Optimal Selection Design with Investment

Victor Augias   Eduardo Perez-Richet

University of Bonn, Department of Economics
Sciences Po, Department of Economics

European Research Council
Established by the European Commission

Contribution

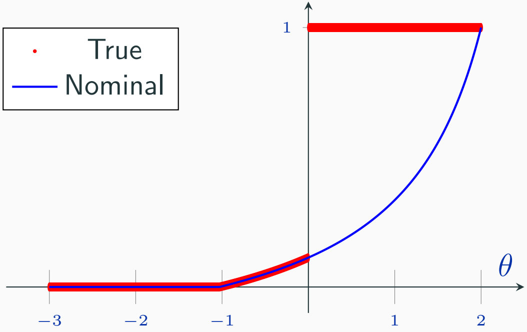
Main theorem

If the cost of investment is quadratic and the principal is selecting in the upper tail, a pass-fail rule is optimal for the principal.

- A possible explanation for the prevalence of pass-fail rules.
- A firmer foundation for the use of such rules.

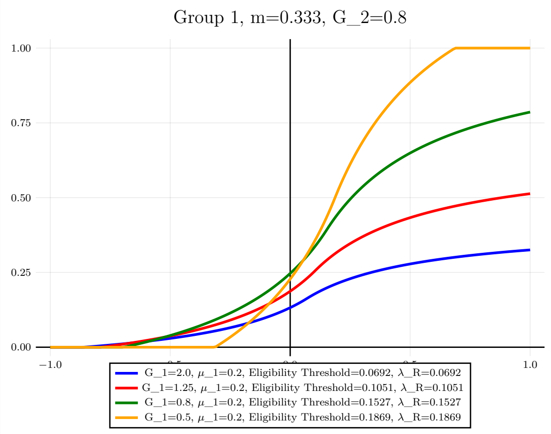
Randomized selection is optimal with falsification

Perez-Richet and Skreta (2022)



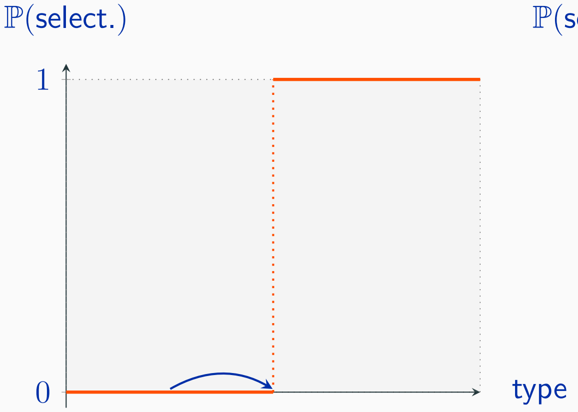
$\gamma c(t|\theta)$ satisfies UTI; $U(-3, 2)$

Perez-Richet and Skreta (2023)

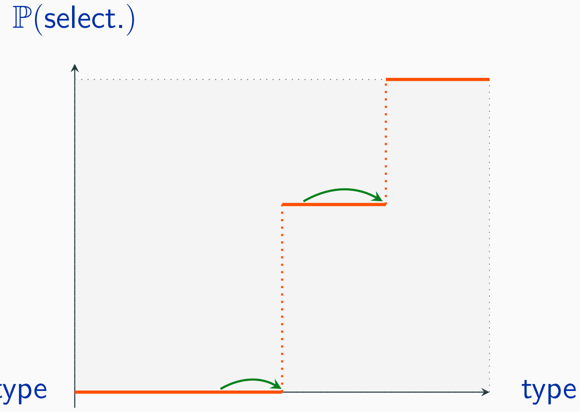


Randomization with investment?

Intuitively, randomization might help by spreading investment incentives (but complicated tradeoff).



(a) Pass-fail



(b) Some randomization.

Extensions

(i) Implementation through information design (less commitment).

