Et si SAT était vraiment difficile? Quelques conséquences des hypothèses ETH et SETH

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But interesting (at least to me) and

somehow fundamental questions

Classical dichotomy

Classical dichotomy

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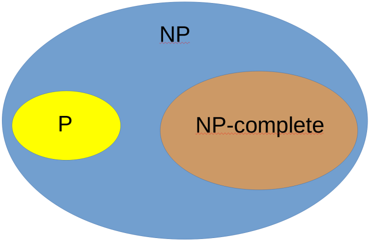

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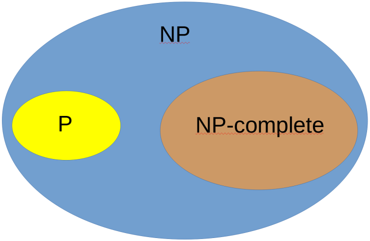

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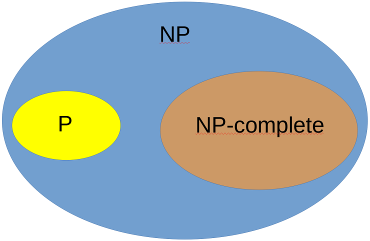

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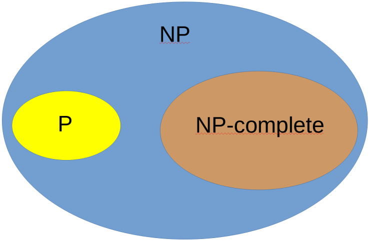

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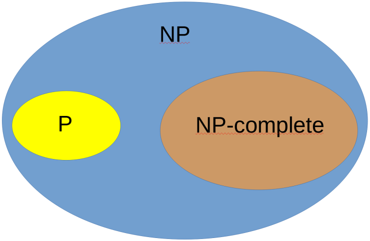

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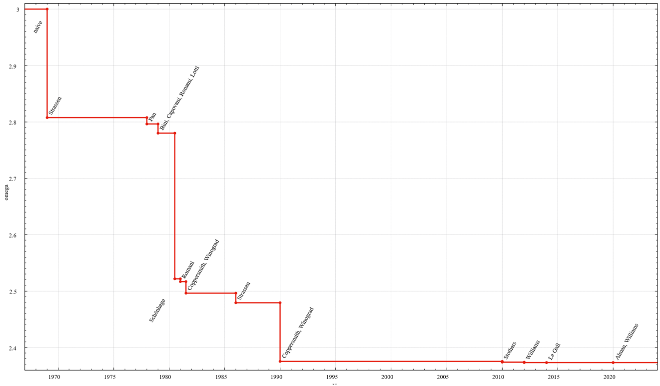
Matrix multiplication: from $O(n^3)$ to $O(n^{2.3728596})$



(source wikipedia)

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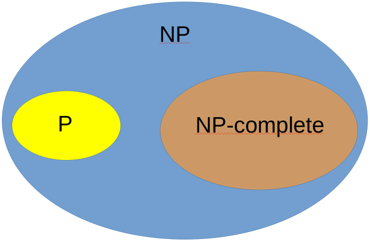

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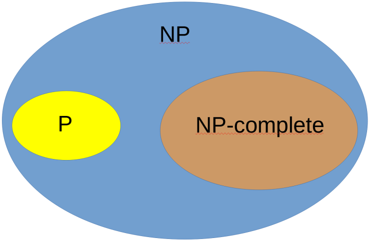

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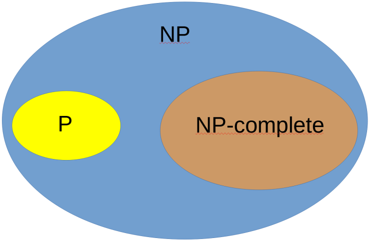

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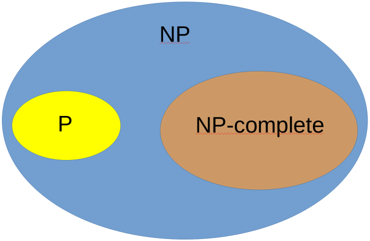

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  $\rightarrow$ Can't we do better?

