Efficiently solving connectivity queries on real algebraic curves

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joint work with Nazrul Islam[2] Adrien Poteaux[3]

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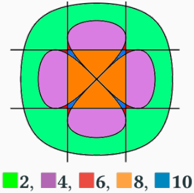
Context and problem statement

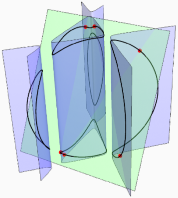
Semi-algebraic (s.a.) sets

Set of **real** solutions of systems of polynomial **equations** and **inequalities**



Physics

[Le, Safey El Din; '22]



Computational geometry

[Le, Manevich, Paulmann; '21]

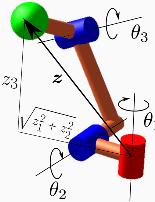
$$\mathrm{S} + \mathrm{E} \underset{\kappa_3}{\overset{\kappa_2}{\rightleftharpoons}} \mathrm{SE} \xrightarrow{\kappa_1} \mathrm{S_p} + \mathrm{E}$$

$$\mathrm{P} + \mathrm{S_p} \underset{\kappa_9}{\overset{\kappa_8}{\rightleftharpoons}} \mathrm{PS_p} \xrightarrow{\kappa_7} \mathrm{P_p} + \mathrm{S_p}$$

$$\mathrm{P} + \mathrm{E} \underset{\kappa_{12}}{\overset{\kappa_{11}}{\rightleftharpoons}} \mathrm{PE} \xrightarrow{\kappa_{10}} \mathrm{P_p} + \mathrm{E}$$

Biochemistry

[Telek, Feliu; '23]



Robotics

[Chablat, P., Safey El Din, Salunkhe, Wenger; '22]

Fundamental problems in effective semi-algebraic geometry

Given a semi-algebraic set S,

- *compute* a real root *classification* of S
- *compute* a *projection* of S: quantifier elimination
- *compute* one point in each *connected component* of S
- *decide* if two points lie in the same *connected component* of S
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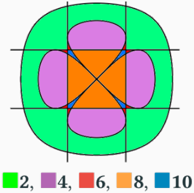

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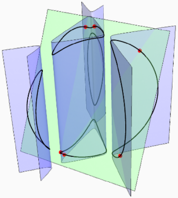

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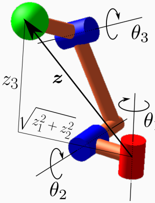

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Reduction to curves: Roadmaps

[Canny, 1988] Compute $\mathscr{R} \subset S$ one-dimensional, sharing its connectivity

Roadmap of $(S, \mathcal{P})$

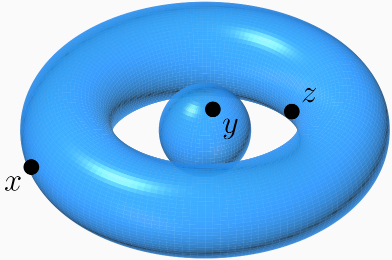
It is a *semi-algebraic curve* $\mathscr{R} \subset S$, containing $\mathcal{P}$ finite and such that for all connected components C of S: $C \cap \mathscr{R}$ is *non-empty* and *connected*

Proposition

$\boldsymbol{y}, \boldsymbol{z} \in \mathcal{P}$ are path-connected in $S \iff$ they are in $\mathscr{R}$

Problem reduction

Arbitrary dimension



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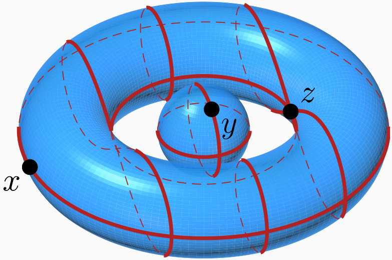

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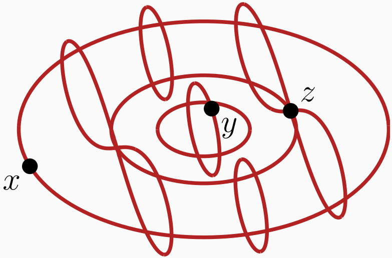
$$\text{Arbitrary dimension} \underset{\textsc{Roadmap}}{\Longrightarrow} \text{Dimension } 1$$



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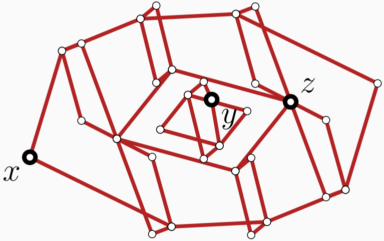
Arbitrary dimension $\underset{\textsc{Roadmap}}{\Longrightarrow}$ Dimension 1 $\underset{\text{Our problem}}{\Longrightarrow}$ Finite graph



Data representation and quantitative estimate

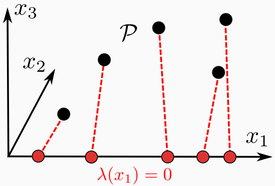
Theorem

In a *generic* system of coordinates there exist...