(ii) Implementation through general mechanisms (more commitment).

(iii) Capacity constrained principal.

(iv) Utilitarian welfare maximization (weight on the agent).

(v) Principal also cares about investment of non-selected agents

Results

- Pass-fail rules remain optimal in (iii), (iv) and (v).
  (i) Capacity constraint lowers the optimal cutoff.
  (ii) Accounting for agents' costs increases the optimal cutoff.
- Weakening the principal's commitment power as in (i) or strengthening it as in (ii) does not modify her optimal payoff.

Model

Setup

Agent

- Initial type: $\theta \in \Theta = [\underline{\theta}, \bar{\theta}]$ with $\underline{\theta} < 0 < \bar{\theta}$.
- Distribution: cdf $F \colon \Theta \to [0, 1]$.
- Final type: $t \in T = \mathbb{R}$.
- Cost: $\gamma c(t, \theta)$ where $\gamma > 0$.

Principal:

- Knows F
- Can observe t but not θ.
- Selection: $a \in \{0, 1\}$.

Payoffs

Agent's payoff

$$a - \gamma c(t, \theta).$$

Principal's payoff

Final type conditional on allocation:

$$v(a, t) = at$$

Timing and incentive-compatibility

1. The principal commits to a selection rule $\sigma \colon T \to [0,1]$ which is publicly revealed.
2. Agent observes θ and chooses an investment strategy $\tau \colon \Theta \to T$.
3. Agent selected with probability $\sigma\big(\tau(\theta)\big)$

Definition (Incentive-compatibility)

- An investment strategy τ is incentive-compatible under selection rule σ if, for all $\theta \in \Theta$:

$$\tau(\theta) \in \arg\max_{t \in T} \sigma(t) - \gamma c(t, \theta)$$

- An investment rule τ is implementable if there exists an allocation rule σ under which τ is incentive-compatible.

Assumptions: Cost

Assumption (Quadratic cost)

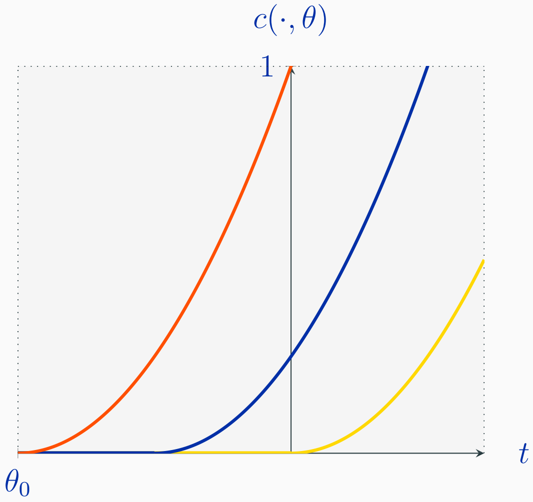
The cost function $c\colon T \times \Theta \to \mathbb{R}_+$ *is given by:*

$$c(t,\theta) = \frac{(t-\theta)^2}{2}\mathbb{1}_{t\geq\theta}.$$

Define θ_0 *by:*

$$\gamma c(0,\theta_0) = 1 \ \Leftrightarrow \ \theta_0 = -\sqrt{2/\gamma}$$

Conjecture: results extend to $c(t,\theta) = g(t-\theta)\mathbb{1}_{t\geq\theta}$ *with* $g', g'' > 0$ *and* $g''' \leq 0$.



