USING STRUCTURED VARIANTS IN LATTICE-BASED CRYPTOGRAPHY

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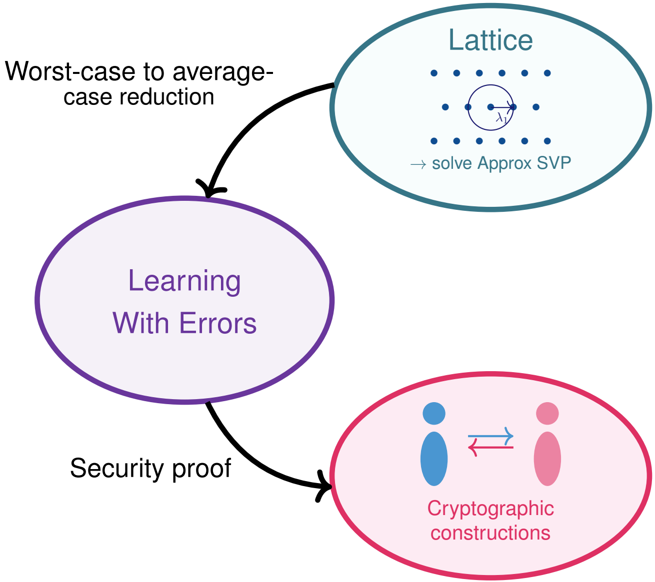
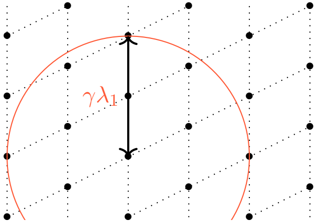
Using LWE to build provable constructions - theory



Approx Shortest Vector Problem (Approx SVP$_\gamma$)

Given a lattice $\mathcal{L}(\mathbf{B})$ of dimension n:

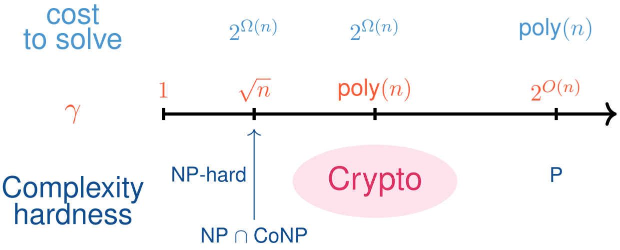
Output: find a non-zero vector $\mathbf{x} \in \mathcal{L}(\mathbf{B})$ such that $\|\mathbf{x}\| \leq \gamma \lambda_1(\mathcal{L}(\mathbf{B}))$



Lattice

$\mathcal{L}(\mathbf{B}) = \{\sum_{1=i}^{n} a_i \mathbf{b}_i, a_i \in \mathbb{Z}\}$, where the $(\mathbf{b}_i)_{1\leq i \leq n}$'s, linearly independent vectors, are a basis of $\mathcal{L}(\mathbf{B})$.

Hardness of Approx SVP$_\gamma$



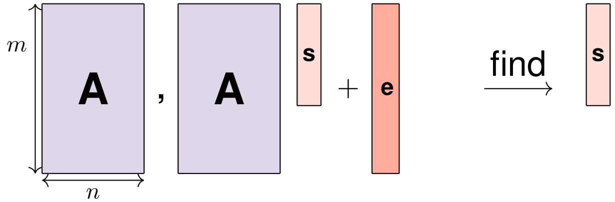
Conjecture

There is no polynomial time algorithm that approximates this lattice problem and its variants to within polynomial factors.

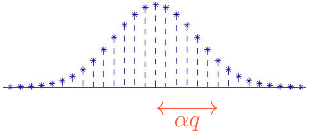
The Learning With Errors problem

$\mathsf{LWE}^n_{\alpha,q}$

Given



- $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{m\times n})$,
- $\mathbf{s} \leftarrow U(\mathbb{Z}_q^{n})$,
- $\mathbf{e} \leftarrow D_{\mathbb{Z}^m,\alpha q}$, small compared to q.



Discrete Gaussian error $D_{\mathbb{Z},\alpha q}$

Search version: Given $(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e})$, find $\mathbf{s}$.
Decision version: Distinguish from $(\mathbf{A}, \mathbf{b})$ with $\mathbf{b}$ uniform.

Regev's encryption scheme

- **Parameters**: $n, m, q \in \mathbb{Z}$, $\alpha \in \mathbb{R}$,
- **Keys**: sk = $\mathbf{s}$ and pk = $(\mathbf{A}, \mathbf{b})$, with $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$
  where $\mathbf{s} \hookleftarrow U(\mathbb{Z}_q^n)$, $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m \times n})$, $\mathbf{e} \hookleftarrow D_{\mathbb{Z}^m, \alpha q}$.
- **Encryption** $(M \in \{0,1\})$: Let $\mathbf{r} \hookleftarrow U(\{0,1\}^m)$,

$$\mathbf{u}^T = \mathbf{r}\,\mathbf{A}, \quad v = \mathbf{r}\,\mathbf{b} + \lfloor q/2 \rceil \cdot M$$

- **Decryption** of $(\mathbf{u}, v)$: compute $v - \mathbf{u}^T\mathbf{s}$,

$$\underbrace{\mathbf{r}\left[\mathbf{A}\,\mathbf{s} + \mathbf{e}\right] + \lfloor q/2 \rceil \cdot M}_{v} - \underbrace{\mathbf{r}\,\mathbf{A}\,\mathbf{s}}_{\mathbf{u}^T\mathbf{s}} = \text{small} + \lfloor q/2 \rceil \cdot M$$

If **close from** 0: return 0, if **close from** $\lfloor q/2 \rceil$: return 1.

