The Nullstellensatz and Positivstellensatz for Sparse Tropical Polynomial Systems

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- Two main concerns: find the 'smallest' suitable witness and be able to deal with sparse systems.

I - Tropical algebra and tropical polynomials

- **Tropical semiring** $\mathbb{R}_\infty = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ with
    - addition $\oplus := \max$;
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    - zero element $\mathbb{0} := -\infty$;
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- Satisfies the usual properties of a field except **no additive inverse**.
- Tropical operations can be extended to vectors and matrices with coefficients in $\mathbb{R}_\infty$ allowing us to perform **tropical linear algebra**.

- A **formal tropical polynomial** p in n variables is a map

$$\begin{array}{ccc} \mathbb{Z}^n & \longrightarrow & \mathbb{R}_\infty \\ \alpha & \longmapsto & p_\alpha \end{array}$$

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- **Support** of p: $\mathrm{supp}(p) := \{\alpha \in \mathbb{Z}^n : p_\alpha \neq \mathbb{0}\}$
- **Polynomial function** associated to p:

$$\hat{p} : \left\{ \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}_\infty \\ x & \longmapsto & \hat{p}(x) := \max_{\alpha \in \mathcal{A}}(p_\alpha + \langle x, \alpha \rangle) \end{array} \right.$$

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Remark : A tropical polynomial function is a convex, affine by parts function with integer slopes.

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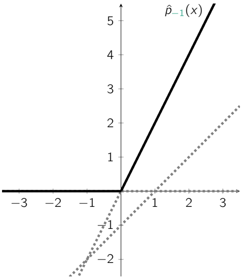

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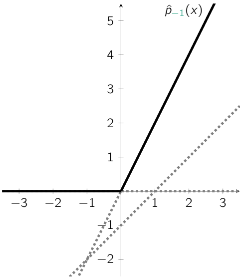

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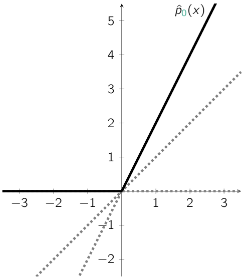

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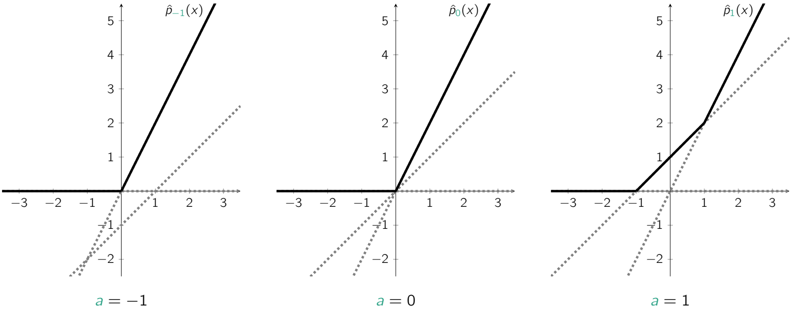

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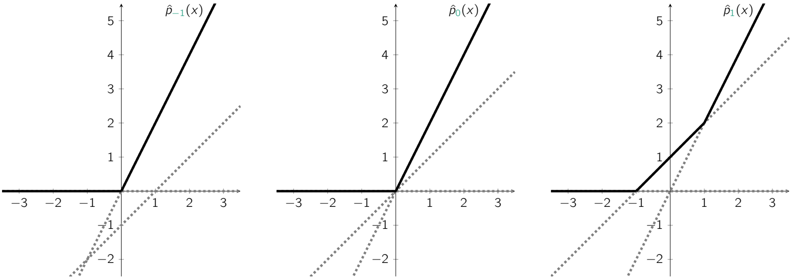

$a = -1$ $a = 0$ $a = 1$

Figure: The graph of $\hat{p}_a$ for different values of a.

Two distinct tropical polynomials can share the same tropical polynomial function!

A point $x \in \mathbb{R}_\infty^n$ is a **root** of a polynomial p whenever the maximum in the expression

$$\hat{p}(x) = \bigoplus_{\alpha \in \mathcal{A}} p_\alpha \odot x^{\odot \alpha} = \max_{\alpha \in \mathcal{A}} (p_\alpha + \langle x, \alpha \rangle)$$

is attained for at least two distinct values of α. This is denoted as $p(x) \nabla \mathbb{0}$.