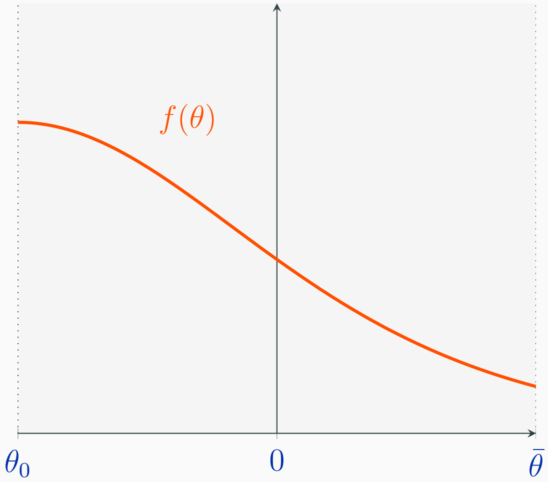
Assumptions: Distribution

Assumption (Selection in the tail)
The cdf F admits a density function f which is:

(i) *strictly positive on Θ;*

(ii) *differentiable;*

(iii) *decreasing: $f'(\theta) \leq 0$ for $\theta \geq \theta_0$.*



Principal's problem

Principal's ex-ante expected payoff is:

$$\int_{\underline{\theta}}^{\bar{\theta}} \tau(\theta)\,\sigma\big(\tau(\theta)\big)\,f(\theta)\,\mathrm{d}\theta$$

Principal's program (P)

$$\begin{aligned} \underset{\sigma,\tau}{\text{maximize}} \quad & \int_{\underline{\theta}}^{\bar{\theta}} \tau(\theta)\,\sigma\big(\tau(\theta)\big)\,f(\theta)\,\mathrm{d}\theta \\ \text{s.t.} \quad & \tau(\theta) \in \arg\max_{t \in T} \sigma(t) - \gamma c(t,\theta), \ \forall \theta \in \Theta \end{aligned}$$

Main result

Main result

Definition

An allocation rule σ is a $\hat{t}$-pass-fail rule if:

$$\sigma(t) = \mathbb{1}_{t \geq \hat{t}}.$$

Theorem

There exists a strictly positive allocation cutoff $t^*_\gamma > 0$ such that the t^*_γ-pass-fail rule is optimal.

Overview of the proof

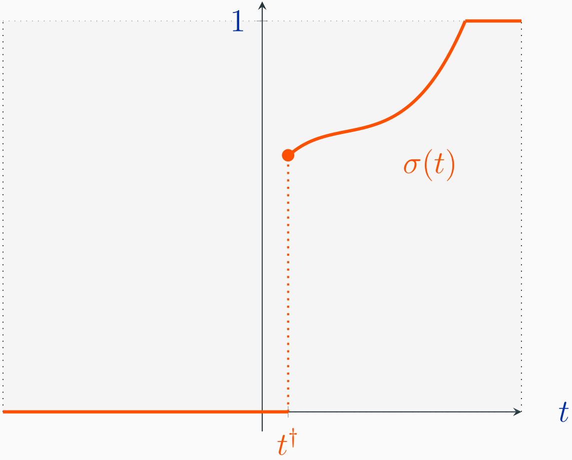
Monotone selection rules

Lemma (Monotone selection)
We can without loss of generality restrict attention to allocation rules such that:

(i) $\sigma(t) = 0$ *for all* $t < 0$, *and;*

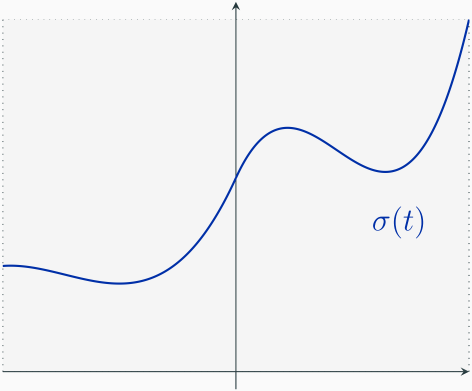
(ii) σ *is non-decreasing;*

Let $t^{\dagger} = \inf\{t \in T \mid \sigma(t) > 0\}$.



Monotone selection rules

Let σ be an arbitrary selection rule.