LWE hard $\Rightarrow$ Regev's scheme is IND-CPA secure.

Using LWE

Hardness of LWE used as a foundation for many constructions.

Learning With Errors

Problem: constructions based on LWE enjoy a nice guarantee of security but are too costly in practice.

Security proof

Cryptographic constructions

Signature, encryption
Advanced schemes
Fully Homomorphic Encryption

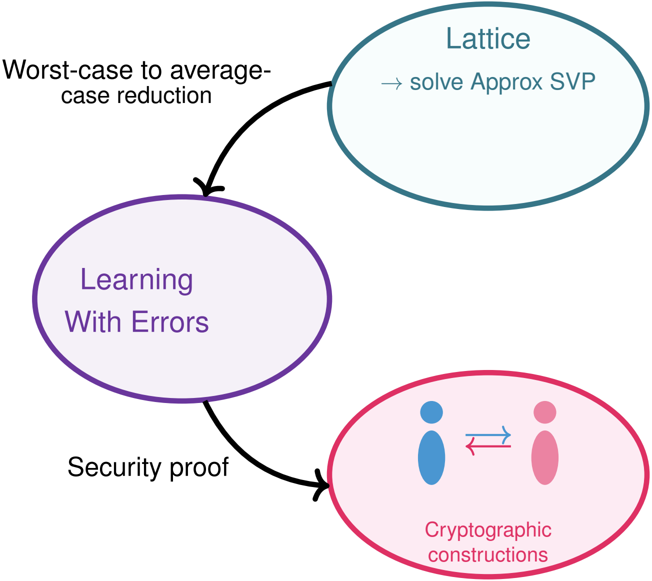
Solutions used today?

Lattice-based NIST finalists

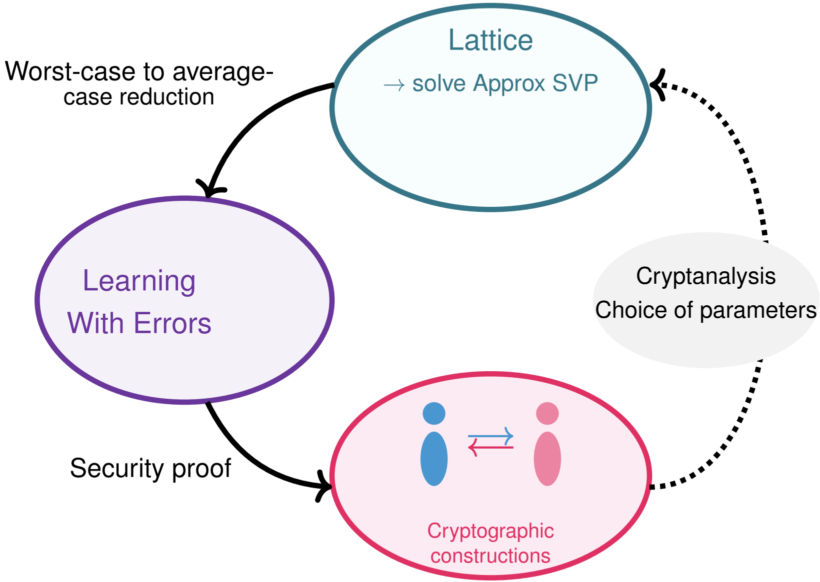
Among the 5 lattice-based finalists, 3 of them are based on (possibly structured) variants of LWE.

- Public Key Encryption
  - **Crystals - Kyber**: Module-LWE with both secret and noise chosen from a centered binomial distribution.
  - **Saber:** Module-LWR (deterministic variant).
  - **NTRU**
  - **FrodoKEM** (as alternate candidate): LWE but with smaller parameters.
- Signature
  - **Crystals - Dilithium**: Module-LWE with both secret and noise chosen in a small uniform interval, and Module-SIS.
  - **Falcon**: Ring-SIS on NTRU matrices.