Zero-dimensional parametrization of $\mathcal{P} \subset \mathbb{C}^n$ finite

...polynomials $(\lambda, \vartheta_2, \ldots, \vartheta_n) \subset \mathbb{Z}[x_1]$ s.t.

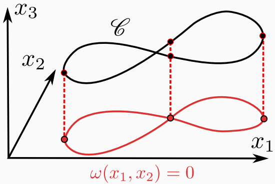
$$\mathcal{P} = \left\{ \left(\boldsymbol{x}_1, \frac{\vartheta_2(\boldsymbol{x}_1)}{\lambda'(\boldsymbol{x}_1)}, \ldots, \frac{\vartheta_n(\boldsymbol{x}_1)}{\lambda'(\boldsymbol{x}_1)} \right) \middle| \lambda(\boldsymbol{x}_1) = 0 \right\}$$



One-dimensional parametrization of $\mathscr{C} \subset \mathbb{C}^n$ algebraic curve

...polynomials $(\omega, \rho_3, \ldots, \rho_n) \subset \mathbb{Z}[x_1, x_2]$ s.t.

$$\mathscr{C} = \overline{\left\{ \begin{array}{c} \left(\boldsymbol{x}_1, \boldsymbol{x}_2, \frac{\rho_3(\boldsymbol{x}_1, \boldsymbol{x}_2)}{\partial_{x_2}\omega(\boldsymbol{x}_1, \boldsymbol{x}_2)}, \ldots, \frac{\rho_n(\boldsymbol{x}_1, \boldsymbol{x}_2)}{\partial_{x_2}\omega(\boldsymbol{x}_1, \boldsymbol{x}_2)} \right) \\ \text{s.t. } \omega(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0 \quad \text{and} \quad \partial_{x_2}\omega(\boldsymbol{x}_1, \boldsymbol{x}_2) \neq 0 \end{array} \right\}}$$



Magnitude of a polynomial

$\boldsymbol{f} \in \mathbb{Z}[x_1, \ldots, x_n]$ has *magnitude* (δ, τ) if

$\deg(\boldsymbol{f}) \leq \delta \quad$ and $\quad |\mathrm{coeffs}(\boldsymbol{f})| \leq 2^{\tau}$

Soft-O notation

$\tilde{O}(N) = O(N \log(N)^a),\ a > 0$

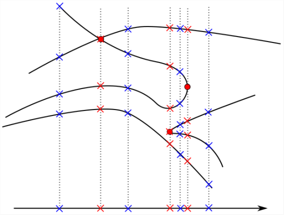
Results

Data

- $\mathscr{R} \subset \mathbb{Z}[x_1, x_2]$ of magnitude (δ, τ), encoding an algebraic curve $\mathscr{C} \subset \mathbb{C}^{n}$;
- $\mathscr{P} \subset \mathbb{Z}[x_1]$ of magnitude (δ, τ), encoding a finite $\mathcal{P} \subset \mathscr{C}$;

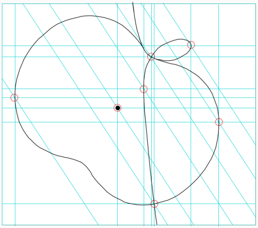
Computing topology

Ambient dimension	Bit complexity	Reference
$n = 2$	$\tilde{O}(\delta^5(\delta + \tau))$	[Kobel, Sagraloff; '15] [D.Diatta, S.Diatta, Rouiller, Roy, Sagraloff; '22]



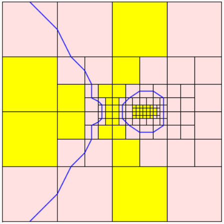
Cylindrical Algebraic Decomposition

[Collins, '75] [Kerber, Sagraloff; '12]



Multiple projections

[Seidel, Wolpert; '05]



Subdivision

[Burr, Choi, Galehouse, Yap; '05]

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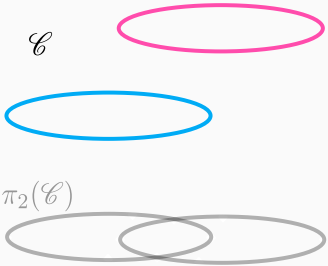

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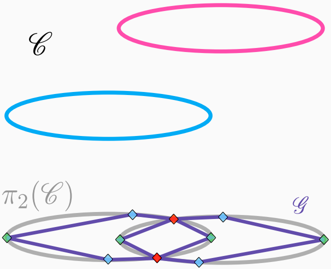

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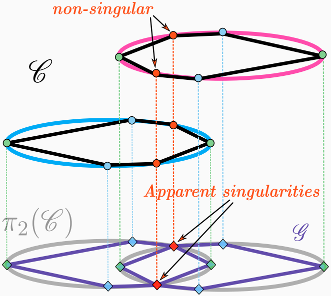

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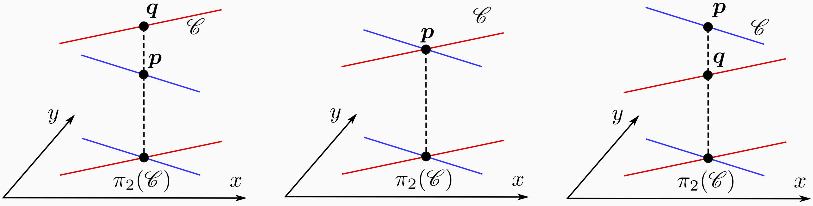
Computing connectivity - Main Result

Ambient dimension	Bit complexity	Reference
$n \geq 2$	$\tilde{O}(\delta^5(\delta + \tau))$ NEW!	[Islam, Poteaux, P.; 2023]

Avoid computation of the complete topology!