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- $(-1, 1)$ is not a root of f_1 since the maximum $\hat{f}_1(-1, 1) = 2$ is attained only by the monomial $1x_2$.

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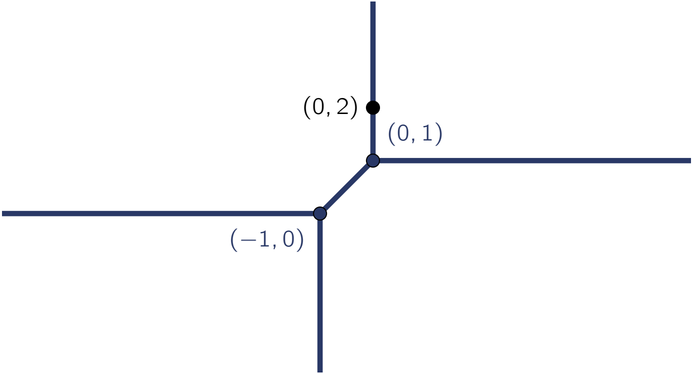
Example : The tropical hypersurface associated to the polynomial $f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2$ is the following:



Likewise, $y \in \mathbb{R}^m$ is said to be in the **tropical right null space** or **kernel** of a $\ell \times m$ matrix $A = (a_{ij})$ whenever for all $1 \leq i \leq \ell$, the maximum in the expression

$$\bigoplus_{j=1}^{m} a_{ij} \odot y_j = \max_{1 \leq j \leq m} (a_{ij} + y_j)$$

is achieved at least twice. This is also denoted as $A \odot y \nabla \mathbb{0}$.

II - Position of the problem

In the following, we fix a collection $f = (f_1, \ldots, f_k)$ of k formal tropical polynomials in n variables, with respective supports $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ and degrees $(d_1, \ldots, d_k)$.

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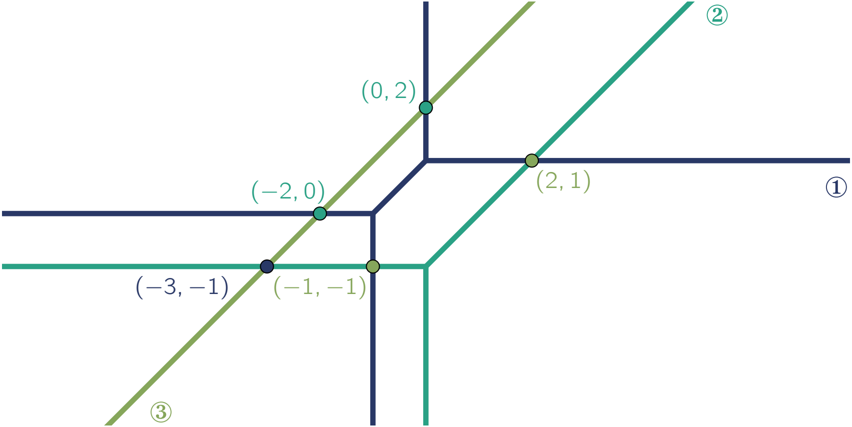
Remark: The same question exists for solution in $\mathbb{R}^n_\infty$. It reduces to the $\mathbb{R}^n$ by considering the support of the solutions.

Figure: The arrangement of tropical varieties of the polynomials from the system

$$(E_1) : \begin{cases} f_1 = 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2 \\ f_2 = 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 = 2x_1 \oplus 0x_2 \end{cases} ,\quad .$$