Find lower bounds, using a stronger hypothesis …
…on Sat! ($P \neq NP \Leftrightarrow \text{Sat} \notin P$)

Sat, k-Sat, ETH and SETH

Sat

- t boolean variables $(x_1, \dots, x_t)$
- m clauses $C_1, \dots, C_m$ $(C_1 = (x_2 \vee \overline{x_4} \vee x_5), \dots)$
- Is there a truth value which satisfies all clauses?

k-Sat: every clause has (exactly/at most) k literals.

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$\rightarrow$ Solvable in $2^t poly(t, m) = O^*(2^t)$.
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3-Sat

$$C = (x_1 \vee x_2 \vee \overline{x_3}) \quad \rightarrow \quad \text{only 7 possibilities}$$

$$\text{(all but } x_1 = x_2 = F,\ x_3 = T)$$

$$\rightarrow \quad \text{test all of them}$$

$$\rightarrow \quad T(t) = 7T(t-3)$$

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3-Sat solvable in $O^*(c_3^t)$, with $c_3 = 7^{1/3} = 1.9... < 2$.

k-Sat

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$\rightarrow$ Can we do better? yes and no ...
"Yes" for-k Sat.
Significantly better? Subexponential (in t) time?
And for Sat?

Sat, k-Sat, ETH and SETH

Subexponential time? Seems very hard to get, even for 3-Sat $\rightarrow$ ETH.

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Definition

Let $\mu_k = \inf\{c \geq 0 : k\text{-Sat solvable in } O^*(2^{ct})\}$.

$\mu_k > 0 \rightarrow$ exponential time is needed.

ETH - Exponential Time Hypothesis (Impagliazzo, Paturi, Ramamohan (1999))

$\mu_3 > 0$.

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$\mu_3 > 0$.

And for Sat? No $O^*(2^{ct})$ algorithm with $c < 1$ is known!

SETH - Strong Exponential Time Hypothesis

$\mu_k \rightarrow_{k\rightarrow\infty} 1$.

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Lower bounds for hard problems: subexponential time

ETH $\rightarrow$ 3-Sat non solvable in subexponential time (wrt t=# variables).

Question

Can we show exponential lower bounds under ETH?

Independent set

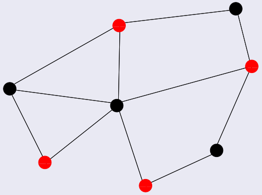

Figure: Indep. set: set of pairwise non adjacent vertices

- Input: (G, k)
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Lower bounds for hard problems: subexponential time

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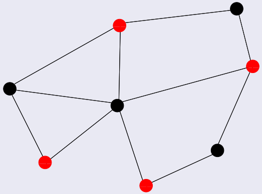

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- Input: (G, k)
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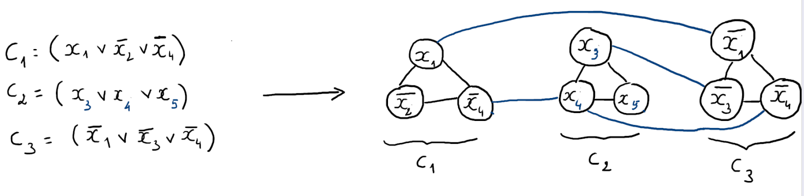
- Solvable in $O^*(2^n)$ ($n = \#$ vertices)
  $\rightarrow$ Not in subexponential time, under ETH?

Reduction 3-Sat ≤ Independent Set



I satisfiable iff $\alpha(G(I)) \geq m$

$\rightarrow$ lower bound $2^{\epsilon n}$ for Independent Set (under ETH)?