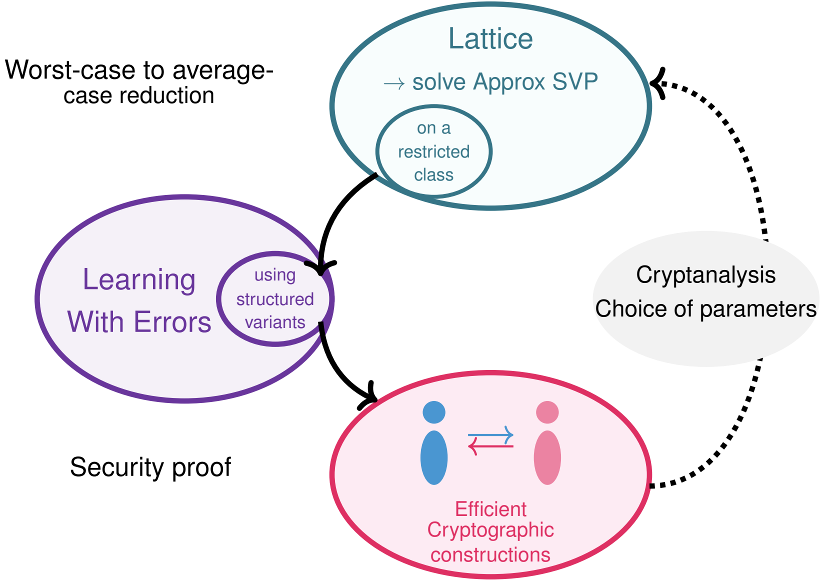
Using LWE to build constructions



Using LWE to build constructions in practice



Using LWE to build constructions in practice

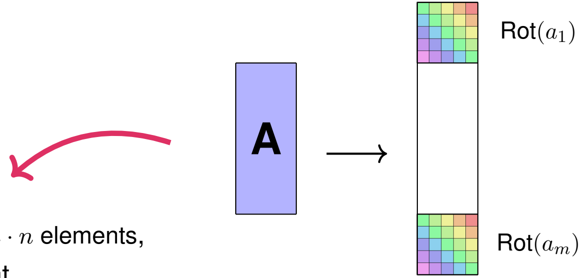


From SIS/LWE to structured variants

Problem: constructions based on LWE enjoy a nice guaranty of security but are too costly in practice.

$\rightarrow$ replace $\mathbb{Z}^n$ by a Ring, for example $R = \mathbb{Z}[x]/\langle x^n + 1\rangle$ $(n = 2^k)$.

- Ring variants since 2006:



- Structured **A** $\in \mathbb{Z}_q^{m \cdot n \times n}$ represented by $m \cdot n$ elements,
- Product with matrix/vector more efficient,
- Hardness of Ring-SIS, [Lyubashevsky and Micciancio 06] and [Peikert and Rosen 06]
- Hardness of Ring-LWE [Lyubashevsky, Peikert and Regev 10].

Idea: replace $\mathbb{Z}^n$ by $R = \mathbb{Z}[x]/\langle x^n + 1\rangle$

where $n = 2^k$ then the polynomial $x^n + 1$ is irreducible.
Elements of this ring are polynomials of degree less than n.

R is a **cyclotomic ring.** R is also the ring of integer $\mathcal{O}_K$ of an number field K:

- $K = \mathbb{Q}[x]/\langle x^n + 1\rangle$: K is a cyclotomic field,
- $R = \mathbb{Z}[x]/\langle \phi_m(x)\rangle$ where ϕ_m is the mth cyclotomic polynomial of degree $n = \varphi(m)$. Its roots are the mth roots of unity $\zeta_m^j \in \mathbb{C}$, with $\zeta_m = e^{\frac{2i\pi}{m}}$. (For $m = 2^{k+1}$, we have $\phi_m(x) = x^n + 1$.)
- Canonical embedding: $\sigma_K : \alpha \in K \mapsto ((\sigma(\alpha))_\sigma = (\alpha(\zeta_m^j))_j$.

Idea: replace $\mathbb{Z}^n$ by $R = \mathbb{Z}[x]/\langle x^n + 1\rangle$

R is isomorph to $\mathbb{Z}^n$

Let $a \in R$, we have $a(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$,
the isomorphism $R \to \mathbb{Z}^n$ associate

the polynomial $a \in R$ to the vector $\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} \in \mathbb{Z}^n$.

Idea: replace $\mathbb{Z}^n$ by $R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$

Let's look at the product of two polynomials $x^n + 1$

- $a(x) = a_0 + a_1 \cdot x + \ldots + a_{n-1} \cdot x^{n-1}$
- $s(x) = s_0 + a_1 \cdot x + \ldots + a_{n-1} \cdot x^{n-1}$

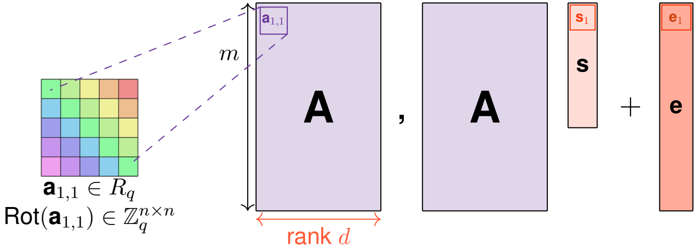
Using matrices, it gives the following block:

$$\begin{bmatrix} a_0 & -a_{n-1} & \cdots & -a_2 & -a_1 \\ a_1 & a_0 & \cdots & -a_3 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2} & a_{n-3} & \cdots & a_0 & -a_{n-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{bmatrix}$$

Module LWE

Let K be a number field of degree n with R its ring of integers.
Think of K as $\mathbb{Q}[x]/(x^n+1)$ and of R as $\mathbb{Z}[x]/(x^n+1)$ for $n = 2^k$.

Replace $\mathbb{Z}$ by R, and $\mathbb{Z}_q$ by $R_q = R/qR$.



- $\mathbf{A} \leftarrow U(R_q^{m\times d})$,
- $\mathbf{s} \leftarrow U(R_q^d)$,
- $e \in R^m$ small compared to q.

Special case $d = 1$
is Ring-LWE

Module SIS and LWE

$R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$ and $R_q = R/qR$.

Module-SIS$_{q,m,\beta}$
Given $\mathbf{a}_1, \ldots, \mathbf{a}_m \in R_q^d$ independent and uniform, find $z_1, \ldots, z_m \in R$ such that $\sum_{i=1}^m \mathbf{a}_i \cdot z_i = 0 \bmod q$ and $0 < \|\mathbf{z}\| \leq \beta$.