Apparent singularities

Projection induces new singularities: the case of nodes



Below Same Above

Apparent singularities: key idea

Apparent singularities

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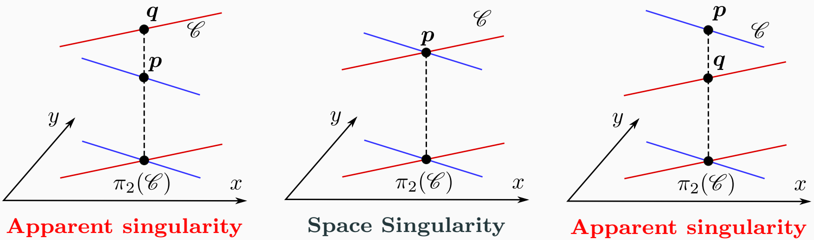

Key idea

Local connectivity does not depend of the relative position

Only two cases to consider!

Genericity assumptions

Data

$\mathscr{C} \subset \mathbb{C}^n$ algebraic curve

$\pi_3 : \mathbb{C}^n \to \mathbb{C}^3$ projection on a *generic 3D space*
$\pi_2 : \mathbb{C}^n \to \mathbb{C}^2$ projection on a *generic plane*

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(H_1) $\pi_2 : \mathscr{C} \to \pi_2(\mathscr{C})$ is **birational**

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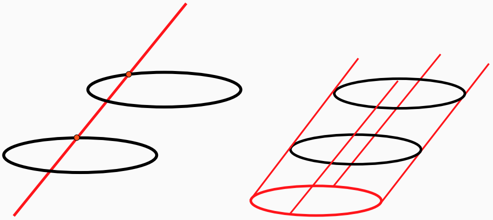
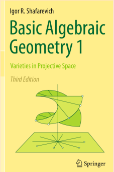


Secants are exceptional lines



[Shafarevich, '13]

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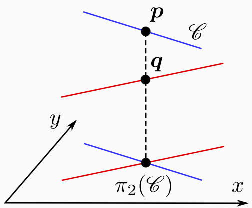
(H_4) Overlaps introduce only **nodes**



Nodal apparent singularity

Genericity assumptions

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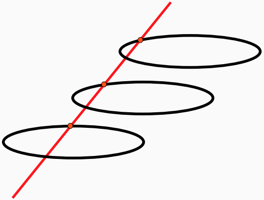

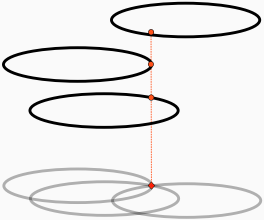

TriSecants are exceptional secants

Proof: Trisecant lemma for singular projective curves


Journal of Mathematical Sciences, Vol. 149, No. 2, 2008

TRISECANT LEMMA FOR NONEQUIDIMENSIONAL VARIETIES

J. Y. Kaminski, A. Kanel-Belov, and M. Teicher UDC 512.7

[Kaminski Kanel-Belov Teicher; '08]

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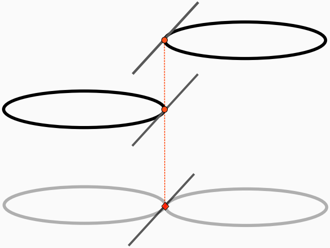
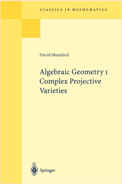


Secants with coplanar tangents are exceptional secants

Proof: Generalize results from literature



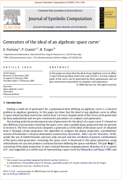
CLASSICS IN MATHEMATICS

David Mumford

Algebraic Geometry I
Complex Projective Varieties

Springer

[Mumford; '76]



Journal of Symbolic Computation

Generators of the ideal of an algebraic space curve

E. Fortuna, P. Gianni, B. Trager

[Fortuna, Gianni Trager; '09]

Algorithm

Input

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- $\mathscr{P} \subset \mathbb{Z}[x_1]$ of magnitude (δ, τ), encoding a finite $\mathcal{P} \subset \mathscr{C}$;
- $\mathscr{C}$ satisfies genericity assumptions w.r.t. $\mathcal{P}$

Output

A partition of $\mathcal{P} \cap \mathbb{R}^n$ w.r.t. the s.a. connected components of $\mathscr{C} \cap \mathbb{R}^n$.