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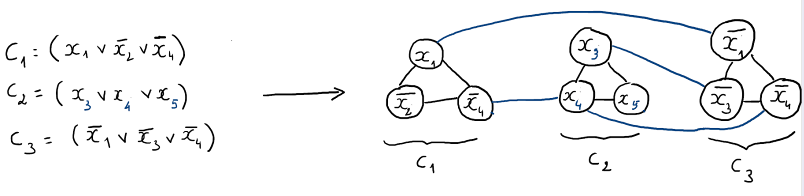

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No! (well, not yet)

$G(I)$ has $n = 3m$ vertices. $2^{o(n)}$ for IS does *not* give a $2^{o(t)}$ for 3-Sat (contradicting ETH), but a $2^{o(m)}$ (NOT contradicting ETH (yet)).

Lower bounds for hard problems: subexponential time

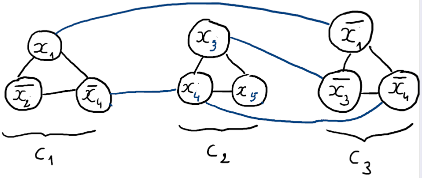
Reduction 3-Sat $\leq$ Independent Set

$C_1 : (x_1 \vee \bar{x}_2 \vee \bar{x}_4)$

$C_2 : (x_3 \vee x_4 \vee x_5)$

$C_3 : (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4)$

$\longrightarrow$



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We need a reduction where $n = O(t)$ ... or to work with 3-Sat instances with $m = O(t)$ clauses.

Sparsification lemma (Impagliazzo et al. (2001))

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$$\text{3-Sat} \rightarrow \text{3-Sat}(B) \rightarrow \text{Independent Set}$$

Hardness of Independent Set

Under ETH, there exists $\epsilon > 0$ such that Independent is not solvable in $2^{\epsilon n}$ (with $n = \#$ vertices).

And the same for many other problems (3-colorability, Hamiltonian path,...)

Lower bounds for hard problems: subexponential time

Question

Can we show exponential lower bounds under ETH?

Answer

Yes we can

. . . well, this was expected, but it was not that direct

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Well...

...but at least you should compete for the Godel prize if you disprove it!

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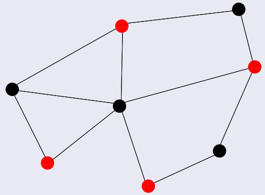

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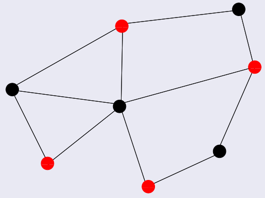

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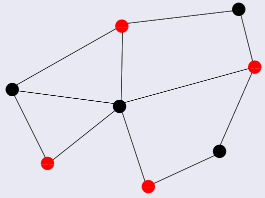

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Lower bounds for hard problems: paramaterized complexity

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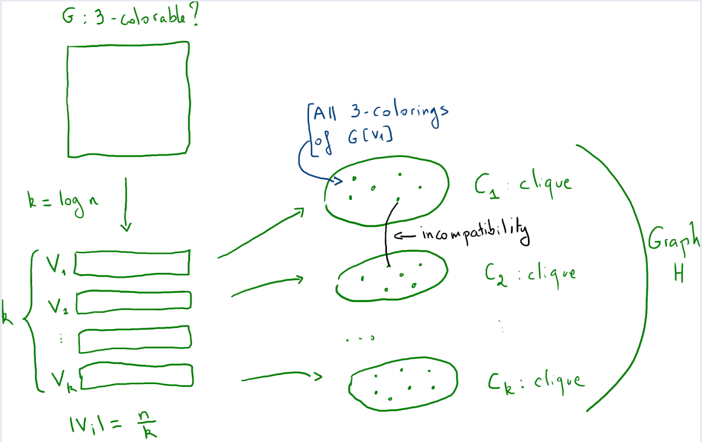
If I.S. solvable in $O(n^{o(k)})$ then 3-coloring solvable in $O(2^{o(n)})$.