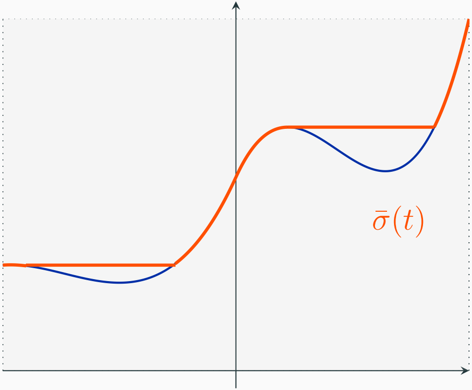


Monotone selection rules

Principal prefers

$$\bar{\sigma}(t) = \inf\left\{ \hat{\sigma} \geq \sigma \,|\, \hat{\sigma} \text{ non-decreasing } \right\}$$

to σ.

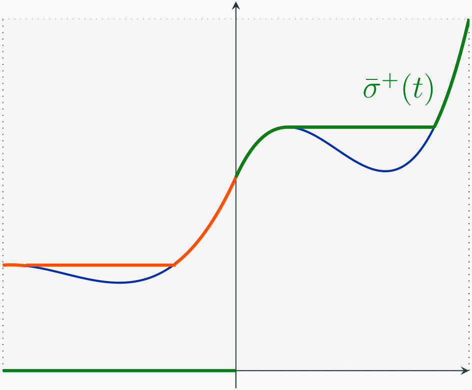


Monotone selection rules

Principal prefers

$$\bar{\sigma}^+(t) = \bar{\sigma}(t)\mathbb{1}_{t\geq 0}$$

to $\bar{\sigma}$.



Agent's pseudo utility

- Indirect utility and pseudo utility:

$$U(\theta) = \max_{t \in T} \sigma(t) - \gamma c(t, \theta)$$
$$= -\frac{\theta^2}{2} + \gamma \underbrace{\left\{ \max_{t \geq \theta} \frac{\sigma(t)}{\gamma} - \frac{t^2}{2} + t\theta \right\}}_{u(\theta)}$$

- Then $u(\theta)$ is convex and the Envelope theorem implies

$$u'(\theta) = \tau(\theta)$$

- Furthermore, we have:

$$\sigma\big(\tau(\theta)\big) = \gamma \left(u(\theta) + \frac{u'(\theta)^2}{2} - \theta u'(\theta) \right)$$

Variational program

Principal's program

$$\max_{u \in \mathcal{U}} V(u) = \int_{\theta_0}^{\bar{\theta}} u'(\theta) \left(u(\theta) + \frac{u'(\theta)^2}{2} - \theta u'(\theta) \right) f(\theta) \, \mathrm{d}\theta$$

$\mathcal{U}$ is the set of implementable pseudo-utilities, and is characterized by $u \in \mathcal{U}$ iff.:

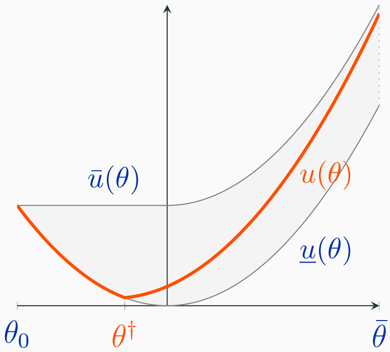
(i) u is a convex function.

(ii) $\underline{u}(\theta) = \frac{\theta^2}{2} \leq u(\theta) \leq \bar{u}(\theta) = \frac{\gamma}{2} + \frac{\theta^2}{2} \mathbb{1}_{\theta \geq 0}$

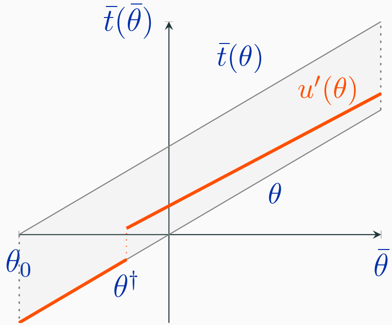
(iii) $u'(\theta) \geq \theta$ and, if $u(\theta) \neq \underline{u}(\theta)$, $u'(\theta) \geq 0$.