1. $\mathscr{D}, \mathscr{Q} \leftarrow \mathsf{Proj2D}(\mathscr{R}), \mathsf{Proj2D}(\mathscr{P})$;
2. $\mathscr{G} \leftarrow \mathsf{Topo2D}(\mathscr{D}, \mathscr{Q})$;
3. $\mathscr{Q}_{\mathrm{app}} \leftarrow \mathsf{ApparentSingularities}(\mathscr{R})$
4. $\mathscr{G}' \leftarrow \mathsf{NodeResolution}(\mathscr{G}, \mathscr{Q}_{\mathrm{app}})$;
5. `return` $\mathsf{ConnectGraph}(\mathscr{Q}, \mathscr{G}')$;

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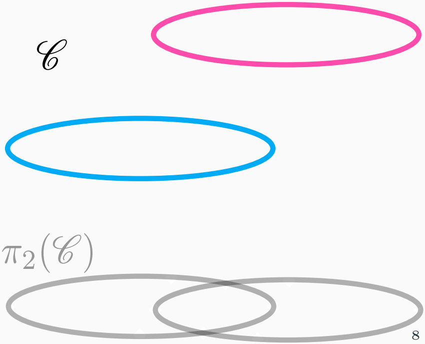

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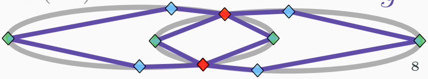
$\mathscr{C}$

