Effective birational deformation of trilinear volumes

Pablo González-Mazón

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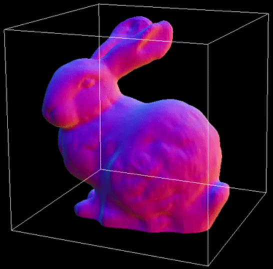
Marie Skłodowska-Curie Actions

Motivation

- Rational maps are used intensively for the generation and manipulation of 2D and 3D shapes



- For instance, free-form deformations have applications to computer-aided geometric design [Hoffmann, 1989], shape optimization [Manzoni et al., 2012], character animation [Chadwick et al., 1989], ...

State of the art: constructive 2D birational maps

- [Sederberg and Zheng, 2015] derived an effective method to construct birational maps $\phi : (\mathbb{P}^1_{\mathbb{R}})^2 \dashrightarrow \mathbb{P}^2_{\mathbb{R}}$ of degree 1×1

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- Birationality criteria for maps of degree $1 \times n$ are given in [Sederberg et al., 2016]
- Recently, methods for the construction of birational maps $\phi : \mathbb{P}^2_{\mathbb{R}} \dashrightarrow \mathbb{P}^2_{\mathbb{R}}$ with quadratic entries have been studied in [Wang et al., 2021]

3D trilinear rational maps

$$\begin{array}{rcl} \phi : \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} & \dashrightarrow & \mathbb{P}^3_{\mathbb{R}} \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) & \mapsto & (f_0 : f_1 : f_2 : f_3) \ , \end{array}$$

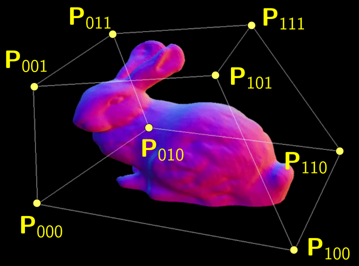
where

$$\mathbf{f} := (f_0\,,\ f_1\,,\ f_2\,,\ f_3)^T = \sum_{0 \le i,j,k \le 1} w_{ijk}\, \mathbf{P}_{ijk}\, B_i(s_0, s_1)\, B_j(t_0, t_1)\, B_k(u_0, u_1)$$

and

$$w_{ijk} \in \mathbb{R}_{\geq 0}\ ,\ \ \mathbf{P}_{ijk} = (1, x_{ijk}, y_{ijk}, z_{ijk})^T$$

$$B_0(x_0, x_1) = x_0 - x_1\ ,\ \ B_1(x_0, x_1) = x_1$$



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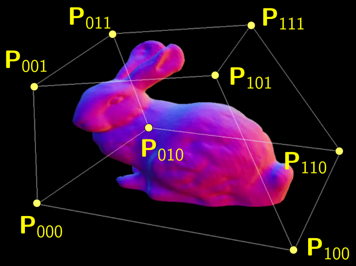
- In general, ϕ is 6-to-1.



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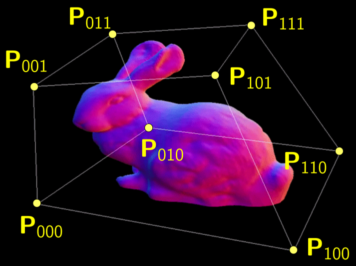

Goal: Given $\mathbf{P}_{ijk}$'s, compute w_{ijk}'s so ϕ is birational

Parametric and boundary surfaces

- Given $(\lambda_0 : \lambda_1) \in \mathbb{P}^1_{\mathbb{R}}$ the (closure of the) restriction of ϕ to $(s_0 : s_1) = (\lambda_0 : \lambda_1)$ is an s-surface