Let $\alpha > 0$ and $\mathbf{s} \in (R_q)^d$, the distribution $A^{(M)}_{\mathbf{s}, D_{R,\alpha q}}$ is:

- $\mathbf{a} \in (R_q)^d$ uniform,
- e sampled from $D_{R,\alpha q}$,

Outputs: $(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e)$.

Module-LWE$_{q,\nu_\alpha}$
Let $\mathbf{s} \in (R_q)^d$ uniform, distinguish between an arbitrary number of samples from $A^{(M)}_{\mathbf{s}, D_{R,\alpha q}}$ or the same number from $U((R_q)^d \times R_q)$.

Ideals and modules

$R = \mathbb{Z}[x]/\langle x^n + 1\rangle$ and $R_q = R/qR$.

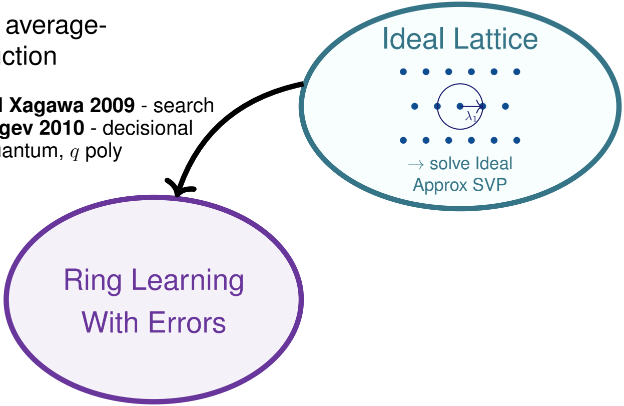
- An ideal I of R is an additive subgroup of R closed under multiplication by every elements of R.
- As R is isomorph to $\mathbb{Z}^n$, any ideal $I \in R$ defines an integer lattice $\Lambda(\mathbf{B})$ where $\mathbf{B} = \{g \bmod x^n + 1 : g \in I\}$.

- A subset $M \subseteq K^d$ is an R-module if it is closed under addition and multiplication by elements of R.
- A finite-type R-module: $M \subseteq R^d : \sum_{i=1}^{D} R \cdot \mathbf{b}_i$, $(\mathbf{b}_i) \in R^d$,
- $M = \sum_{i=1}^{d} I_i \cdot \mathbf{b}_i$ where I_i are ideals of R and $(I_i, \mathbf{b}_i)$ is a pseudo-basis of M.
- As ideals, any module defines an integer module lattice.

Hardness of Ring Learning With Errors problem

Worst-case to average-case reduction

- **Stehlé, Steinfeld, Tanaka and Xagawa 2009** - search
- **Lyubashevsky, Peikert, Regev 2010** - decisional reduction both quantum, q poly



Self reductions

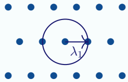
- **Applebaum, Cash, Peikert, Sahai 2009** - same error and secret

Hardness of Module Learning With Errors problem

Worst-case to average-case reduction

- **Langlois Stehlé 2015** - quantum, q poly
- Folklore: adapting **Peikert 2009** gives classical reduction but q exp and only search variant
- **Boudgoust, Jeudy, Roux-Langlois, Wen 2021** classical, q poly, decisional, linear rank

Module Lattice



$\rightarrow$ solve Module Approx SVP

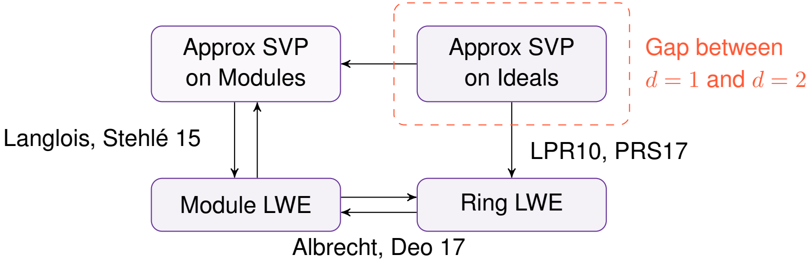
Module Learning With Errors

Self reductions

- **Applebaum, Cash, Peikert, Sahai 2009** - same error and secret
- **Boudgoust, Jeudy, Roux-Langlois, Wen 2022**: short error and secret distributions

Module or Rings?

- Hardness of the problem



Module or Rings?

- Choice of parameters
  - Example of Ring $R_q = \mathbb{Z}_q[x]/\langle x^n + 1\rangle$
  - Constraints on parameters $n = 2^k$, $q = 1 \bmod 2n$ ...
- An example of parameter set:
  - $n = 512 \Rightarrow$ 60 bits of security,
  - $n = 1024 \Rightarrow$ 140 bits of security,
  - $(n = 256, d = 3)$ gives $nd = 768$ which is ”in between”.
- Module LWE allows more flexibility.

NIST competition

From 2017 to 2024, NIST competition to develop new standards on post-quantum cryptography

2022 first results: **3 over 4 new standards** are lattice-based

- Kyber - encryption scheme based on Module-LWE,
- Dilithium - signature scheme based on Module SIS and LWE,
- Falcon - signature scheme based on NTRU and Ring-SIS.

Encryption scheme based on Ring-LWE

[Lyubashevsky, Peikert, Regev 2011]

KeyGen : The secret key is a small $s \in R$
The public key is $(a, b) = (a, b = a \cdot s + e) \in R_q^2$,
with $a \leftarrow U(R_q)$ and a small $e \in R$.