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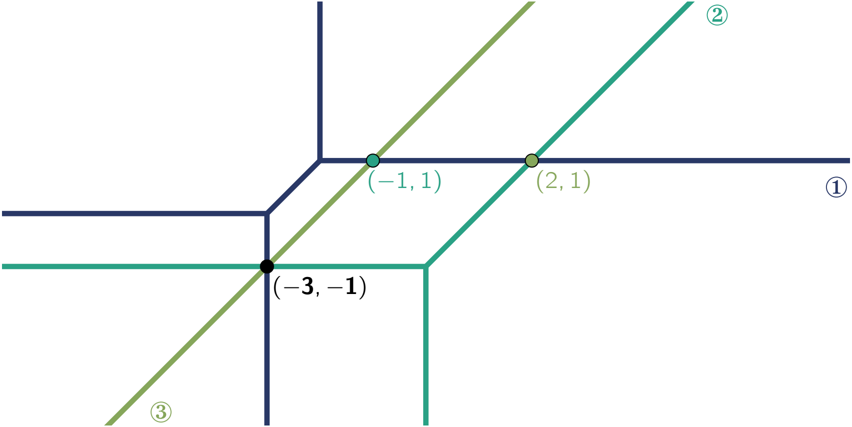
$$(E_2) : \begin{cases} f_1 & = & 1 \oplus 4x_1 \oplus 1x_2 \oplus 3x_1x_2 \\ f_2 & = & 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 & = & 2x_1 \oplus 0x_2 \ , \end{cases} .$$



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- A finite subset $\mathcal{E}$ of $\mathbb{Z}^n$ yields a (finite) submatrix $\mathcal{M}_\mathcal{E}$ of $\mathcal{M}$ obtained by taking only the rows whose support is included in $\mathcal{E}$ and the columns indexed by $\mathcal{E}$.
- For $\mathcal{E} = \{\alpha \in \mathbb{N}^n : \alpha_1 + \cdots + \alpha_n \leq N\}$, we denote $\mathcal{M}_N := \mathcal{M}_\mathcal{E}$.

A tropical Nullstellensatz was established by Grigoriev and Podolskii (2018) for full polynomials. It uses the submatrix $\mathcal{M}_N$ of the Macaulay matrix $\mathcal{M}$ obtained by truncating it to the degree $N = (n+2)(d_1 + \cdots + d_k)$.

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Tropical Dual Nullstellensatz [Grigoriev and Podolskii (2018)]

The polynomials of f have a common root $x \in \mathbb{R}^n$ iff there exists a vector $y \in \mathbb{R}^m$ with $m = \binom{N+n}{n}$ in the tropical right null space of the truncated Macaulay matrix $\mathcal{M}_N$ for

$$N = (n+2)(d_1 + \cdots + d_k) \ .$$

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❶ - Contextualisation of the result

This work improves on Grigoriev and Podolskii's result by taking into account the sparse structure of the polynomials, and connects the tropical Nullstellensatz with classical elimination theory. In particular, it relies on a construction by Canny and Emiris (1993) and Strumfels (1994).

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This result in an improved truncation degree and allows us to deal better with sparse polynomials.

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- If $h := h_1 \square \cdots \square h_k$ where $\square$ denotes the sup-convolution, then $\mathrm{hypo}(h) = \mathrm{hypo}(h_1) + \cdots + \mathrm{hypo}(h_k)$ and moreover the supports of h is $Q = Q_1 + \cdots + Q_k$.

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- The projection of $\text{hypo}(h)$ onto Q yields a **coherent mixed subdivision** of Q.
- **Canny-Emiris set** associated to f: $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^n$ with δ a generic vector in the linear space directing the affine hull of Q.

The Newton polytopes associated to both systems (E_1) and (E_2) and their Minkowski sum are as follow.