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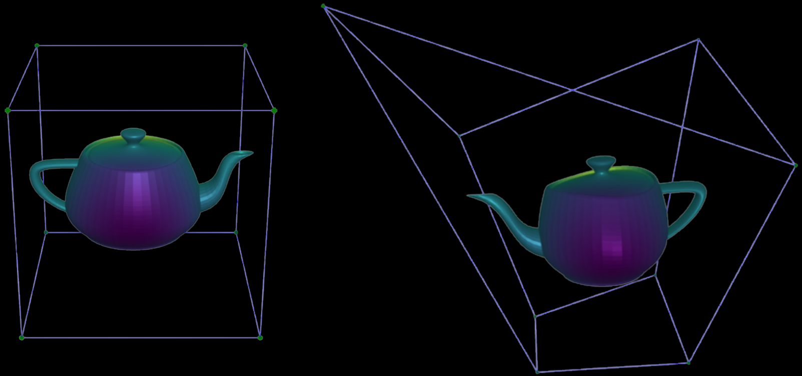
- **Definition:** The type of ϕ is $(a, b, c) \in \mathbb{Z}^3$ where a (resp. b, c) is the degree of the s- (resp. t-, u-) surfaces
- The s-surfaces (resp. t-, u-) determined by the facets of the unit cube are the boundary surfaces $\mathbf{\Sigma}_0, \mathbf{\Sigma}_1$ (resp. $\mathbf{T}_j$, $\mathbf{Y}_k$)

Example: a rational map of type $(1, 1, 2)$



Birational maps of type $(1,1,1)$

- Let $R = \mathbb{R}[s_0, s_1] \otimes_{\mathbb{R}} \mathbb{R}[t_0, t_1] \otimes_{\mathbb{R}} \mathbb{R}[u_0, u_1]$

Theorem: Let ϕ be dominant. TFAE:

1. ϕ is birational of type $(1,1,1)$
2. The minimal graded free resolution of $I = (f_0, f_1, f_2, f_3)$ is

$$0 \to \begin{array}{c} R(-2,-1,-1) \\ \oplus \\ R(-1,-2,-1) \\ \oplus \\ R(-1,-1,-2) \end{array} \to R(-1,-1,-1)^4 \to I \to 0$$

3. $\mathbf{f}$ has syzygies of degree $1 \times 0 \times 0$, $0 \times 1 \times 0$, and $0 \times 0 \times 1$

Geometric constraints: (1,1,1)

- Let $\mathbf{X} = (x_0, x_1, x_2, x_3)^T$ be homogeneous variables in $\mathbb{P}^3_{\mathbb{R}}$

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Property A.1: For each $i = 0, 1$, $\mathbf{\Sigma}_i$ is the plane defined by $\langle \boldsymbol{\sigma}_i, \mathbf{X} \rangle = 0$ for some $\boldsymbol{\sigma}_i = (\sigma_{0i}, \sigma_{1i}, \sigma_{2i}, \sigma_{3i}) \in \mathbb{R}^4$

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Property A.2: For each $j = 0, 1$, $\mathbf{T}_j$ is the plane defined by $\langle \boldsymbol{\tau}_j, \mathbf{X} \rangle = 0$ for some $\boldsymbol{\tau}_j = (\tau_{0j}, \tau_{1j}, \tau_{2j}, \tau_{3j}) \in \mathbb{R}^4$

Property A.3: For each $k = 0, 1$, $\mathbf{Y}_k$ is the plane defined by $\langle \boldsymbol{\upsilon}_k, \mathbf{X} \rangle = 0$ for some $\boldsymbol{\upsilon}_k = (\upsilon_{0k}, \upsilon_{1k}, \upsilon_{2k}, \upsilon_{3k}) \in \mathbb{R}^4$

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- If ϕ is birational of type $(1, 1, 1)$ Properties A.1, A.2 and A.3 are satisfied

