JNCF2023 – Computing periods of hypersurfaces

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Summary of presentation

The period matrix of a smooth projective algebraic variety X is a matrix that describes the isomorphism between the homology and cohomology of X, as given by the DeRham theorem. The coefficients of this matrix play a fundamental role in the study of the geometry of X: Torelli-type theorems establish that in certain cases, the isomorphism class of the variety is entirely characterized by these numbers. Furthermore, in theoretical physics, periods appear as scattering amplitudes of particle interactions.

In this presentation, we will introduce an algorithm to compute numerical approximations of these periods.

For complex projective hypersurfaces, these numbers can be expressed as integrals of rational fractions over some integration domain without boundary. In 2019, Sertöz gave a method for computing the periods of any smooth hypersurface by introducing a parameter in the rational fraction to study the deformation of the periods with respect to this parameter. This deformation is described by a differential equation known as the Picard-Fuchs equation of the period. Using this tool, one may recover the periods by deforming them from a variety for which the periods have known analytic formulae.

However, this method requires numerically integrating high order and degree differential equations, which can be costly, and it is not easily generalizable to broader types of varieties, as it relies on knowledge of the periods of a specific variety of the same type.

The new method we will present tackles both these drawbacks by computing a description of the homology of the variety that is well-suited to performing numerical integration of differential operators. The main ingredient for this computation is Picard-Lefschetz theory.

This new method allows computing the periods of generic cubic surfaces in a few minutes, and the holomorphic periods of generic quartic surfaces in an hour, a task that the previous method could not manage in reasonable time. Moreover, it enables the computation of the periods of (the desingularization of) certain singular hypersurfaces.

Details

We will start by illustrating the method with the example of an elliptic curve X, which is the zero set of a homogeneous polynomial of degree 3 in 3 variables. Note that in the case of elliptic curves (or even algebraic curves in general, see Deconinck-Van Hoeij, 2001) this method is nothing new, but it serves as a good introductory example.

By parameterizing one of the variables in terms of the others, we may turn this variety into a threefold cover of the projective line $X \to \mathbb{P}^1$, with ramification at the critical points of the fibration. If the choice of parametrization is generic enough, there will be 6 critical points, and the behavior at these points is well understood. The fiber above some $t \in \mathbb{C}$ is a collection of three points which move as t moves in the complex plane. When t loops around one of the critical points, two points of the fiber get permuted.

Using this observation, we wish to find a closed path to integrate on in X. One way to achieve this is to find a loop in $\mathbb{C}$, that avoids the critical points and lifts to a loop in X. For this, we need to make sure we land back to the point we started from when following the loop.

Here is an algorithmic way of finding all such paths: fix a regular base point $b \in \mathbb{C}$, and look only at loops pointed at this point. Such a loop is always a combination of the six loops $\ell_1, \dots, \ell_6$ around the critical points. If for each critical point we compute the permutation of the fiber $X_b = f^{-1}(b)$ induced by deformation along ℓ_i, we get building blocks for finding paths that lift. Indeed, this yield a path in X of which the boundary is a difference of points of X_b. Such paths, called thimbles, serve as building blocks for the total homology space. All that remains is then to find linear combination of thimbles with empty boundary, which is a simple linear algebra problem. In fact, Picard-Lefschetz theory gives a more precise description of the thimbles: for each critical point, there is (up to a multiple) precisely one thimble that is non trivial (i.e. that does not contract to a point), which is the thimble which permutes the two points of the fiber.

In the case of elliptic curves, we have 6 thimbles, of which we know the boundary (which is a difference of 2 points, so the space of boundaries has rank 2). Thus canceling boundaries, we obtain 4 closed paths in X. It turns out that 2 of these paths are actually contractible at infinity (the elliptic curve is not defined over $\mathbb{C}$ but over the entire projective line), and we need to remove them. In the end we recover the 2 cycles that we expect for elliptic curves. Computing the periods then amounts to computing a path integral, which can be done thanks to numerical analytic continuation (see Marc Mezzarobba 2016).

One main advantage of this method is that it can be generalized to higher dimensions, which I will showcase in dimension 2. One may turn a (hyper)surface Y in $\mathbb{P}^3$ into a Lefschetz fibration, i.e. give a map $f : Y \to \mathbb{P}^1$ which has very similar properties to the previous map. The main differences are that the fiber is no longer a collection of points, but a curve. What's more, we are no longer looking for closed paths, but for closed surfaces in Y. The similarities are that the fiber still deforms continuously, there are still finitely many critical points, and the way the fiber deforms when looping around these critical points is well understood.

Our goal is now to find the image of a deformation of a closed path in the fiber along a loop in the complex plane, which has no boundary. It turns out that when the fiber loops around a critical point, there are two such paths that permute. The deformation of one of the permuting cycles along one of the loops pointed at b gives rise to a thimble (which is now a surface, of which the boundary is a path in Y_b), and we may apply the same machinery as with the elliptic curve. We glue linear combinations of thimble with zero boundary, and remove some extra cycles, and we then recover the closed surface we want to integrate on. Then the actual integration can be done in two steps: first computing the period of the fiber (which is just a curve, see the elliptic curve case above), and then computing the integral of that period along the deformation path, which is again a line integral.

The main ingredient for this method to work is to be able to compute the monodromy matrices. This can be done using the same deformation methods used in the aforementioned algorithm by Sertöz, except we are no longer applying it directly to the variety, but to one of its fiber, which has dimension one less. Thus the operators we integrate have lower order, and are thus integrated faster.