(iv) $0 \leq u(\theta) + u'(\theta)^2/2 - \theta u'(\theta) \leq 1/\gamma$

Feasible pseudo utilities: illustration



(a) An admissible pseudo-utility u.



(b) Its induced investment rule u'.

Solutions as extreme points

$$\text{Let: } \mathcal{U}(\theta^\dagger) = \Big\{ u \in \mathcal{U} \;:\; \theta^\dagger = \sup\{\theta \;:\; u(\theta) = \underline{u}(\theta)\} \Big\}.$$

Principal's program

$$\underset{\theta^\dagger, u \in \mathcal{U}(\theta^\dagger)}{\text{maximize}} \quad V_{\theta^\dagger}(u) = \int_{\theta^\dagger}^{\bar{\theta}} u'(\theta) \left(u(\theta) + \frac{u'(\theta)^2}{2} - \theta u'(\theta) \right) f(\theta) \, \mathrm{d}\theta$$

Lemma

- *For any $\theta^\dagger$, the set $\mathcal{U}(\theta^\dagger)$ is convex and compact.*
- *The functional $V_{\theta^\dagger} : \mathcal{U}(\theta^\dagger) \to \mathbb{R}$ is upper-semicontinuous, and, if f is decreasing, it is also convex.*
- *Therefore $V_{\theta^\dagger}(u)$ has a maximizer that is an extreme point of $\mathcal{U}(\theta^\dagger)$.*

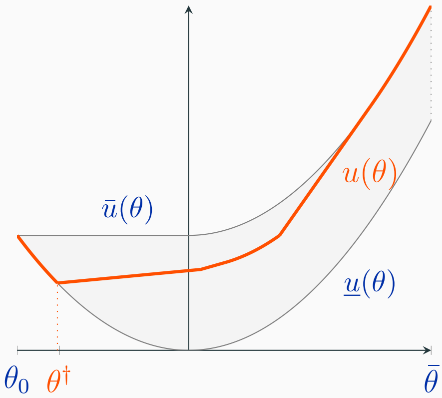
Extreme points: necessary conditions

Proposition

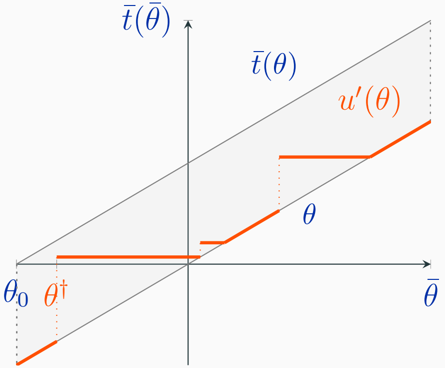
If u is an extreme point of $\mathcal{U}(\theta^{\dagger})$, it must be a sequence of affine and quadratic pieces. Moreover, u cannot be quadratic below zero.

- Affine: $u(\theta) = a\theta + b$ for some a, b.
- Quadratic: $u(\theta) = \dfrac{\theta^2}{2} + c$ for some $c \in [0, \gamma/2]$.
- Let $\mathcal{S}(\theta^{\dagger})$ be the set of pseudo-utilities satisfying these necessary conditions.

Extreme points: example



(a) An extreme point u.



(b) Its induced investment rule u'.

Extreme points: Intuition

Extreme points are such that one or more constraints are binding:

- If $u''(\theta) = 0$ then u' is constant and positive so u is affine increasing.
- If $u'(\theta) = \theta$ then u is quadratic.
- If $u(\theta) + u'(\theta)^2/2 - \theta u'(\theta) = 0$ or $\frac{1}{\gamma}$ then u is solution to Clairaut's differential equation.
    - When $= 0$ the only solution belonging to $\mathcal{U}$ is $u(\theta) = \underline{u}(\theta)$ so u is quadratic.
    - When $= \frac{1}{\gamma}$ then either $u(\theta) = \overline{u}(\theta)$, so u is quadratic, or u is affine and tangent to $\bar{u}(\theta)$.