Linear syzygies 1: algebra

Lemma: Assume Property A.1 and ϕ dominant. Then, $\mathbf{f}$ has a syzygy of degree $1 \times 0 \times 0$ iff the matrix

$$\begin{pmatrix} w_{100} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{100} \rangle & w_{110} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{110} \rangle & w_{101} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{101} \rangle & w_{111} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{111} \rangle \\ w_{000} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{000} \rangle & w_{010} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{010} \rangle & w_{001} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{001} \rangle & w_{011} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{011} \rangle \end{pmatrix}$$

has rank one. In particular, we find $\alpha \in \mathbb{R}$ such that

$$- w_{1jk} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle = \alpha \, w_{0jk} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle$$

for each $0 \leq j, k \leq 1$, and any syzygy of degree $1 \times 0 \times 0$ of $\mathbf{f}$ is proportional to

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(s_0, s_1) = B_0(s_0, s_1) \, \boldsymbol{\sigma}_0 + \alpha \; B_1(s_0, s_1) \, \boldsymbol{\sigma}_1$$

Linear syzygies 1: geometry

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- The pullback of $\boldsymbol{\Sigma}_{\hat{i}}$ is defined by the vanishing of

$$\langle \boldsymbol{\sigma}_{\hat{i}}, \mathbf{f} \rangle = B_i(s_0, s_1) \sum_{0 \leq j,k \leq 1} w_{ijk} \langle \boldsymbol{\sigma}_{\hat{i}}, \mathbf{P}_{ijk} \rangle \, B_j(t_0, t_1) \, B_k(u_0, u_1)$$

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- $\mathbf{f}$ admits a syzygy of degree $1 \times 0 \times 0$ iff for each $i = 0, 1$ ϕ contracts to $\boldsymbol{\Sigma}_0 \cap \boldsymbol{\Sigma}_1$ the surface in $(\mathbb{P}^1_{\mathbb{R}})^3$ defined by

$$\sum_{0 \le j,k \le 1} w_{ijk} \langle \boldsymbol{\sigma}_{\hat{i}}, \mathbf{P}_{ijk} \rangle B_j(t_0, t_1) B_k(u_0, u_1) = 0$$

Linear syzygies 2 and 3

- Similarly, $\mathbf{f}$ has syzygies of degree $0 \times 1 \times 0$ and $0 \times 0 \times 1$ iff

$$\begin{pmatrix} w_{010} \left\langle \boldsymbol{\tau}_0, \mathbf{P}_{010} \right\rangle & w_{110} \left\langle \boldsymbol{\tau}_0, \mathbf{P}_{110} \right\rangle & w_{011} \left\langle \boldsymbol{\tau}_0, \mathbf{P}_{011} \right\rangle & w_{111} \left\langle \boldsymbol{\tau}_0, \mathbf{P}_{111} \right\rangle \\ w_{000} \left\langle \boldsymbol{\tau}_1, \mathbf{P}_{000} \right\rangle & w_{100} \left\langle \boldsymbol{\tau}_1, \mathbf{P}_{100} \right\rangle & w_{001} \left\langle \boldsymbol{\tau}_1, \mathbf{P}_{001} \right\rangle & w_{101} \left\langle \boldsymbol{\tau}_1, \mathbf{P}_{101} \right\rangle \end{pmatrix}$$

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- The same observations hold after the obvious modifications

Configuration A

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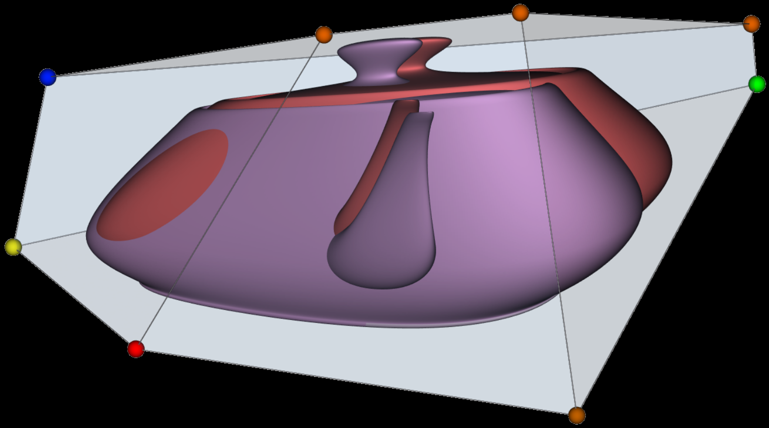
$$\langle \boldsymbol{\sigma}_i, \mathbf{P}_{\hat{i}jk} \rangle > 0 \quad , \quad \langle \boldsymbol{\tau}_j, \mathbf{P}_{i\hat{j}k} \rangle > 0 \quad , \quad \langle \boldsymbol{\upsilon}_k, \mathbf{P}_{ij\hat{k}} \rangle > 0$$