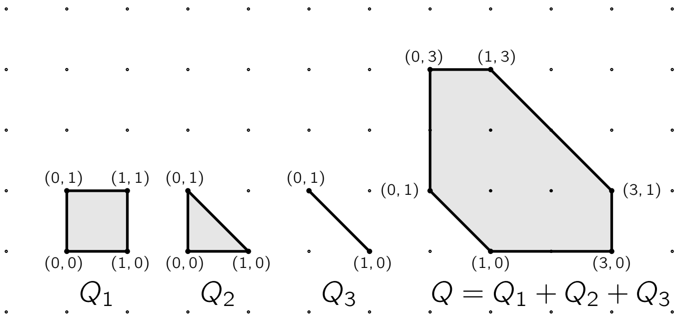
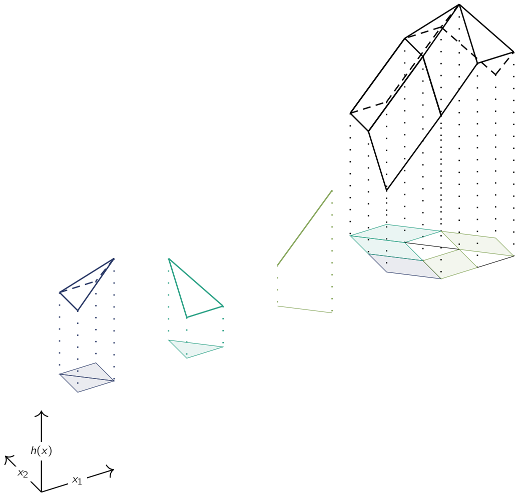
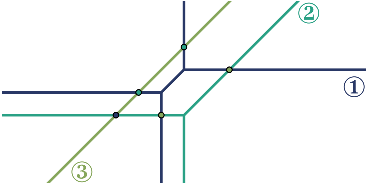


Figure: The subdivision of Q associated to (E_1) arises from the projection of the Minkowski sum of the hypographs of the h_i.



(a) The arrangement of tropical varieties of the polynomials from the system (E_1).



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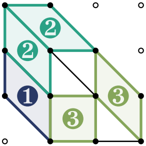
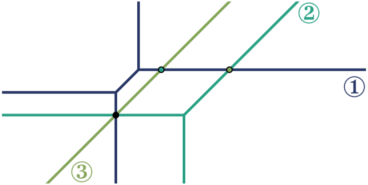
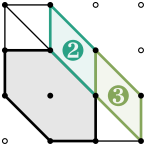


(c) The arrangement of tropical varieties of the polynomials from the system (E_2).



(d) The subdivision of Q associated to (E_2).



Example: Considering again the systems (E_1) and (E_2), for

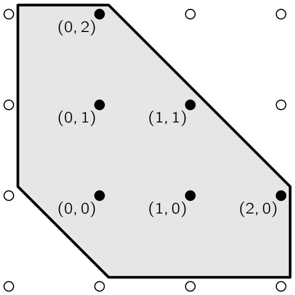
$$\delta = (-1+\varepsilon, -1+\varepsilon)$$

with $\varepsilon > 0$ sufficiently small, we obtain the Canny-Emiris set

$$\mathcal{E} := (Q+\delta) \cap \mathbb{Z}^n = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$$

corresponding to the set of monomials $\{1, x_1, x_2, x_1^2, x_1 x_2, x_2^2\}$.

Figure: The polytope $Q + \delta$ with $\delta = (-0.9, -0.9)$.



III - Our contribution: the tropical Nullstellensatz and Positivstellensatz for sparse polynomial systems

❷ - The tropical Nullstellensatz

Nullstellensatz for Sparse Tropical Polynomial Systems

The system $f \; \nabla \; \mathbb{0}$ *has a solution* $x \in \mathbb{R}^n$ *iff there exists a vector* $y \in \mathbb{R}^{\mathcal{E}'}$ *in the tropical right null space of the submatrix* $\mathcal{M}_{\mathcal{E}'}$ *of* $\mathcal{M}$*, where* $\mathcal{E}'$ *is any subset of* $\mathbb{Z}^n$ *containing a nonempty Canny-Emiris set* $\mathcal{E}$*.*

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Corollary: *The system $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$ if and only if the truncated Macaulay tropical linear system $\mathcal{M}_N \odot y \nabla \mathbb{0}$ has a solution $y \in \mathbb{R}^m$ for*

$$N = d_1 + \cdots + d_k - 1 \ ,$$

where $d_i = \deg(f_i)$ for all $1 \leq i \leq k$.

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$$N = d_1 + \cdots + d_k - 1 \ ,$$

where $d_i = \deg(f_i)$ for all $1 \leq i \leq k$. Moreover, if Q has full dimension, then one can take $N = d_1 + \cdots + d_k - n$ in the previous statement.