Enc : Given $m \in \{0, 1\}^n$, a message is a polynomial in R with coordinates in $\{0, 1\}$. Sample small r, e_1, e_2 in R and output

$$(a \cdot r + e_1, b \cdot r + e_2 + \lfloor q/2 \rfloor \cdot m) \in R_q \times R_q.$$

Dec : Given $(u, v) \in R_q \times R_q$, compute

$$v - u \cdot s = (r \cdot e - s \cdot e_1 + e_2) + b\lfloor q/2 \rfloor \cdot m$$

For each coordinate of m, the plaintext is 0 if the result is closer from 0 than $\lfloor q/2 \rfloor$, and 1 otherwise.

Kyber

[Avanzi, Bos, Ducas, Kiltz, Lepoint, Lyubashevsky, Schanck, Schwabe, Seiler, Stehle]

- Kyber relies on Module-LWE,
- Use $R_q = \mathbb{Z}_q[x]/\langle x^{256}+1\rangle$ with $q = 7681$.

- The small elements follow a binomial distribution B_η:
  For some positive integer η, sample $\{(ai, bi)\}_{i=1}^{\eta} \leftarrow (\{0,1\}^2)^\eta$ and output $\sum_{i=1}^{\eta}(a_i - b_i)$.

- The uniform public key is generated given a $seed$ and a function PARSE,
- Multiplication operations uses NTT - Number Theoretic Transform - which is a variant of the FFT in rings,
- Size of ciphertext is compressed by keeping only high order bits.

Performances

Current timings (ECDH)
Public key around 32 bytes
Efficiency comparable in terms of cycles.

Kyber-512

Sizes (in bytes)		Haswell cycles (ref)		Haswell cycles (avx2)	
sk:	1632	gen:	122684	gen:	33856
pk:	800	enc:	154524	enc:	45200
ct:	768	dec:	187960	dec:	34572

Kyber-768

Sizes (in bytes)		Haswell cycles (ref)		Haswell cycles (avx2)	
sk:	2400	gen:	199408	gen:	52732
pk:	1184	enc:	235260	enc:	67624
ct:	1088	dec:	274900	dec:	53156

Kyber-1024

Sizes (in bytes)		Haswell cycles (ref)		Haswell cycles (avx2)	
sk:	3168	gen:	307148	gen:	73544
pk:	1568	enc:	346648	enc:	97324
ct:	1568	dec:	396584	dec:	79128

Choice of parameters

- Parameters used by Kyber:
  - $n = 256$ and $d = 2, 3, 4$ giving three levels of security: $512, 768, 1024$,
  - $q = 7681$

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  - By considering the LWE instance with dimension nd,
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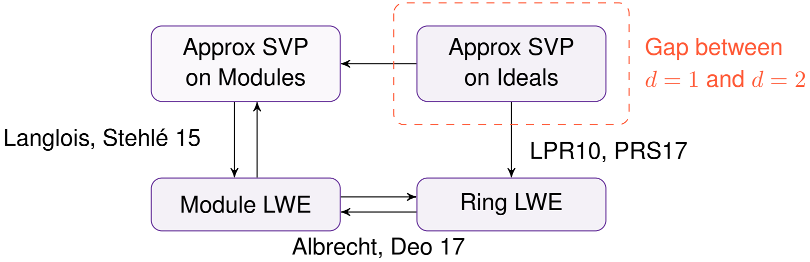
- How do they choose the parameters?
  - By considering the LWE instance with dimension nd,
  - and the ”lattice estimator” [Albrecht, Player, Scott 2015],
- There is no consideration of the structure!
  - Why?
  - Because we don’t know how...

Approx Ideal SVP seems to be the easiest

- Hardness of the problem



Solving Approx Ideal SVP[1]

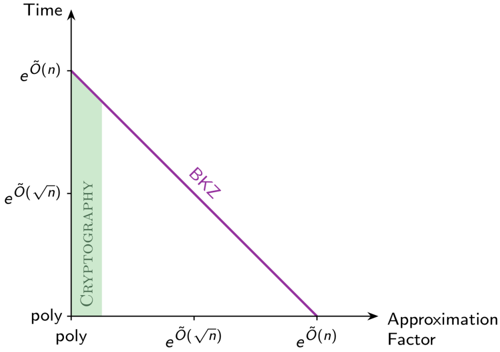
- For a long time, no algorithm manages to exploit the structure of Ideal SVP.

- 2014: Quantum algorithm computing ($\mathcal{S}$-)units, class groups in polynomial time!
  [EHKS14,BS16]

- Followed by a long series of cryptanalysis works.
  [CGS14,CDPR16,CDW17/21,PHS19,BR20,BLNR22,BL21,BEFHY22]

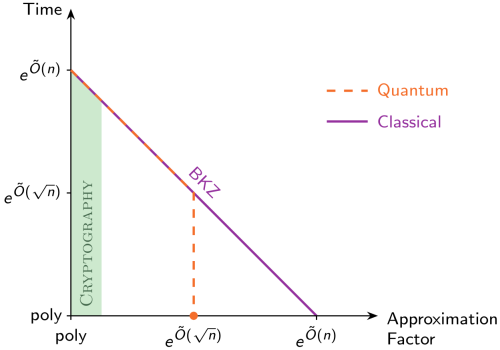
[1] Thanks to Olivier Bernard and Andrea Lesavourey for part of the slides (particularly to Olivier for the `tikz` picture!)