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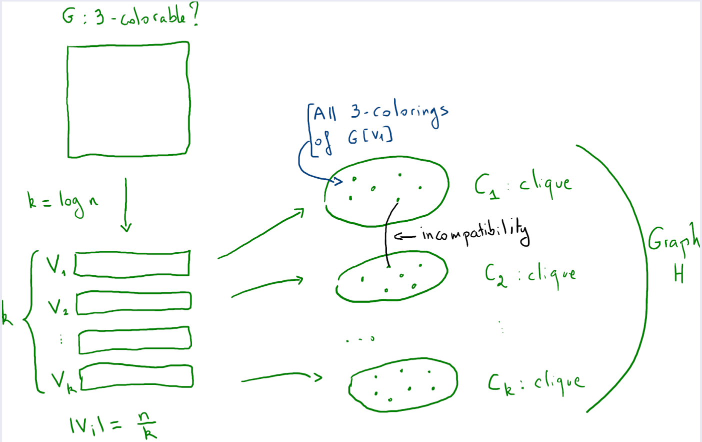

G 3-colorable iff $\alpha(H) = k$.

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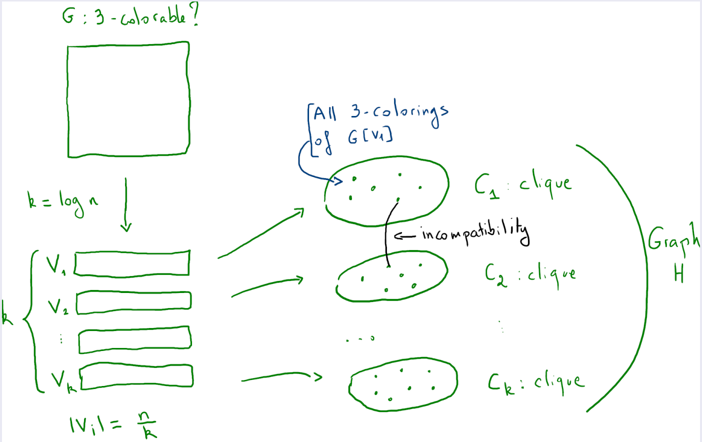

G 3-colorable iff $\alpha(H) = k$.

- $|C_i| \leq 3^{n/k} \rightarrow H$ has $N \leq k3^{n/k}$ vertices.
- $\alpha(H) = k$? Time $N^{o(k)} \leq k^{o(k)} 3^{n.o(k)/k} = 2^{o(n)}$ $(k = \log n)$.

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The same occurs for other problems (e.g., dominating set).

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$\rightarrow$ Remark: use of non-polytime reduction.

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  $\rightarrow$ No under ETH
  In time n^{ck} for some $c < 1$?
  $\rightarrow$ Well, doable for IS ... but not for other problems under SETH (even no $n^{k-\epsilon}$).

We can

$\rightarrow$ Find lower bounds beyond polytime:

- under ETH (no $2^{o(n)}$, no $n^{o(k)}$, ...),
- under SETH, sharp bounds,

both in classical and parameterized complexity frameworks.

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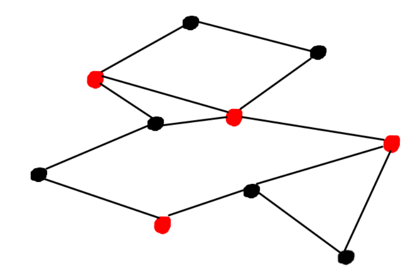

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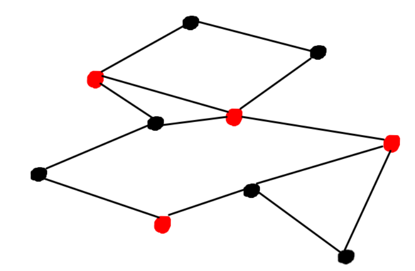

Dominating Set: S such that every vertex not in S has a neighbor in S.

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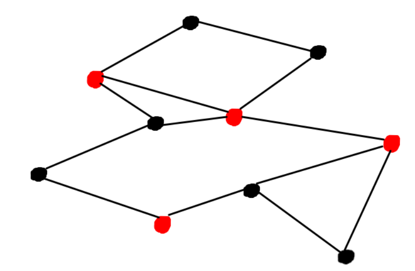

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- k-DS: does G has a D.S. of size k?

Enumerating all subsets of size $k \rightarrow n^k$.
Can I avoid this? Can I solve 3-DS in $n^{3-\epsilon}$? k-DS in $n^{k-\epsilon}$ for some/any k?