Example: The matrix associated with system (E_1) is

$$
\mathcal{M}_{\mathcal{E}}^{(1)} = \begin{array}{r} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\
\begin{pmatrix}
1 & 2 & 1 & & 1 & \\
0 & 0 & 1 & & & \\
 & 0 & & 0 & 1 & \\
 & & 0 & & 0 & 1 \\
 & 2 & 0 & & & \\
 & & & 2 & 0 & \\
 & & & & 2 & 0
\end{pmatrix}
\end{array} .
$$

There is no finite vector in its tropical right null space and thus there is no finite solution to the equation $f \,\nabla\, \mathbb{0}$.

Example: The matrix associated with system (E_2) is

$$
\mathcal{M}_{\mathcal{E}}^{(2)} = \begin{array}{r} \\ f_1 \\ f_2 \\ x_1 f_2 \\ x_2 f_2 \\ f_3 \\ x_1 f_3 \\ x_2 f_3 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\
\left(\begin{array}{cccccc}
1 & 4 & 1 & & 3 & \\
0 & 0 & 1 & & & \\
 & 0 & & 0 & 1 & \\
 & & 0 & & 0 & 1 \\
 & 2 & 0 & & & \\
 & & & 2 & 0 & \\
 & & & & 2 & 0
\end{array}\right)
\end{array} .
$$

The vector $y = \mathrm{ver}(-3,-1) = (0,-3,-1,-6,-4,-2)$ is finite and is in the tropical right null space of the previous matrix, hence there is a finite solution to the equation $f \ \nabla \ \mathbb{0}$, which is indeed given by $(-3,-1)$.

Outline of the proof

A $d \times d$ tropical matrix $A = (a_{ij})_{1 \leq i,j \leq d}$ is **tropically diagonally dominant** whenever

$$a_{ii} > a_{ij}$$

for all $1 \leq i, j \leq d$ such that $i \neq j$.

Lemma: *If A is tropically diagonally dominant, then the only solution $y \in \mathbb{R}_\infty^d$ to the equation $A \odot y \nabla \mathbb{0}$ is $y = \mathbb{0}$.*

Proof: Consider $y_i = \max_{1 \leq j \leq n} y_j$, then if $y_i > -\infty$ then the inequalities $a_{ii} > a_{ij}$ and $y_i \geq y_j$ imply that

$$a_{ii} + y_i > a_{ij} + y_j \quad \text{for all} \quad 1 \leq i \neq j \leq n \enspace ,$$

thus contradicting the assumption that $A \odot y \nabla \mathbb{0}$.

Outline of the proof

- If $f \nabla \mathbb{0}$ has a solution $x \in \mathbb{R}^n$, then the **Veronese embedding** $y = \text{ver}(x) := (x^p)_{p \in \mathcal{E}'}$ of x is a solution to $\mathcal{M}_{\mathcal{E}'} \odot y \nabla \mathbb{0}$.
- Otherwise we apply a construction from Canny and Emiris (1993) and Sturmfels (1994) but in a potentially **non generic** case to show that there is no finite vector $y \in \mathbb{R}^{\mathcal{E}'}$ in the tropical right null space of $\mathcal{M}_{\mathcal{E}'}$.

Outline of the proof

- If $p \in \mathcal{E}$, then $(p-\delta, h(p-\delta))$ is in the **relative interior** of a facet F of hypo(h), and F can be written as $F_1 + \cdots + F_k$ with F_i faces of hypo(h_i).
- Since f does not have a common root, at least one F_i is a singleton. Consider the maximal index j such that $F_j = \{a_j\}$ is a **singleton**. The couple (j, a_j) is called the **row content** of p.
- If $p \in \mathcal{E}$ and if (j, a_j) is its row content, then the support of the polynomial $X^{p-a_j} f_j$ is included in $\mathcal{E}$. This allows us to construct a square submatrix $\mathcal{M}_{\mathcal{E}\mathcal{E}} = (m_{pp'})_{(p,p')\in\mathcal{E}\times\mathcal{E}}$ of $\mathcal{M}_{\mathcal{E}}$.