Extreme points can be implemented by step selection rules

Definition (Bunching and inaction)

An interval is a bunching region if $u'(\theta) = t$ on that interval and it is a an inaction region if $u'(\theta) = \theta$ on that interval.

Definition (Step selection rules)

A monotonic selection rule σ is step selection rule if there exists a countable collection of intervals such that σ is constant over each of those intervals.

Implementation by step selection rules

If $u \in \mathcal{S}(\theta^\dagger)$ then it must induce a (countable) sequence of inaction and bunching regions. Moreover, any $u \in \mathcal{S}(\theta^\dagger)$ can be implemented by a step selection rule. Let us call $\mathcal{S}(\theta^\dagger)$ the set of simple pseudo-utility functions.

Implementation by step selection rules: Illustration

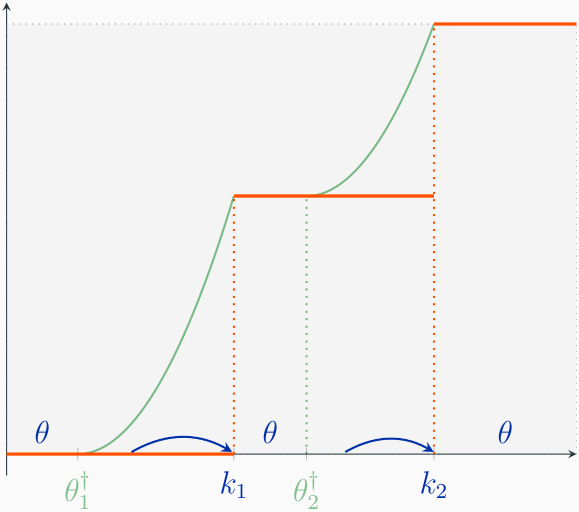
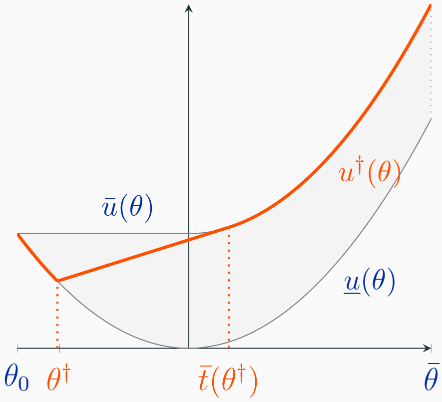
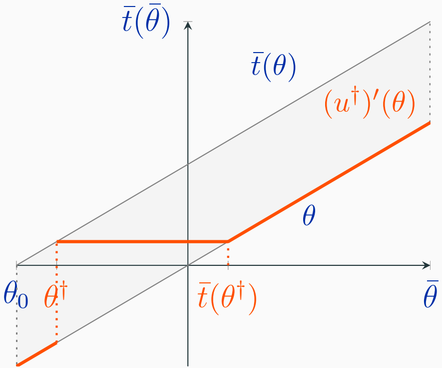


Figure 4: Implementation by a simple rule.

Our candidate: pass-fail



(a) Pseudo-utility under pass-fail.



(b) Investment under pass-fail.