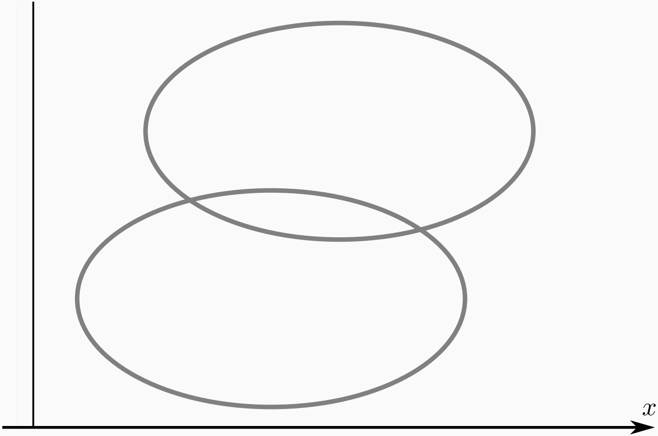
Planar topology computation

Bit complexity: $\tilde{O}(\delta^5(\delta + \tau))$

$\pi_2(\mathscr{C})$ $\mathscr{G}$



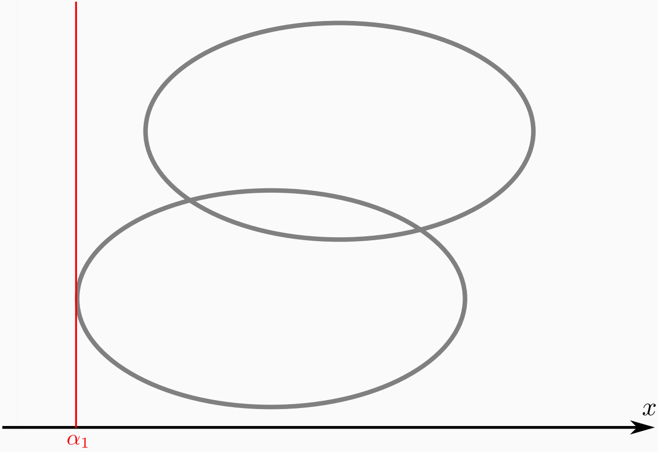
Computing the topology of plane curves

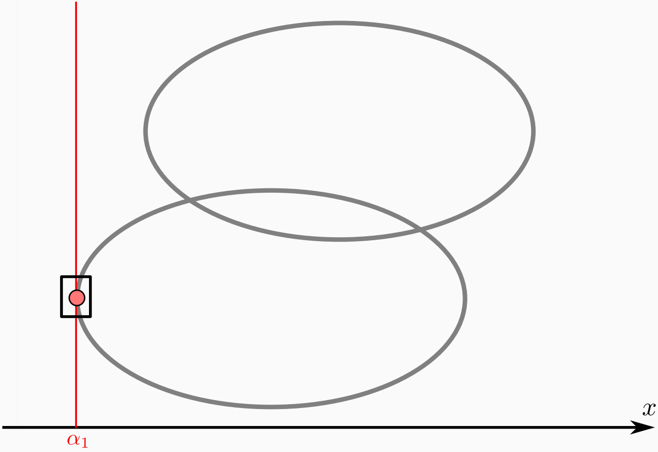


Cylindrical algebraic decomposition

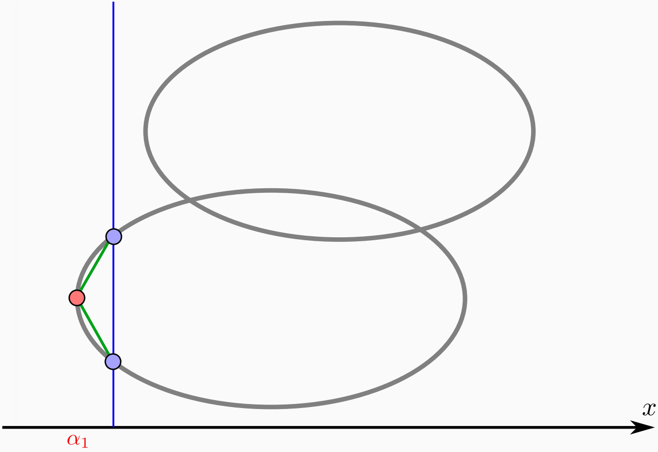
Method introduced by G. Collins in 1975

Computing the topology of plane curves




x
α_1

Computing the topology of plane curves



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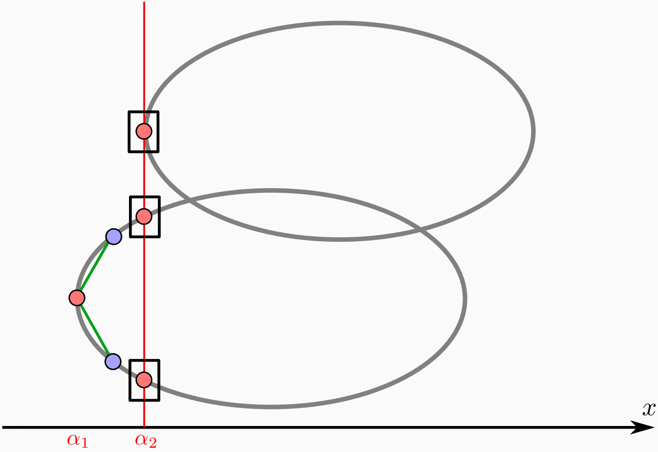

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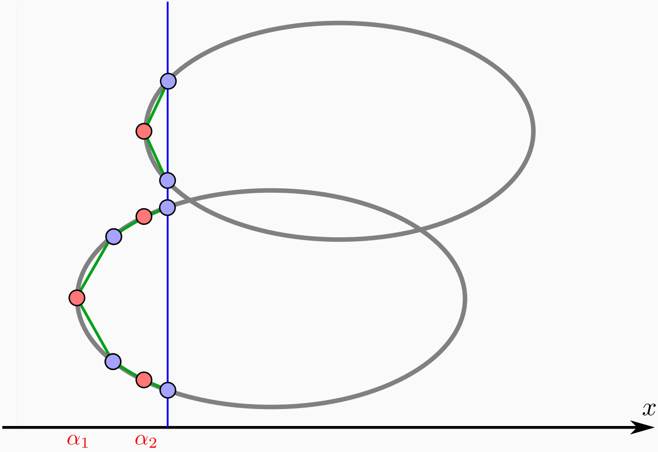

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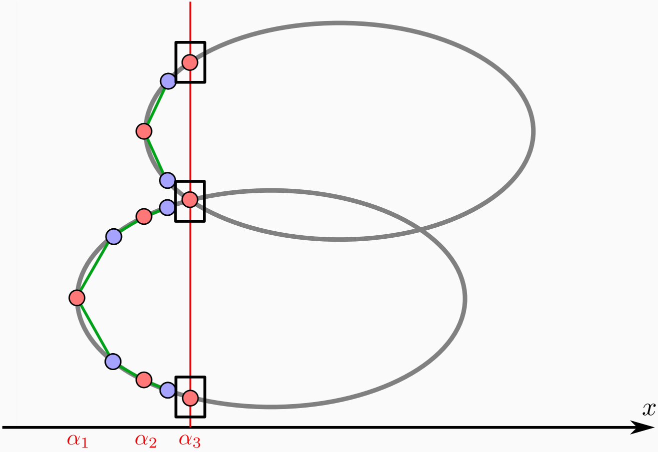

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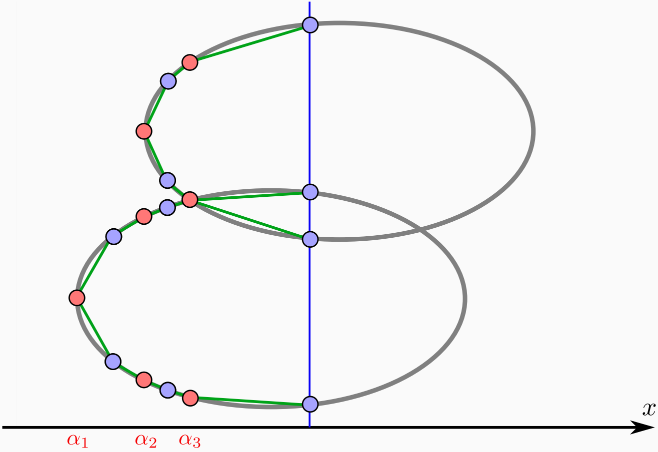