Outline of the proof

- The matrix $\widetilde{\mathcal{M}}_{\mathcal{E}\mathcal{E}} = (\widetilde{m}_{pp'})_{(p,p')\in\mathcal{E}\times\mathcal{E}}$ obtained by setting $\widetilde{m}_{pp'} = m_{pp'} - h(p' - \delta)$ is **tropically diagonally dominant**.
- Therefore its tropical right null space is reduced to $\{\mathbb{0}\}$, and thus this is also the case for $\mathcal{M}_{\mathcal{E}\mathcal{E}}$.
- Hence there does not exist $y \in \mathbb{R}^{\mathcal{E}}$ such that $\mathcal{M}_{\mathcal{E}} \odot y \,\nabla\, \mathbb{0}$.
- Finally, since $\mathcal{M}_{\mathcal{E}'}$ can be written by block as

$$\mathcal{M}_{\mathcal{E}'} = \begin{array}{c} \begin{array}{cc} \mathcal{E} & \mathcal{E}' \setminus \mathcal{E} \end{array} \\ \begin{pmatrix} \mathcal{M}_{\mathcal{E}} & \mathbb{0} \\ * & * \end{pmatrix} \end{array},$$

  we deduce that there does also not exist $y \in \mathbb{R}^{\mathcal{E}'}$ such that $\mathcal{M}_{\mathcal{E}'} \odot y \,\nabla\, \mathbb{0}$.

III - Our contribution: the tropical Nullstellensatz and Positivstellensatz for sparse polynomial systems

❸ - The tropical Positivstellensatz

- Let $f^{\pm} = (f_1^{\pm}, \ldots, f_k^{\pm})$ be two collections of tropical polynomials.

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- For $1 \leq i \leq k$, denote by $\mathcal{A}_i^{\pm}$ the support of $f_i^{\pm}$ and let $f_i = f_i^{+} \oplus f_i^{-}$, with support $\mathcal{A}_i = \mathcal{A}_i^{+} \cup \mathcal{A}_i^{-}$.

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- We denote by $f^+(x) \rhd f^-(x)$ the system

$$\max_{\alpha \in \mathcal{A}_i^+} \left(f_{i,\alpha}^+ + \langle \alpha, x \rangle\right) \rhd_i \max_{\alpha \in \mathcal{A}_{i^-}} \left(f_{i,\alpha}^- + \langle \alpha, x \rangle\right) \text{ for all } 1 \le i \le k$$

of unknown $x \in \mathbb{R}_\infty^n$.

- Let $\mathcal{M}^{\pm}$ be the Macaulay matrices associated to $f^{\pm}$ – *i.e.* with entries $f^{\pm}_{i,\beta-\alpha}$.

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- For any subset $\mathcal{E}$ of $\mathbb{Z}^n$, denote by $\mathcal{M}^{\pm}_{\mathcal{E}}$ the submatrices of $\mathcal{M}^{\pm}$ by taking only the row indices $(i,\alpha) \in [k] \times \mathbb{Z}^n$ such that the supports of the rows (i,α) of both $\mathcal{M}^{+}$ and $\mathcal{M}^{-}$ and $\mathcal{M}^{-}_{\mathcal{E}}$ is included in $\mathcal{E}$ and the column indices given by $\mathcal{E}$.

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- Finally, denote by $\mathcal{M}^{+}_{\mathcal{E}} \odot y \rhd \mathcal{M}^{-}_{\mathcal{E}} \odot y$ the following system of tropical linear inequalities:

$$\max_{\beta\in\mathcal{E}} \left(\mathcal{M}^{+}_{(i,\alpha),\beta} + y_\beta\right) \rhd_i \max_{\beta\in\mathcal{E}} \left(\mathcal{M}^{-}_{(i,\alpha),\beta} + y_\beta\right) \text{ for all } 1 \leq i \leq k.$$

Let $Q = Q_1 + \cdots + Q_k$, where $Q_i = \mathrm{conv}(\mathcal{A}_i)$ for $i = 1, \ldots, k$.

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We now call **Canny-Emiris subsets** of $\mathbb{Z}^n$ associated to the pair of collections (f^+, f^-) any set $\mathcal{E}$ of the form

$$\mathcal{E} := ((n+1)Q + \delta) \cap \mathbb{Z}^n \quad ,$$

where δ is a generic vector in $V + \mathbb{Z}^n$, with V the direction of the affine hull of Q.