Pass-fail optimality

- Let $\mathrm{D}V_{\theta^\dagger}(u)(h)$ be the Gâteaux derivative of $V_{\theta^\dagger}$ at u in direction h.
- By convexity of $V_{\theta^\dagger}$, for any $u \in \mathcal{C}(\theta^\dagger)$:

$$V_{\theta^\dagger}(u^\dagger) - V_{\theta^\dagger}(u) \geq \mathrm{D}V_{\theta^\dagger}(u)(u^\dagger - u).$$

- We prove that, for any $u \in \mathcal{C}(\theta^\dagger)$:

$$\mathrm{D}V_{\theta^\dagger}(u)(u^\dagger - u) = \int_{\theta^\dagger}^{\bar{\theta}} \big(\underbrace{\alpha(\theta)}_{\geq 0} f(\theta) + \underbrace{\beta(\theta)}_{\leq 0} f'(\theta)\big) \underbrace{\big(u^\dagger(\theta) - u(\theta)\big)}_{\geq 0} \,\mathrm{d}\theta \geq 0.$$

$\square$

Optimal Pass-Fail

- Then the optimal pass-fail is obtained by solving:

$$\max_{\hat{t}\in[0,\bar{\theta}]} \hat{t}\Big\{F(\hat{t}) - F\big(\hat{t} - \sqrt{2/\gamma}\big)\Big\} + \int_{\hat{t}}^{\bar{\theta}} \theta f(\theta) d\theta$$

- Unique solution t^* under the decreasing density assumption.

Comparative statics

- As γ increases (higher investment cost):
    - Optimal cutoff $t^*(\gamma)$ decreases.
    - Take-off type increases (investment interval shrinks).
    - Principal's payoff decreases.
    - Extreme (high or low) agent types are not affected.
    - High types in the investment region are better off whereas low investing types are worse off.

Extensions

Less commitment: information design

- Suppose the selection decision is delegated to a DM who is roughly aligned with the principal (e.g. $w(a,t) = a(t+b)$, b small).
- The principal can only commit to information structure about agent's final type t.
- Then the principal can obtain the same outcome.

More commitment: Myersonian mechanism design

- Standard mechanism design (direct by revelation principle):
  1. Principal commits to a selection rule based on report, recommendation and realized action, and to a recommendation rule based on reports..
  2. Agent (truthfully) reports type to principal.
  3. Principal recommends action to agent.
  4. Agent obeys.
  5. The rule is applied

Proposition ("Taxation principle")

If the mechanism is incentive compatible and makes deterministic recommendations, then the same outcome can be implemented by an action-based rule.

Utilitarian welfare

Consider the problem of a utilitarian social planner seeking to maximize weighted social welfare:

$$\max_{\sigma,\tau} \quad \int_{\underline{\theta}}^{\bar{\theta}} \tau(\theta)\,\sigma\big(\tau(\theta)\big)\,f(\theta)\,\mathrm{d}\theta + \alpha \int_{\underline{\theta}}^{\bar{\theta}} \Big(\sigma\big(\tau(\theta)\big) - \gamma c\big(\tau(\theta),\theta\big)\Big)\,f(\theta)\,\mathrm{d}\theta$$

$$\text{s.t.} \quad \tau(\theta) \in \arg\max_{t \in T} \sigma(t) - \gamma c(t,\theta)$$

where $\alpha > 0$ is the Pareto weight the planner assigns to the welfare of agents.

Utilitarian welfare

We can again recast the program of the planner as a variational program:

$$\max_{u\in\mathcal{U}} \underbrace{\int_{\underline{\theta}}^{\bar{\theta}} u'(\theta)\left(u(\theta)+\frac{u'(\theta)^2}{2}-\theta u'(\theta)\right) f(\theta)\,\mathrm{d}\theta}_{\text{convex in } u} + \alpha \underbrace{\int_{\underline{\theta}}^{\bar{\theta}} \left(u(\theta)-\frac{\theta^2}{2}\right) f(\theta)\,\mathrm{d}\theta}_{\text{linear in } u}.$$

Proposition

Pass-or-fail is again optimal. Moreover, the welfare-optimal selection cutoff is greater whenever $\alpha > 0$ than when $\alpha = 0$.