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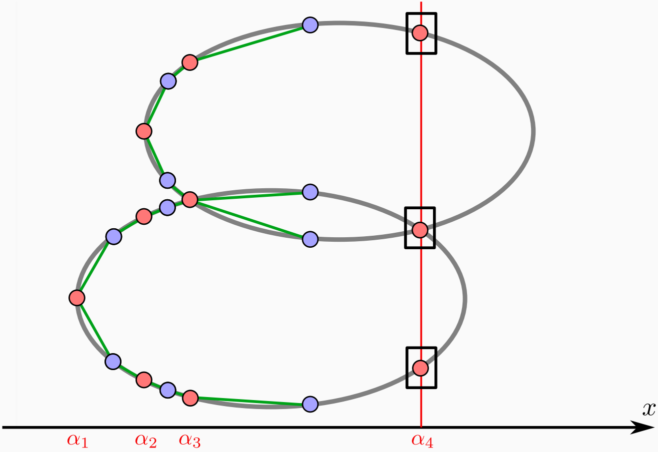
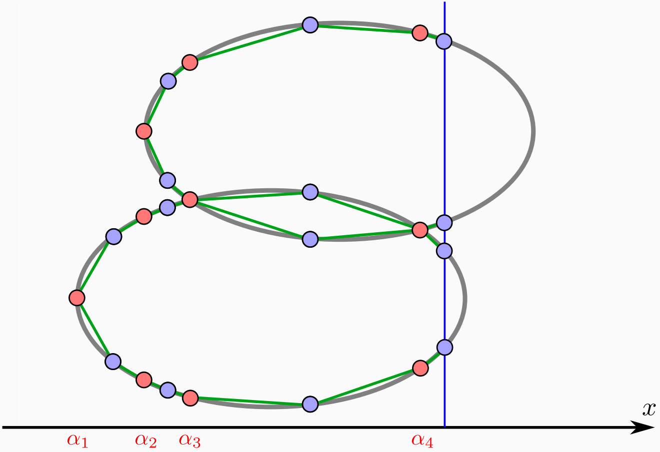
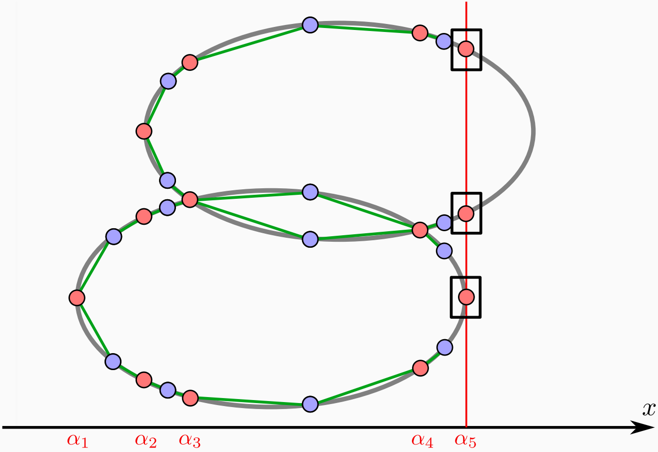
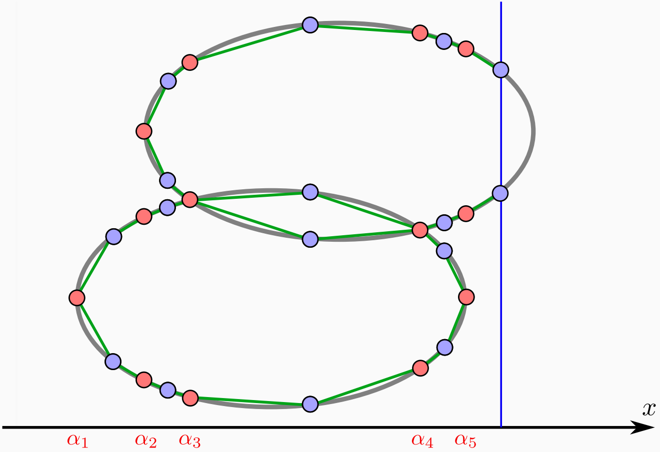
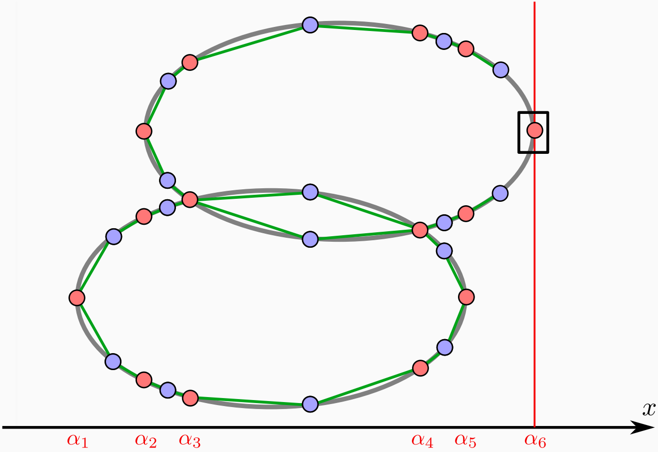