Lower bounds for polynomial problems

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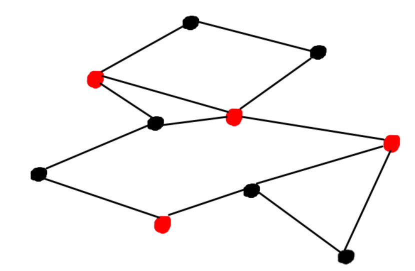

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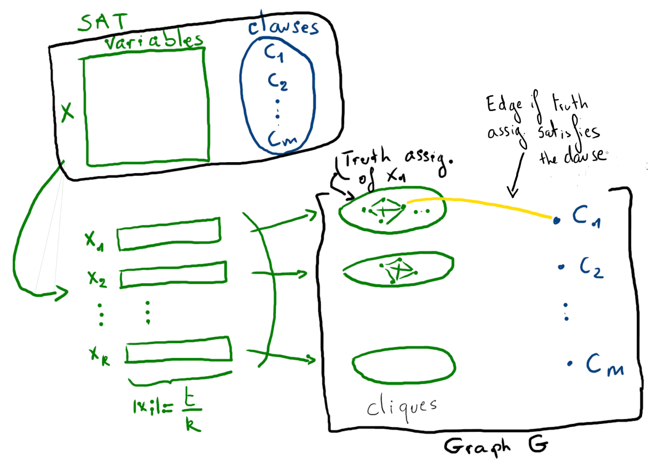
No, under SETH! No $n^{3-\epsilon}$ for 3-DS; $\forall k$, no $n^{k-\epsilon}$ for k-DS!!

$\forall k \geq 3, \epsilon > 0$: if k-DS is solvable in $O(n^{k-\epsilon})$ then SETH is false.



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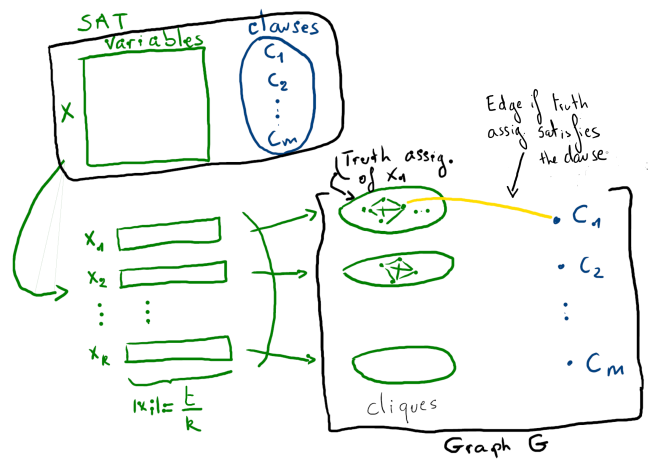

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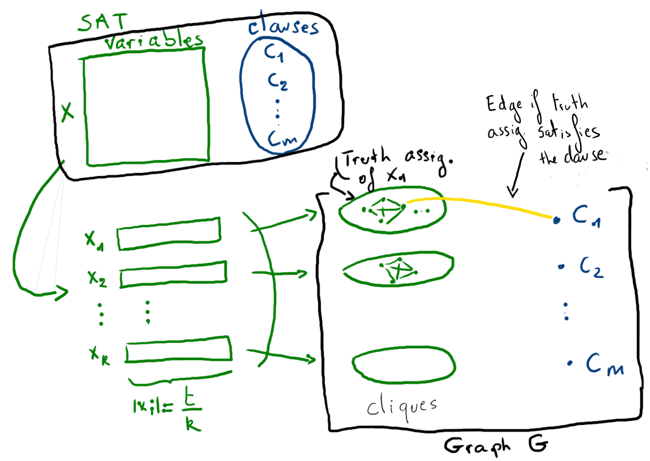

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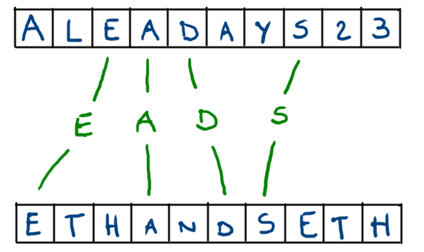
G has $n \leq k2^{t/k} + m$ vertices $\rightarrow$ G has a D.S. of size k?

