On Covering Spheres with a Q-arrangement of Simplices

An inductive Characterization of Q-matrices

by Khalil Ghorbal and Christelle Kozaily (Inria, France)
on March 9, 2023. JNCF

Linear Complementarity

» Linear Complementarity Systems (LCS)

A LCS is defined as the following system

$$
\begin{aligned}
q' &= Aq + Bz + e \\
w &= q + Mz \\
0 &\le w \perp z \ge 0
\end{aligned}
$$

* $q \in \mathbb{R}^n$ (state), $w \in \mathbb{R}^m$ (output), $z \in \mathbb{R}^m$ (input)
* q' stands for the time derivative of q
* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M \in \mathbb{R}^{m\times m}$, $e \in \mathbb{R}^n$
* This is a semi-explicit non-smooth DAE (unilateral constraints)

» Sufficient Condition For State Continuous Solution

$$q' = Aq + Bz + e$$
$$w = q + Mz$$
$$0 \leq w \perp z \geq 0$$

Heemels and Schumacher 2000

If the principle minors of M are all positive then the LCS admits a unique solution continuous in x.

» Linear Complementarity Problem (LCP)

A LCP(q,M) is defined as the following system

$$\cancel{q' = Aq + Bz + e}$$
$$w = q + Mz$$
$$0 \leq w \perp z \geq 0$$

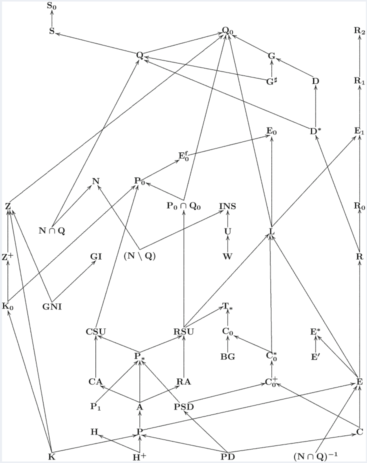
* $w, z \in \mathbb{R}^n$ are unknown
* M is a P-matrix: $\mathrm{LCP}(q, M)$ has a unique solution $\forall q$.
* M is a Q-matrix: $\mathrm{LCP}(q, M)$ has a solution $\forall q$.

» State-of-the-art

somehow unsatisfactory...

5 Matrix class inclusion map

See Fig. 1.



» Complementary Cones

$$\begin{aligned} w &= q + Mz \\ 0 \leq w &\perp z \geq 0 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} q &= \begin{pmatrix} I & -M \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \\ w, z &\geq 0 \wedge w.z = 0 \end{aligned}$$

* $C = \langle a_1, \dots, a_n \rangle$, $a_i \in \{I_i, -M_i\}$
* 2^n complementary cones C_k
* The cones 'are' the vertices of the hypercube Q_n
* Longest Hamiltonian cycle of Q_n has a length of 2^n
* Cones are sewed along their common $(n-1)$-facets
* M is a P_0-matrix if all cones partition $\mathbb{R}^n$
* (and if all cones are non-degenerate, M is a P-matrix)
* M is a Q-matrix if all cones cover $\mathbb{R}^n$, i.e. $\mathbb{R}^n \subset \cup_k C_k$

» Partition

Characterizing P-matrices [1958]

M is a P-matrix if and only if the 2^n principle minors of *M* are positive.

* A principle minor is the determinant of a well-oriented complementary subset of $(I \quad -M)$
* Positivity ensures that the orientation of all cones is preserved
* ... and that all cones are non-degenerate

» Partition

Characterizing P-matrices [1958]

M is a P-matrix if and only if the 2^n principle minors of *M* are positive.

- A principle minor is the determinant of a well-oriented complementary subset of $(I \quad -M)$
- Positivity ensures that the orientation of all cones is preserved
- … and that all cones are non-degenerate

Characterizing Q-matrices [?]

M is a Q-matrix if and only if …

Planar Case

» Planar Q-Covering

For S^1 to be covered, the point e_2 must be surrounded in one of the following ways:

1. $S^0(e_2) \subseteq [e_1, -M_1]$ (local problem of lower dimension),
2. $e_2 \in \langle e_1, -M_2 \rangle^\circ$ (e_2 is in the interior of $\langle e_1, -M_2 \rangle$),
3. $e_2 \in \langle -M_1, -M_2 \rangle^\circ$ (e_2 is in the interior of $\langle -M_1, -M_2 \rangle$),
4. $e_2 = -M_1 \wedge S^0(e_2) \subseteq [e_1, -M_2]$ (e_2 coincides with $-M_1$).