Algebraic cryptanalysis of Ideal-SVP



1. Schnorr's hierarchy (*unstructured*)

Algebraic cryptanalysis of Ideal-SVP



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2. CDW algorithm [Cramer, Ducas, Wesolowski 17/21]: uses short *Stickelberger* relations.

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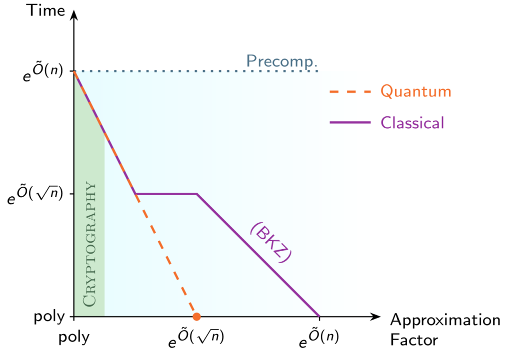

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3. PHS and Twisted-PHS [Pellet-Mary, Hanrot, Stehlé 19, Bernard, Roux-Langlois 20, Bernard, Lesavouvey, Nguyen, Roux-Langlois 22]: *$\mathcal{S}$-unit attacks*.

Solving Approx Ideal SVP

Consider an intermediate problem.

Short Generator Principal ideal Problem (SG-PIP):
Given a principal ideal $I = (g)$ such that g is short, retrieve g.

[2] $\text{Log}_K : x \mapsto (\ln|\sigma_1(x)|, \ldots, \ln|\sigma_n(x)|)$

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1. Find a generator $h = gu$ of I ($u \in \mathcal{O}_K^\times$)
   Can be done in polynomial time with a quantum computer

2. Find g given h.
   Use the Log-embedding[2] and the Log-unit lattice $\mathrm{Log}(\mathcal{O}_K^\times)$

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- [Cramer, Ducas, Peikert, Regev 2016] quantum polynomial-time or classical $2^{n^{2/3+\epsilon}}$-time algorithm to solve SG-PIP over cyclotomic fields.

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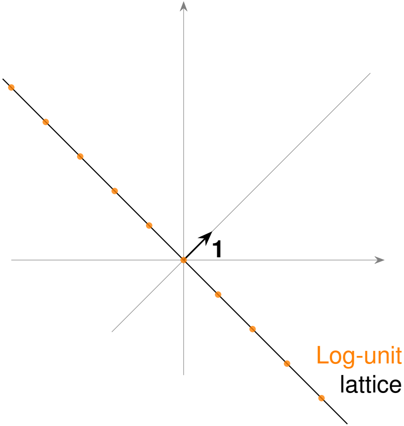
View of the algorithm



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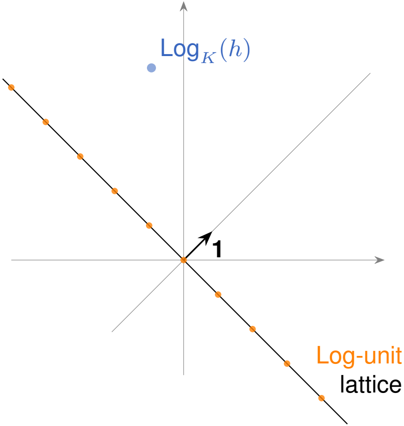

Let I be a challenge ideal.

1. Quantum decomposition
   Apply Log_K
   $\mathrm{Log}_K(h) = \mathrm{Log}_K(g) + \mathrm{Log}_K(u) \in \mathrm{Log}_K(g) + \mathrm{Log}_K(\mathcal{O}_K^\times)$

$$h = g \cdot u$$

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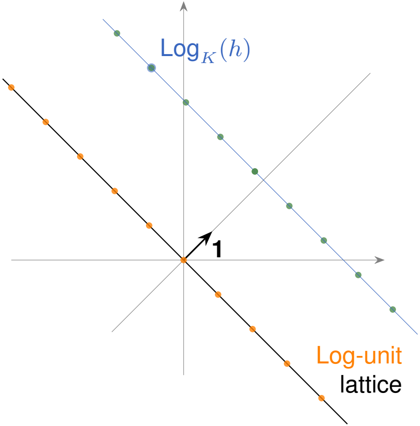

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2. *Short* coset representative ?

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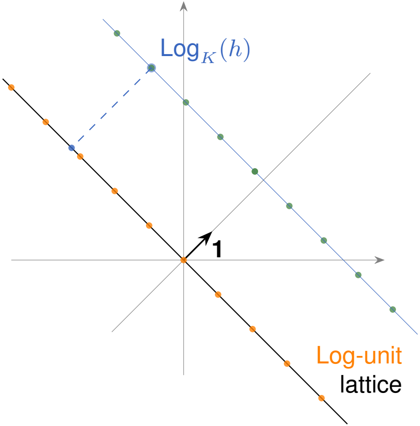

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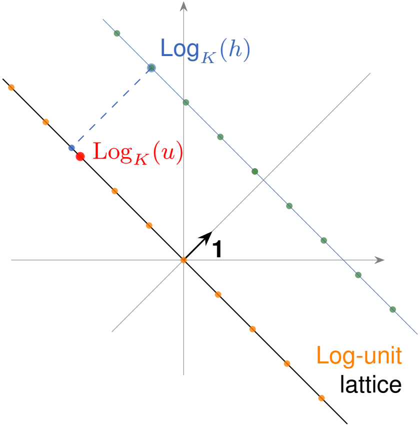