x
α_1
α_2
α_3
α_4

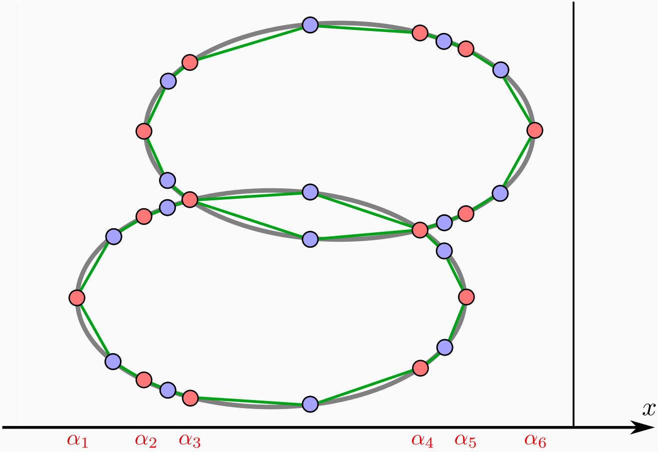

x
α_1
α_2
α_3
α_4
α_5

Computing the topology of plane curves




x
α_1
α_2
α_3
α_4
α_5
α_6

Computing the topology of plane curves



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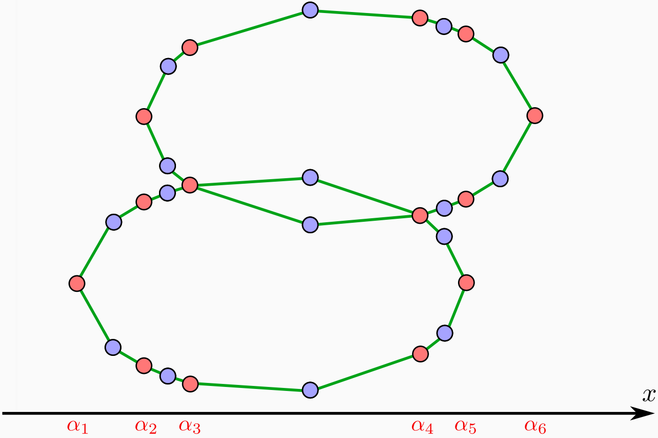

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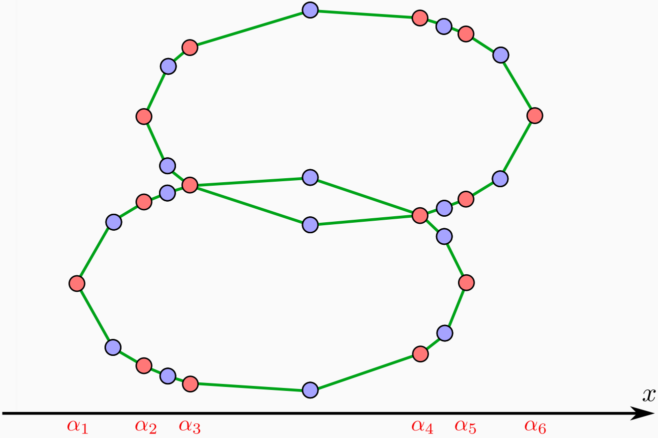

Complexity bottleneck→ Isolating critical boxes

Isolation of some univariate polynomials with algebraic coefficients

$\boxed{\text{Complexity: } \tilde{O}(\delta^5(\delta + \tau))}$ [Kobel, Sagraloff; '15] [D.Diatta, S.Diatta, Rouiller, Roy, Sagraloff; '22]

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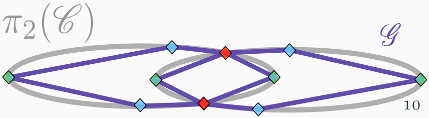

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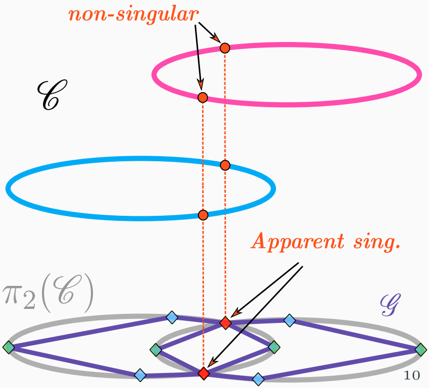
Identification of apparent sing.