Principal cares about non-selected agents

- Suppose the payoff of the principal is:

$$v(a,t) = at + \lambda(1-a)t, \qquad \lambda \in [0,1]$$

- Then the problem of the principal is:

$$\max_{u \in \mathcal{U}} \lambda \int_{\theta_0}^{\bar{\theta}} u'(\theta) f(\theta) d\theta + (1-\lambda) \int_{\theta_0}^{\bar{\theta}} u'(\theta) \left(u(\theta) + \frac{u'(\theta)^2}{2} - \theta u'(\theta) \right) f(\theta)\, \mathrm{d}\theta$$

Capacity and quota constraints

- Each agent belongs to an observable group $i \in I = \{1, \ldots, n\}$ (non manipulable characteristic of the agent).
- Each group represents a proportion π_i of the total population , where $\pi_i > 0$ and $\sum_{i \in I} \pi_i = 1$.
- Types in each group are distributed according to f_i and each group has a cost parameter $\gamma_i > 0$.
- The principal can select at most a total mass $\kappa < 1$ of agents and at least a fraction q_i of agents from group i, where $q_i \geq 0$ and $\sum_{i \in I} q_i = 1$.
- Assumptions:
  - The principal can condition selection on the group (color-sighted affirmative action).
  - Quadratic cost and selection in the tail are satisfied for all groups.

Capacity and quota constraints

The constrained selection problem is given by:

$$
\begin{aligned}
\underset{(\sigma_i)_{i\in I}}{\text{maximize}} \quad & \sum_{i\in I} \pi_i \int_{\underline{\theta}}^{\bar{\theta}} \tau_i(\theta)\, \sigma_i\big(\tau_i(\theta)\big)\, f_i(\theta)\, \mathrm{d}\theta \\
\text{subject to} \quad & \tau_i(\theta) \in \arg\max_{t\in T} \sigma_i(t) - \gamma_i c(t,\theta),\ \forall i \in I \\
\text{and} \quad & \sum_{i\in I} \pi_i \int_{\underline{\theta}}^{\bar{\theta}} \sigma_i\big(\tau_i(\theta)\big) f_i(\theta)\, \mathrm{d}\theta \leq \kappa \\
\text{and} \quad & \pi_i \int_{\underline{\theta}}^{\bar{\theta}} \sigma_i\big(\tau_i(\theta)\big) f_i(\theta)\, \mathrm{d}\theta \geq q_i \kappa
\end{aligned}
$$

Proposition

Setting a pass-or-fail selection rule for each group is optimal.

Capacity and quota constraints

Letting $\lambda \geq 0$ and $(\mu_i)_{i\in I} \in \mathbb{R}^n_+$ be the Lagrange multipliers, we obtain:

$$\begin{aligned} \min_{\lambda,(\mu_i)_{i\in I}} \max_{(\sigma_i)_{i\in I}} \quad & \sum_{i\in I} \pi_i \int_{\underline{\theta}}^{\bar{\theta}} \left(\tau_i(\theta) - \lambda + \mu_i\right) \sigma_i\big(\tau_i(\theta)\big)\, f_i(\theta)\,\mathrm{d}\theta + \lambda\kappa - \sum_{i\in I} \mu_i q_i \kappa \\ \text{s.t.} \quad & \tau_i(\theta) \in \arg\max_{t\in T} \sigma_i(t) - \gamma_i c(t,\theta),\ \forall i \in I \end{aligned}$$

The inner maximization program is separable across groups and corresponds to our original problem with a shifted eligibility threshold:

$$\begin{aligned} \max_{\sigma_i} \quad & \int_{\underline{\theta}}^{\bar{\theta}} \left(\tau_i(\theta) - \lambda + \mu_i\right) \sigma_i\big(\tau_i(\theta)\big)\, f_i(\theta)\,\mathrm{d}\theta \\ \text{s.t.} \quad & \tau_i(\theta) \in \arg\max_{t\in T} \sigma_i(t) - \gamma_i c(t,\theta), \end{aligned}$$