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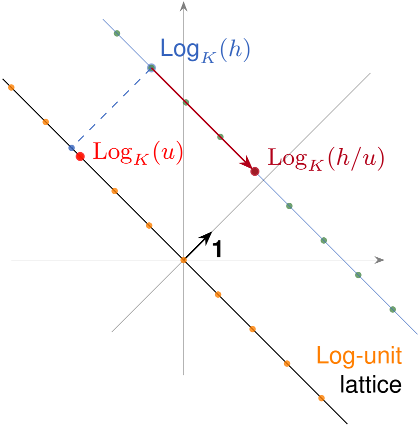
View of the algorithm



Let I be a challenge ideal.

1. Quantum decomposition
   Apply Log_K
   $\mathrm{Log}_K(h) = \mathrm{Log}_K(g) + \mathrm{Log}_K(u) \in \mathrm{Log}_K(g) + \mathrm{Log}_K(\mathcal{O}_K^\times)$

2. *Short* coset representative ?

3. Hope this is *short* in I.

$$h = g \cdot u$$

$$(h/u) = g$$

SVP of general ideals

Consider K a number field, I an ideal and S a set of prime ideals.

1. Compute a S-generator of I, i.e. h s.t. $(h) = I \prod_{\mathfrak{p}\in S} \mathfrak{p}^{v_\mathfrak{p}}$
2. Reduce h

Two variants for step 2.

1. First reduce $\prod_\mathfrak{p} \mathfrak{p}^{v_\mathfrak{p}}$; then find a generator with the Log-embedding.
   $\rightarrow$ [Cramer, Ducas, Wesolowski 2017] cyclotomic fields, subexponential approximation factor

2. Use the Log-S-embedding[3] to reduce everything.
   $\rightarrow$ [Pellet-Mary, Hanrot, Stehlé 2019] all number fields, exponential preprocessing, subexponential approximation factor
   $\rightarrow$ [Bernard, Roux-Langlois 2020] other def. of $\mathrm{Log}_{K,S}$, same asymptotic results, **good results in practice for cyclotomics up to dimensions 70.**

[3] $\mathrm{Log}_{K,S} : x \mapsto (\ln|\sigma_1(x)|, \ldots, \ln|\sigma_n(x)|, -v_{\mathfrak{p}_1}(x)\ln(N(\mathfrak{p}_1)), \ldots, -v_{\mathfrak{p}_r}(x)\ln(N(\mathfrak{p}_r)))$

Bernard, Lesavourey, Nguyen, Roux-Langlois (2022)

Can we extend these good results to higher dimensions ?

Two major obstructions for experiments:

- Decomposition $(h) = I \cdot \prod_{\mathfrak{p}\in S} \mathfrak{p}^{v_\mathfrak{p}}$
- Group of S-units $(s) = \prod_{\mathfrak{p}\in S} \mathfrak{p}^{e_\mathfrak{p}}$

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- Obtain explicit short basis of S_m
- It is constructive: the associated generators can be computed efficiently
- Free family of short S-units

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Allows us to approximate $\mathrm{Log}(\mathcal{O}_{K,S}^{\times})$ with a full-rank sublattice

- Cyclotomic units
- Explicit Stickelberger generators
- Real $S \cap K_m^{+}$-units $\rightarrow$ only part sub-exponential; dimension $n/2$
- 2-saturation to reduce the index

Experimental results[4]

Cyclotomic fields with almost all conductors, up to dimension 210.

Simulated targets in the Log-space

[4] Code available at https://github.com/ob3rnard/Tw-Sti.

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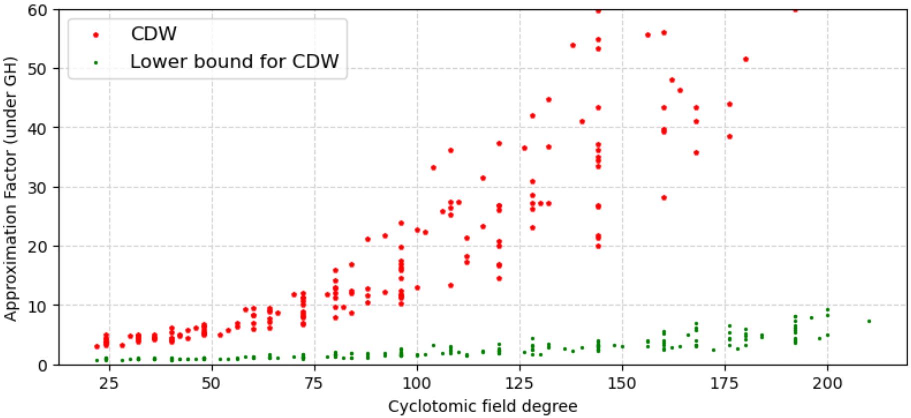