Bit complexity: $\tilde{O}(\delta^5(\delta + \tau))$

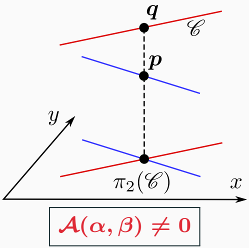


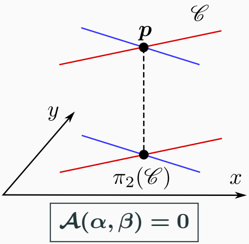
Witness apparent singularities

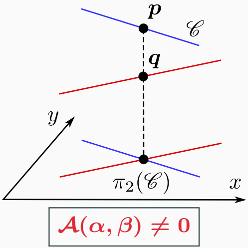
- $\mathscr{R} = (\boldsymbol{\omega}, \boldsymbol{\rho_3}, \ldots, \rho_n) \subset \mathbb{Z}[x, y]$ encoding $\mathscr{C} \subset \mathbb{C}^n$ in generic position;
- $\mathcal{A}(x, y) = \partial^2_{x_2}\boldsymbol{\omega} \cdot \partial_{x_1}\boldsymbol{\rho_3} - \partial^2_{x_1 x_2}\boldsymbol{\omega} \cdot \partial_{x_2}\boldsymbol{\rho_3} \in \mathbb{Z}[x, y]$

Proposition - Generalization of [El Kahoui; '08]

A **node** (α, β) is an **apparent singularity** if and only if $\mathcal{A}(\alpha, \beta) \neq 0$







Computational aspect

1. Non-vanishing can be tested using **gcd computations**
2. Gcd computations can be done modulo a prime number

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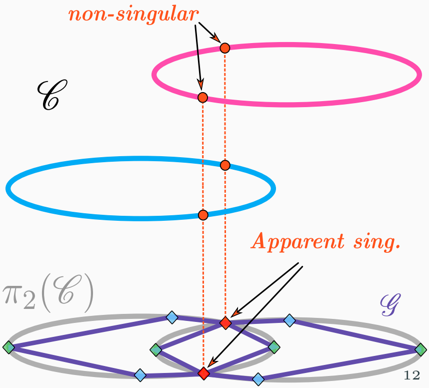

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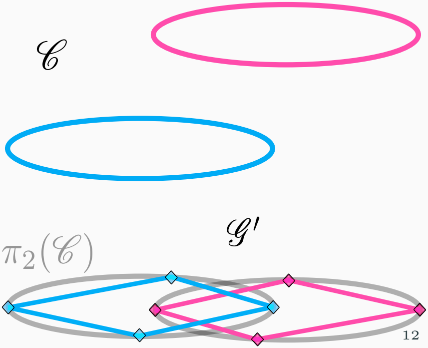

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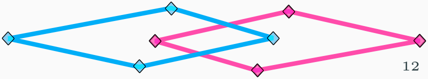
$\mathscr{C}$

$\mathscr{G}'$



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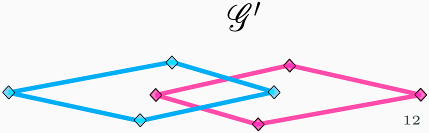
Overall Complexity

$\tilde{O}(\delta^5(\delta + \tau))$

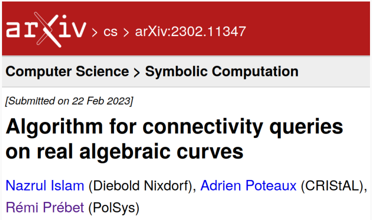


Conclusion

Summary

Connectivity for roughly the *same price* than **planar topology**:

- ✓ Assumptions *generically* holds
- ✓ Identify *apparent singularity* with modular GCD computations
- ✓ Avoid the *costly lifting* step from plane topology


arXiv > cs > arXiv:2302.11347

Computer Science > Symbolic Computation

[Submitted on 22 Feb 2023]

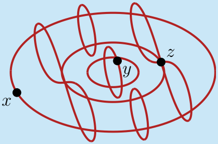
Algorithm for connectivity queries on real algebraic curves

Nazrul Islam (Diebold Nixdorf), Adrien Poteaux (CRIStAL), Rémi Prébet (PolSys)

Future work

Extension to this work:

- Adapt to algebraic curves given as *union*
- Generalize to *semi-algebraic* curves
- Investigate the connectivity of plane curves



Practical aspects:

- Develop an optimized implementation
- Solve challenging applications together with roadmap algorithms

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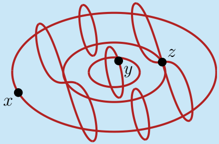

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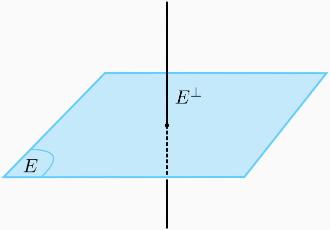
Thank you for your attention!

Generic projection: avoid bad directions

Generic affine subspace

E : of dimension $k \geq 1$
$E^{\perp}$: orthogonal of E (codimension k)



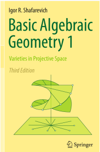
Remark

Projection: on $E \Leftrightarrow$ in the direction of $E^{\perp}$

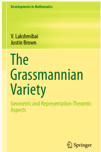
Generic intersection of with varieties

Let D be an algebraic set of dimension $\leq l$ then,

$$E \cap D \text{ is } \begin{cases} \text{finite} & \text{if } l = k \\ \text{empty} & \text{if } l < k \end{cases}$$



[Shafarevitch]



[Lakshmibai et al.]

For plane and space projection

Dim. of bad directions	Bad directions for π_2	Bad directions for π_3
≤ 1	$\emptyset$	$\emptyset$
≤ 2	$< \infty$	$\emptyset$