Given $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, with $m_1^2 + m_3^2 = m_2^2 + m_4^2 = 1$, e_2 is surrounded if and only if one of the following holds

1. $-m_1 < 0$
2. $-m_4 > 0 \wedge m_2 > 0$
3. $-\det(M), m_2, -m_1$ have the same sign
4. $-m_1 = 0 \wedge -m_3 = 1 \wedge -m_4 < 0$

» Q-matrices for $n = 2$

$$
\begin{aligned}
&(m_2 m_3 m_4 < m_1 m_4^2)\\
\vee&(m_2 m_4 > 0 \wedge m_1 = m_2 \wedge m_3 = m_4)\\
\vee&(m_3 > 0 \wedge 1 + m_2 = 0 \wedge m_4 = 0)\\
\vee&(m_2 < 0 \wedge m_4 < 0)\\
\vee&(m_1 m_2 > 0 \wedge \frac{m_2 m_3}{m_1} > m_4)
\end{aligned}
$$

Q-Covering

» Spherical Geometry

$$w = q + Mz$$
$$0 \leq w \perp z \geq 0$$

* If q has a solution, then so does λq for any positive λ.
* $q = 0$ has an obvious solution $(0, 0)$.
* $u \simeq v$ if and only if $u = \lambda v$ for some positive λ
* It suffices to study the covering on $(\mathbb{R}^n \setminus \{0\}) /_{\simeq}$ which is homemorphic to S^{n-1}, the sphere of dimension $n - 1$
* Complementary cones become spherical simplices

» Occam's Razor

* Q-matrix is not necessarily about a linear application
* Positive homogeneity can be dropped
* The vector space structure of $\mathbb{R}^n$ is useless

» Occam's Razor

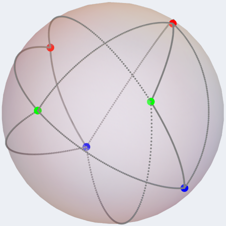
* Q-matrix is not necessarily about a linear application
* Positive homogeneity can be dropped
* The vector space structure of $\mathbb{R}^n$ is useless

Deciding whether M is a Q-matrix is equivalent to a specific simplicial covering of $(n-1)$-dimensional sphere following the longest Hamiltonian path of the hypercube graph Q_n.

» Cells

Definition (Cell)

A *cell* is a closed non-empty connected subset of S^{n-1} delimited by an $(n-2)$-facet in each direction and such that its interior does not intersect any other facet.



» Non-constructive Characterization

Definition (Surrounded Point)

We say that a point $q \in S^{n-1}$ is *surrounded* if and only if there exists an (open) neighborhood $U \subseteq S^{n-1}$ of q such that U is covered, i.e. $U \subset \cup_k C_k$.

» Non-constructive Characterization

Definition (Surrounded Point)

We say that a point $q \in S^{n-1}$ is *surrounded* if and only if there exists an (open) neighborhood $U \subseteq S^{n-1}$ of q such that U is covered, i.e. $U \subset \cup_k C_k$.

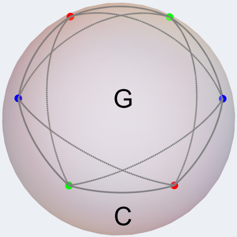
Localization

If each cell has at least one surrounded vertex (0-simplex), then S^{n-1} is covered.

1. How many points one needs to check?
2. How to prove that a given point is surrounded?

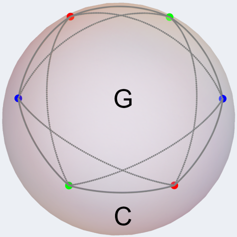
» Ghost Cells

Checking the original list of $2n$ points I_i and $-M_i$ is not enough.



» Ghost Cells

Checking the original list of $2n$ points I_i and $-M_i$ is not enough.



$n = 3$

Ghost cells are covered whenever they exist.

Characterizing Q-matrices for $\boldsymbol{n} = 3$

» Spacial Q-Covering

e_3 could be surrounded in one of the following ways:

1. $S^1(e_3) \subseteq [e_1, -M_1] \oplus [e_2, -M_2]$ (local problem in lower dimension)
2. $e_3 \in \langle a_1, a_2, -M_3 \rangle^\circ$ (interior of a simplex)
3. $e_3 \in \langle e_1, -M_3 \rangle^\circ \wedge S^1(e_3) \subseteq [e_1, -M_3] \oplus [e_2, -M_2]$ (two simplices with $-M_3$)
4. $e_3 = -M_2 \wedge S^1(e_3) \subseteq [e_1, -M_3] \oplus [e_2, -M_2]$ (e_3 coincides with $-M_2$)
5. $e_3 \in \langle e_1, -M_3 \rangle^\circ \wedge -M_2 = -M_3 \wedge S^1(e_3) \subseteq [-M_1, e_2] \oplus [-M_3, e_2]$ (a mix of simplices)

» Q-matrix with non-pointed cone

$n = 3$

No covering is possible with non-pointed complementary cones.

» Q-matrix with non-pointed cone

$n = 3$

No covering is possible with non-pointed complementary cones.

$$\begin{pmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \\ 3 & 0 & -1 \end{pmatrix} .$$

» Ongoing/Future work

- Ghost cells for dimensions ≥ 4
- Is there a better way to detect/count holes? (Homology)

Thanks for attending!

More details available here
`https://arxiv.org/abs/2203.12333`