Toric Sylvester forms: a nice tool for elimination.

Carles Checa, National and Kapodistrian University of Athens, Athena RC

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Abstract

Sylvester forms give a basis for the module $(I^{\mathrm{sat}}/I)_\alpha$ when it is free, for I being an ideal generated by $n+1$ forms in dimension n, I^{sat} is the saturation with respect to the irrelevant ideal and $\alpha \in \mathrm{Pic}(X_\Sigma)$. This notion was introduced by Jouanolou in [3] and extended first to the multiprojective setting by Buse-Chardin-Nemati [1] and recently to a family of smooth toric varieties by Buse and the speaker in [2]. The goal of this talk will be to present very briefly Sylvester forms and show many contexts in which they can be useful in elimination theory.

Let k be an algebraically closed field. Let M be a lattice and $F_0, \ldots, F_n$ be a sparse polynomial system of the form:

$$F_i = \sum_{a \in \mathcal{A}_i} c_{i,\mu} x^\mu \in C = A \otimes R \quad \mathcal{A}_i \subset M. \quad A = k[u_{i,a}]$$

where $I = (F_0, \ldots, F_n)$ and R is the Cox ring of the toric variety X_Σ associated with the normal fan Σ of the polytope $\Delta = \sum_{i=0}^n \mathrm{conv}(\mathcal{A}_i)$ or to some classes $\alpha_i \in \mathrm{Pic}(X_\Sigma)$.

Definition 0.1. Let $\rho : A \to k$ be a specialization map and let $f_i = \rho(F_i)$ for $i = 0, \ldots, n$ a polynomial system in . An elimination matrix $\mathcal{M}$ is a matrix with entries in A such that:

- The rank of $\mathcal{M}(f)$ drops, if and only if, $f_0 = \ldots f_n = 0$ has a solution.
- If the number of solutions of $f_0 = \ldots f_n = 0$ is finite and equal to κ, the corank of $\mathcal{M}(f)$ equals κ.

In our work, we introduced the notion of toric Sylvester forms sylv_μ in order to provide for a basis of $(I^{\mathrm{sat}}/I)_{\delta-\nu}$ for $\delta = \sum \alpha_i - K_X$ with K_X the anticanonical divisor and x^μ a basis of C_ν for some $\nu \in \mathrm{Pic}(X_\Sigma)$. In such cases, elimination matrices arise as the matrices asocciated with the maps

$$\big(\oplus_i C(-\alpha_i) \oplus I^{\mathrm{sat}}/I\big)_\alpha \xrightarrow{(F_0,\ldots,F_n,\mathrm{sylv}_\mu)} C_\alpha.$$

The goal of this talk will be to explore many ways in which these matrices can be useful in elimination theory.

References

[1] Laurent Busé, Marc Chardin, and Navid Nemati. *Multigraded Sylvester forms, Duality and Elimination Matrices.* 2021. arXiv: 2104.08941 [math.AC].

[2] Laurent Busé and Carles Checa. *Toric Sylvester forms and applications in elimination theory.* 2022. DOI: 10.48550/ARXIV.2209.11281. URL: https://arxiv.org/abs/2209.11281.

[3] J. Jouanolou. "Formes d'inertie et résultant: un formulaire". In: *Advances in Mathematics* 126 (1997), pp. 119–250.