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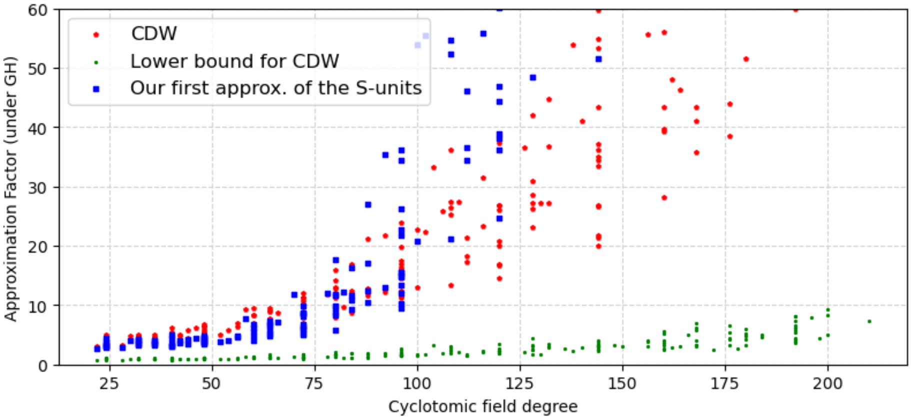

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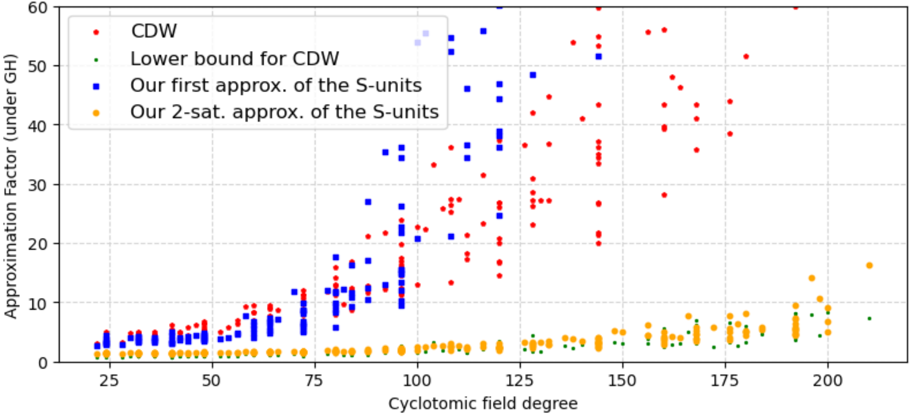

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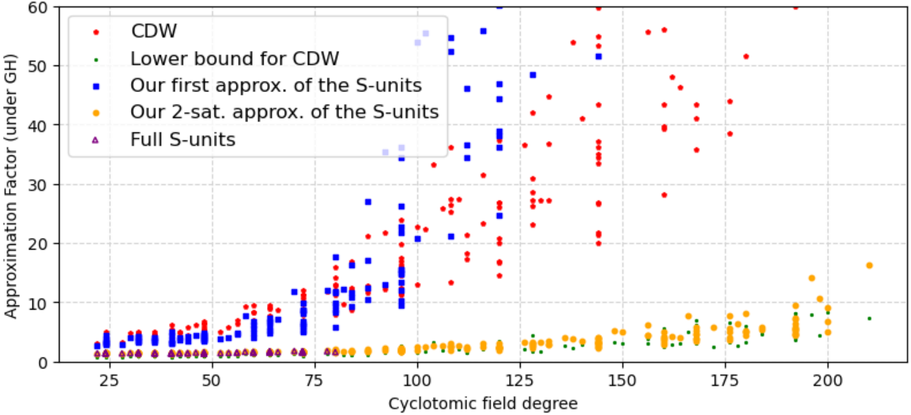

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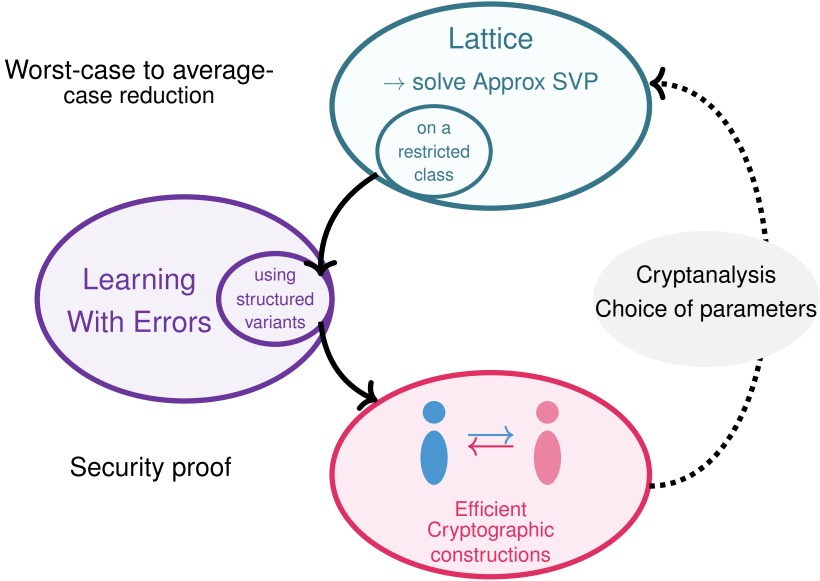
Using LWE to build constructions in practice



Conclusion

- Lattice-based cryptography allows to build efficient constructions such as encryption or signature schemes with a security based on the hardness of difficult algorithmic problems on lattices.

- Three schemes (Kyber, Dilithium and Falcon) will be standardise by the NIST, together with a hash-based signature.
  Two of them are based on Module-LWE.

- Approx Ideal SVP seems to be the easier problem to try to solve $\rightarrow$ the results of recent attacks does not impact the security of lattice-based constructions.