + 3 & t^3 - 3\,t & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We deduce the following generators of $\mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.a)$:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = D \begin{pmatrix} 1 \\ \partial \\ \vdots \\ \partial^4 \end{pmatrix} = \begin{pmatrix} -6 + 6\,t\,\partial + (-3\,t^2 + 3)\,\partial^2 + (t^3 - 3\,t)\,\partial^3 \\ \partial^4 \end{pmatrix}.$$

Plan

Perspectives

1. Extension of the corollary to the matrix case. (The extension of the main theorem is done)

2. To give an effective proof of the characterization of the coherence property, the point 2.ii is, for now, missing:
   $\mathcal{I} \cap \mathcal{J}$ finitely generated where $\mathcal{I}$ and $\mathcal{J}$ are finitely generated.

3. An effective module theory over $\mathbb{I}_1(\mathbb{k})$ yields an effective elimination theory for linear systems of integro-differential equations with polynomial coefficients.

4. A goal is also to develop a `Maple` package dedicated to the effective module theory over $\mathbb{I}_1(\mathbb{k})$.

Remark

$\mathcal{B} = \mathbb{k}[t]$ or $\mathcal{C}^{\infty}(\mathbb{R})$

$\mathcal{B}$ has a structure of $\mathbb{I}_1(\mathbb{k})$-module given by $P\,f := P(f)$ for $P \in \mathbb{I}_1(\mathbb{k})$ and $f \in \mathcal{B}$.

Then we can define $\mathrm{Ann}_{\mathbb{I}_1(\mathbb{k})}(.f) = \{P \in \mathbb{I}_1(\mathbb{k}) \mid P\,f := P(f) = 0\}$.

The general definition of the annihilator in the ring $\mathbb{I}_1(\mathbb{k})$ is $\mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.f) = \{b \in \mathbb{I}_1(\mathbb{k}) \mid b\,f\}$.

The link is :

$$\begin{aligned}
\mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.f\,e) &= \{P \in \mathbb{I}_1(\mathbb{k}) \mid P\,f\,e = 0\} \\
&= \{P \in \mathbb{I}_1(\mathbb{k}) \mid P(f)\,e = 0\} \quad \text{(Lemma)} \\
&= \{P \in \mathbb{I}_1(\mathbb{k}) \mid P(f) = 0\} \quad \text{(Normal form)} \\
&= \mathrm{Ann}_{\mathbb{I}_1(\mathbb{k})}(.f).
\end{aligned}$$

End remark

Fact

- $P \in \mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.I(f)\,e) \iff P\,I \in \mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.f\,e)$.
- $Q \in \mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.f\,e) \iff Q\,\partial \in \mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.I(f)\,e)$.

First,

$$
\begin{aligned}
& P \in \mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.I(f)\,e) \\
\iff & P\,I(f)\,e = 0 \\
\iff & P\,I\,\partial\,I(f)e + PeI(f)\,e = 0 \quad \text{since } 1 = I\,\partial + e \\
\iff & P\,I(I(f)\,\partial + f)\,e \quad \text{since} \quad P\,e\,I(f)\,e = 0 \\
\iff & P\,I\,I(f)\partial\,e + P\,I\,f\,e = 0 \\
\iff & P\,I\,f\,e = 0 \quad \text{since} \quad \partial\,e = 0 \\
\iff & P\,I \in \mathrm{ann}_{\mathbb{I}_1(\mathbb{k})}(.f\,e)
\end{aligned}
$$

Secondly,
$Q\,\partial\,I(f)\,e = 0 \iff (Q\,\partial)(I(f)) = 0 \iff Q(\partial(I(f))) = 0 \iff Q(f) = 0 \iff Q\,f\,e = 0.$