for each $0 \leq i, j, k \leq 1$

Example: Configuration A



Simultaneous drop of rank

- For each $0 \leq i, j, k \leq 1$ define

$$\Delta_{ijk} = \begin{vmatrix} \sigma_{1i} & \sigma_{2i} & \sigma_{3i} \\ \tau_{1j} & \tau_{2j} & \tau_{3j} \\ \upsilon_{1k} & \upsilon_{2k} & \upsilon_{3k} \end{vmatrix}$$

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- We can write

$$\Delta_{0jk} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_0 \wedge \boldsymbol{\tau}_j \wedge \boldsymbol{\upsilon}_k = -\Delta_{1jk} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle$$

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- Thus, the first rank condition is equivalent to

$$\operatorname{rank} \begin{pmatrix} w_{100}\,\Delta_{000} & w_{110}\,\Delta_{010} & w_{101}\,\Delta_{001} & w_{111}\,\Delta_{011} \\ w_{000}\,\Delta_{100} & w_{010}\,\Delta_{110} & w_{001}\,\Delta_{101} & w_{011}\,\Delta_{111} \end{pmatrix} = 1$$

Simultaneous drop of rank

- Similarly, the other two conditions can be rewritten as

$$\operatorname{rank}\begin{pmatrix} w_{010}\,\Delta_{000} & w_{110}\,\Delta_{100} & w_{011}\,\Delta_{001} & w_{111}\,\Delta_{101} \\ w_{000}\,\Delta_{010} & w_{100}\,\Delta_{110} & w_{001}\,\Delta_{011} & w_{101}\,\Delta_{111} \end{pmatrix} = 1$$

$$\operatorname{rank}\begin{pmatrix} w_{001}\,\Delta_{000} & w_{101}\,\Delta_{100} & w_{011}\,\Delta_{010} & w_{111}\,\Delta_{110} \\ w_{000}\,\Delta_{001} & w_{100}\,\Delta_{101} & w_{010}\,\Delta_{011} & w_{110}\,\Delta_{111} \end{pmatrix} = 1$$

Constructive 3D birational maps of type $(1, 1, 1)$

Theorem: Assume Configuration A:

- Choose freely positive values for w_{000}, w_{100}, w_{010}, w_{001}
- Set

$$\alpha = \frac{w_{100}}{\Delta_{100}} \frac{\Delta_{000}}{w_{000}}, \quad \beta = \frac{w_{010}}{\Delta_{010}} \frac{\Delta_{000}}{w_{000}}, \quad \gamma = \frac{w_{001}}{\Delta_{001}} \frac{\Delta_{000}}{w_{000}}$$

- Then, ϕ is birational of type $(1, 1, 1)$ iff there is a non-zero constant $\omega \in \mathbb{R}$ such that for each $0 \leq i, j, k \leq 1$

$$w_{ijk} = \omega \, \alpha^i \, \beta^j \, \gamma^k \, \Delta_{ijk}$$

Constructive 3D birational maps of type $(1,1,1)$

- Moreover, the inverse rational map is given by

$$\frac{s_1}{s_0} = \frac{\langle \boldsymbol{\sigma}_0, \mathbf{X} \rangle}{\langle \boldsymbol{\sigma}_0, \mathbf{X} \rangle - \alpha \langle \boldsymbol{\sigma}_1, \mathbf{X} \rangle} \ , \ \frac{t_1}{t_0} = \frac{\langle \boldsymbol{\tau}_0, \mathbf{X} \rangle}{\langle \boldsymbol{\tau}_0, \mathbf{X} \rangle - \beta \langle \boldsymbol{\tau}_1, \mathbf{X} \rangle} \ ,$$