Time $n^{k-\epsilon} \leq 2^{t(1-\epsilon/k)} poly(m,t)$ $\rightarrow$ SETH is false.

Lower bound for DS, also for other classical problems.

LCS (longest common subsequence)



Solvable in $O(n^2)$ ($n = |U| = |W|$) using DP

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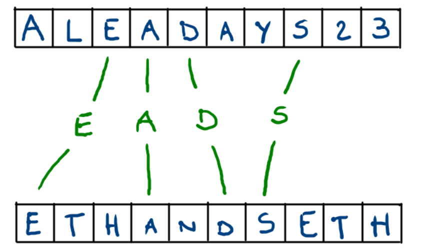

Solvable in $O(n^2)$ ($n = |U| = |W|$) using DP

Theorem ((Abboud et al. 2015))

Under SETH, $\forall \epsilon > 0$, LCS is not solvable in $O(n^{2-\epsilon})$.

We can

$\rightarrow$ Find lower bounds for polytime problems, under SETH: *fine-grained complexity*.

1. Introduction, ETH and SETH
2. Lower bounds for NP-hard problems
   1. Subexponential time
   2. Parameterized complexity
3. Lower bounds for polynomial problems
4. Concluding remarks

Concluding remarks

Other topics:

- Lower bounds for other problems
- Lower bounds for approximation algorithms
- Randomized ETH
- ...

Lower bounds for other problems?

Back to independent set

Theorem

A graph has either an independent set of size $\lfloor \log_2(n)/2 \rfloor$, or a clique of size $\lfloor \log_2(n)/2 \rfloor$.

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Theorem

It is not in P, and even not solvable in $n^{o(\log n)}$, under ETH!

Lower bounds for other problems?

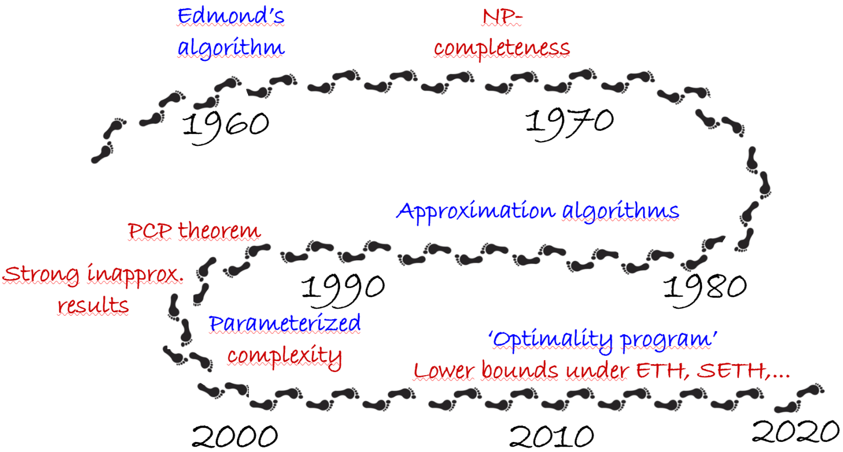
We can

$\rightarrow$ Get hardness results for problems "hard but not NP-complete", under ETH.