Tropical Positivstellensatz

There exists a solution $x \in \mathbb{R}^n$ to the system $f^+(x) \rhd f^-(x)$ if and only if there exists a vector $y \in \mathbb{R}^{\mathcal{E}'}$ satisfying $\mathcal{M}^+_{\mathcal{E}'} \odot y \rhd \mathcal{M}^-_{\mathcal{E}'} \odot y$, where $\mathcal{E}'$ is any subset of $\mathbb{Z}^n$ containing a nonempty Canny-Emiris subset $\mathcal{E}$ of $\mathbb{Z}^n$ associated to the pair (f^+, f^-).

Corollary: Let $f_0^\pm, \dots, f_k^\pm$ be a collection of pairs of tropical polynomials. Then, the following implication holds for all $x \in \mathbb{R}^n$

$$\left(\forall 1 \le i \le k,\ f_i^+(x) \ge f_i^-(x)\right) \quad \Longrightarrow \quad f_0^+(x) \ge f_0^-(x)$$

iff the Macaulay linearization $\mathcal{M}^+_{\mathcal{E}'} \odot y \rhd \mathcal{M}^-_{\mathcal{E}'} \odot y$ associated to the relations $f_i^+(x) \ge f_i^-(x)$ for $i = 1, \le \dots, \le k$ and $f_0^+(x) < f_0^-(x)$, where $\mathcal{E}'$ is as above, has no finite solution y.

IV - Algorithmical aspects

Mean pay-off games (see Akian, Gaubert et Guterman (2012)) :

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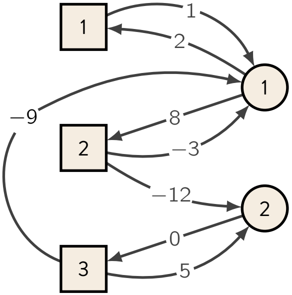
- the winner is the player who gets the highest average payment per turn;
- set $A = (a_{ij})_{(i,j)\in I\times J}$ et $B = (b_{ij})_{(i,j)\in I\times J}$.

Example : Let G be the following graph:



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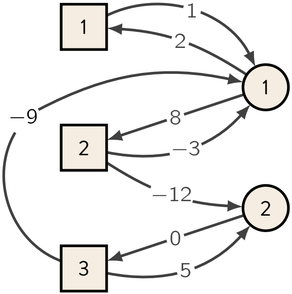

One has $A = \begin{pmatrix} 2 & -\infty \\ 8 & -\infty \\ -\infty & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -\infty \\ -3 & -12 \\ -9 & 5 \end{pmatrix}$.

Theorem [AGG12] : For all $j \in J$, player Max has a winning positional strategy for the *mean pay-off game* given by the payment matrices A and B by playing the initial move j iff there exists a solution $y \in \mathbb{R}_{\infty}^{J}$ of the tropical matrix inequality $A \odot y \leq B \odot y$ such that $y_j \neq \mathbb{0}$.

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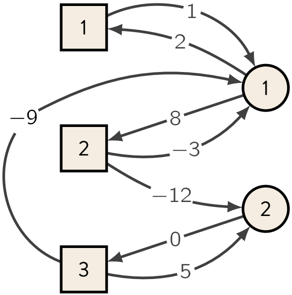
The winning initial moves correspond to the support of the solutions of the inequality $A \odot y \leq B \odot y$.

In the previous example,



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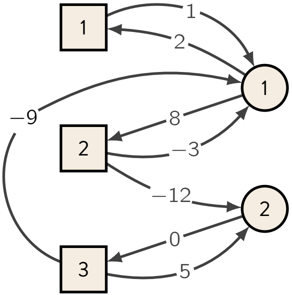

one has $A \odot x \leq B \odot x \iff \begin{cases} 2 + y_1 & \leq & 1 + y_1 \\ 8 + y_1 & \leq & \max(-3 + y_1, -12 + y_2) \\ y_2 & \leq & \max(-9 + y_1, 5 + y_2). \end{cases}$

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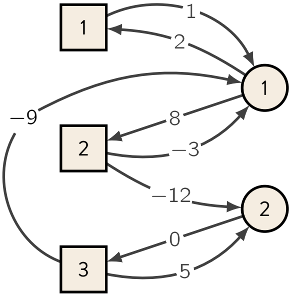

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The first inequality shows that every solution $y \in \mathbb{R}^2_\infty$ must satisfy $y_1 = \mathbb{0}$, which implies that the two other inesualities are satisfied for all values of $y_2 \in \mathbb{R}_\infty$.