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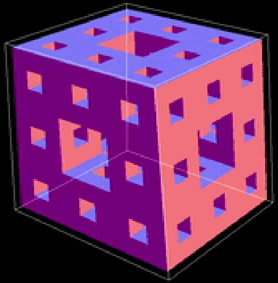
$$\frac{u_1}{u_0} = \frac{\langle \boldsymbol{\upsilon}_0, \mathbf{X} \rangle}{\langle \boldsymbol{\upsilon}_0, \mathbf{X} \rangle - \gamma \langle \boldsymbol{\upsilon}_1, \mathbf{X} \rangle}$$

- Only the constants α, β, γ depend on the w_{ijk}'s!

Birational deformation of a Menger sponge



Birational maps of type $(1, 1, 2)$

Theorem: Let ϕ be dominant. TFAE:

1. ϕ is birational of type $(1, 1, 2)$
2. The minimal graded free resolution of $I = (f_0, f_1, f_2, f_3)$ is

$$0 \to R(-2,-2,-2) \to \begin{array}{c} R(-2,-1,-1) \\ \oplus \\ R(-1,-2,-1) \\ \oplus \\ R(-2,-1,-2) \\ \oplus \\ R(-1,-2,-2) \end{array} \to R(-1,-1,-1)^4 \to I \to 0$$

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Property C: Assume Properties A.1 and A.2. The planes $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Sigma}_1$, $\mathbf{T}_0$, $\mathbf{T}_1$ intersect at a point $\mathbf{V}$ (in $\mathbb{P}^3_{\mathbb{R}}$)

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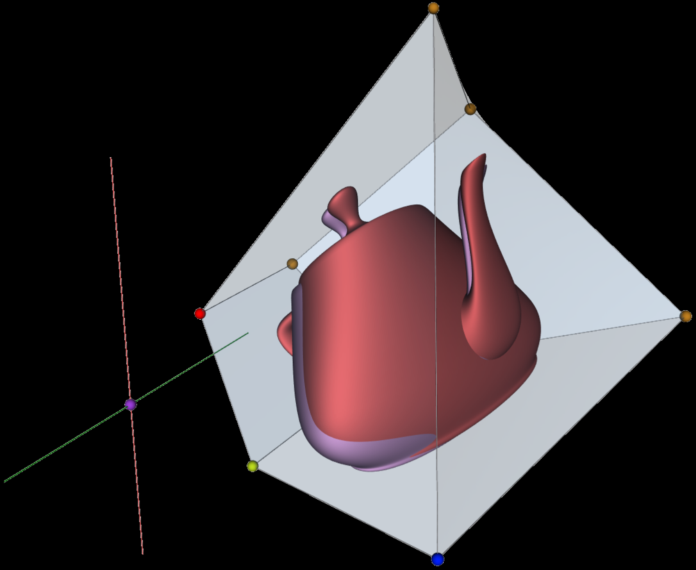
Lemma: If ϕ is birational of type $(1, 1, 2)$ Property C is satisfied

Example: Property B.3 + Property C



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- Additionally, for each $0 \leq i, j, k \leq 1$ we set

$$\Delta_{ijk} = \begin{vmatrix} \sigma_{1i} & \sigma_{2i} & \sigma_{3i} \\ \tau_{1j} & \tau_{2j} & \tau_{3j} \\ \upsilon_{1ik} & \upsilon_{2ik} & \upsilon_{3ik} \end{vmatrix} , \ \lambda_i = \langle \boldsymbol{\sigma}_i, \mathbf{B} \rangle \ , \ \nu_{ik} = \langle \boldsymbol{\upsilon}_{ik}, \mathbf{V} \rangle$$