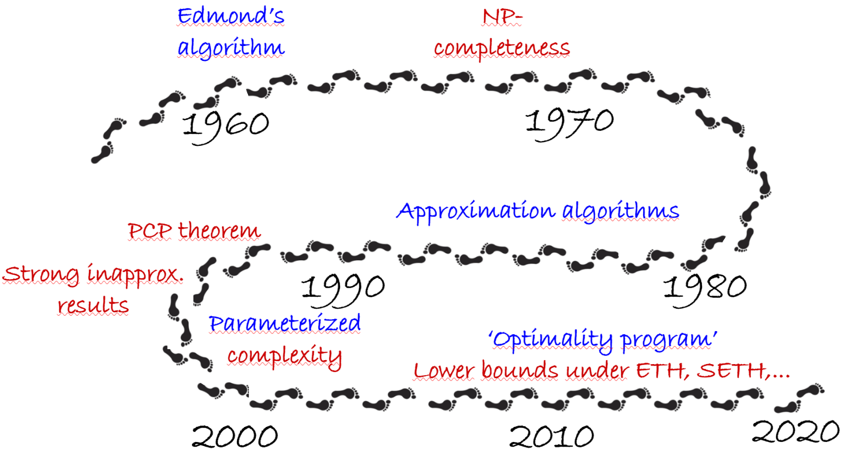
Concluding remarks

- Lower bounds for other problems
- Lower bounds for approximation algorithms
- Randomized ETH
- ...

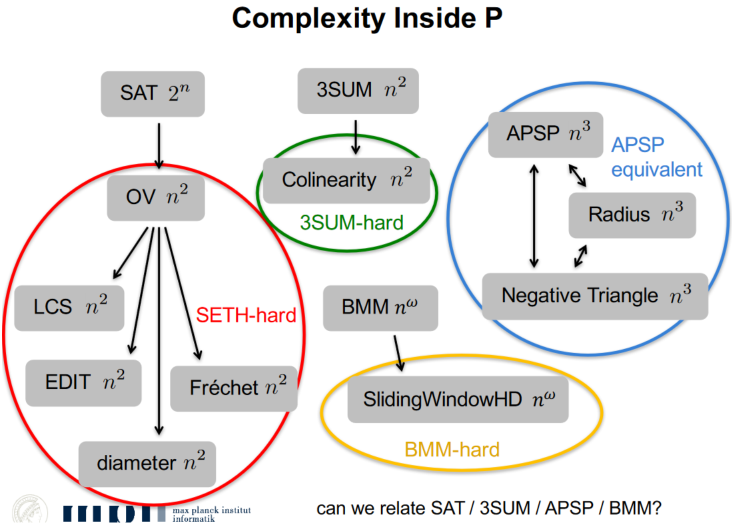
Concluding remarks



Concluding remarks



Merci de votre attention!



From a lecture of Karl Bringmann, `https://www.cs.sbg.ac.at/~forster/courses/polycomp/slides/polycomp11.pdf`.

Some tight results under SETH

Under SETH

- I.S. is not solvable in $(2-\epsilon)^{tw} n^c$ with tw=treewidth (Lokshtanov, Marx, and Saurabh 2010). For D.S.: no $(3-\epsilon)^{tw} n^c$.
- Many tight bounds for other parameters (pathwidth, cliquewidth,...) in parameterized complexity.
- No $(2-\epsilon)^n$ algorithm for hitting set.
- Diameter of a graph, under SETH: no $m^{2-\epsilon}$ (exact) algorithm (Roditty and Williams 2013), no $(2-\epsilon)$-approximation in $m^{1+o(1)}$ (even in sparse graphs), (Li'21, Dalirrooyfard Wei'20)

Subexponential time lower bounds under ETH: There is no $2^{o(\sqrt{n})}$ algorithm for Vertex Cover, 3-Colorability, and Hamiltonian Path for planar graphs.

(In)approximability

In polynomial time:

- $\forall c > 0$: no c-approximation algorithm
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Randomized ETH

Definition r-ETH (from Dell et al. 2012)

There is a constant $c > 0$ such that no randomized algorithm can decide 3-Sat in time 2^{ct} with error probability at most $1/3$.

Negative results under r-ETH:

- Computing the permanent of a 0-1 matrix of size $n \times n$ cannot be done in $2^{o(n)}$, and not even in time $2^{o(m)}$ where m is the number of non-zero elements.
- Some (tight) lower bounds for approximation ratios in subexponential time, e.g. in Katsikarelis, Lampis, Paschos 2019.

Also #-ETH: $\exists c$ s.t. counting the number of sat. assignments for 3-SAT cannot be done in 2^{ct}.