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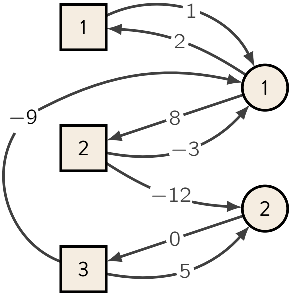

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This translates into the fact that the move 1 is a losing move for player Max, while the move 2 is a winning move.

- **Shapley operator** associated to a mean payoff game

$$T : \begin{array}{ccc} \mathbb{R} \cup \{\pm\infty\} & \longrightarrow & \mathbb{R} \cup \{\pm\infty\} \\ y = (y_j)_{j \in J} & \mapsto & \left(\min_{i \in I} -a_{ik} + \left(\max_{j \in J} b_i j + y_j\right)\right)_{k \in J} \end{array}$$

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- **value** of the game: $\chi(T) = \lim_{n \to +\infty} \frac{T^n(0)}{n}$

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Corollary: $\exists y\in\mathbb{R}^n$ such that $A\odot y\leq B\odot y$ iff $\chi(f)\geq 0$.

- **Value iteration algorithm:** polynomial time status is open but it is a practically fast method.

Value iteration algorithm

For two vectors $u, v \in (\mathbb{R} \cup \{+\infty\})^n$, we write $v \ll u$ if for all i such that $u_i < +\infty$, we have $v_i < u_i$. Moreover, for $\lambda \in \mathbb{R}$, we denote $\lambda + u$ the vector with coordinates $\lambda + u_i$. Algorithm 1 exploits the

Algorithm 1 Value iteration algorithm with widening.

procedure VALUEITERATION(T)
    ▹ *T a Shapley operator from $(\mathbb{R} \cup \{+\infty\})^m$ to $(\mathbb{R} \cup \{+\infty\})^m$*
    $u := 0 \in \mathbb{R}^m$, $v := 0 \in \mathbb{R}^m$, and $\varepsilon \geqslant 0$.
repeat
    $u := v$, $v := u \wedge (u + T(u))/2$ ▹ *value iteration step*
    $I := \{i \mid v_i \geqslant -\varepsilon + u_i\}$
    $\tilde{u} := (\tilde{u}_i) \in (\mathbb{R} \cup \{+\infty\})^m$ with $\tilde{u}_i = +\infty$ if $i \in I$ and $\tilde{u}_i = u_i$ otherwise ▹ *widening step*
    $\tilde{v} := T(\tilde{u})$
until $v \geqslant -\varepsilon + u$ or $v \ll -\varepsilon + u$ or $\tilde{v} \ll -\varepsilon + \tilde{u}$
    if $v \ll -\varepsilon + u$ or $\tilde{v} \ll -\varepsilon + \tilde{u}$ **then return** "Unfeasible"
    ▹ *There is no finite vector y such that $-\varepsilon + y \leqslant T(y)$.*
    else return u
    ▹ *We have $T(u) \geqslant -\varepsilon + u$.*
    end
end

Table: Average number of columns in the Macaulay matrix respectively in the full and sparse case for random systems of k inequations in n variables among 10 samples.

		k					
		2	3	4	5	6	7
n	2	24.5 - 12.9	44.2 - 26.2	63.4 - 42.3	124.9 - 80.5	162.6 - 110.4	238.5 - 149.5
	3	75.3 - 28.0	251.8 - 74.0	491.1 - 203.3	802.9 - 343.6	1276.0 - 624.6	1902.4 - 948.1
	4	248.5 - 39.0	604.8 - 90.8	1728.3 - 421.7	2481.7 - 649.2	6481.1 - 1666.3	12542.4 - 4034.1
	5	704.4 - 46.9	2184.7 - 87.8	5195.4 - 571.7	13731.3 - 1700.5	32362.5 - 4261.9	×
	6	1380.6 - 53.2	5726.7 - 114.5	×	×	×	×

Incoming work:

- Effective implementation (work in progress)

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Thank you for your attention!