Constructive 3D birational maps of type $(1, 1, 2)$

Theorem: Assume Properties B.3 + C:

- Choose freely positive values for w_{000}, w_{100}, w_{010}, w_{001}
- Set

$$\alpha = \frac{w_{100}\,\nu_{10}}{\Delta_{100}}\,\frac{\Delta_{000}}{w_{000}\,\nu_{00}}\;,\;\; \beta = \frac{w_{010}}{\Delta_{010}}\,\frac{\Delta_{000}}{w_{000}}\;,\;\; \gamma = \frac{w_{001}\,\nu_{01}}{\Delta_{001}}\,\frac{\Delta_{000}}{w_{000}\,\nu_{00}}$$

- Then, ϕ is birational of type $(1, 1, 2)$ iff there is a non-zero constant $\omega \in \mathbb{R}$ such that for each $0 \le i, j, k \le 1$

$$w_{ijk} = \omega\;\alpha^i\,\beta^j\,\gamma^k\,\frac{\Delta_{ijk}}{\nu_{ik}}$$

Constructive 3D birational maps of type $(1, 1, 2)$

Theorem:

- Moreover, the inverse rational map is given by

$$\frac{s_1}{s_0} = \frac{\langle \boldsymbol{\sigma}_0, \mathbf{X} \rangle}{\langle \boldsymbol{\sigma}_0, \mathbf{X} \rangle - \alpha \langle \boldsymbol{\sigma}_1, \mathbf{X} \rangle} \quad , \quad \frac{t_1}{t_0} = \frac{\langle \boldsymbol{\tau}_0, \mathbf{X} \rangle}{\langle \boldsymbol{\tau}_0, \mathbf{X} \rangle - \beta \langle \boldsymbol{\tau}_1, \mathbf{X} \rangle} \; ,$$

$$\frac{u_1}{u_0} = \frac{\begin{vmatrix} \frac{1}{\epsilon_1} \langle \boldsymbol{\sigma}_1, \mathbf{X} \rangle & \frac{1}{\epsilon_0} \langle \boldsymbol{\sigma}_0, \mathbf{X} \rangle \\ \frac{1}{\nu_{10}} \langle \boldsymbol{\upsilon}_{10}, \mathbf{X} \rangle & \frac{1}{\nu_{00}} \langle \boldsymbol{\upsilon}_{00}, \mathbf{X} \rangle \end{vmatrix}}{\begin{vmatrix} \frac{1}{\lambda_1} \langle \boldsymbol{\sigma}_1, \mathbf{X} \rangle & \frac{1}{\lambda_0} \langle \boldsymbol{\sigma}_0, \mathbf{X} \rangle \\ \frac{1}{\nu_{10}} \langle \boldsymbol{\upsilon}_{10}, \mathbf{X} \rangle & \frac{1}{\nu_{00}} \langle \boldsymbol{\upsilon}_{00}, \mathbf{X} \rangle \end{vmatrix} - \gamma \begin{vmatrix} \frac{1}{\lambda_1} \langle \boldsymbol{\sigma}_1, \mathbf{X} \rangle & \frac{1}{\lambda_0} \langle \boldsymbol{\sigma}_0, \mathbf{X} \rangle \\ \frac{1}{\nu_{11}} \langle \boldsymbol{\upsilon}_{11}, \mathbf{X} \rangle & \frac{1}{\delta_{01}} \langle \boldsymbol{\upsilon}_{01}, \mathbf{X} \rangle \end{vmatrix}}$$

Constructive 3D birational maps of larger type

- If ϕ has type $(1, 2, 2)$, the resolution of $I = (f_0, f_1, f_2, f_3)$ is

$$0 \to R^2 \to R^5 \to R^4 \to I \to 0$$

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- We still do not know how to construct these maps ...