# From Stochastic matching models on graphs to Machings on large random graphs

Pascal Moyal
IECL, Université de Lorraine

Joint works with M.H.Dialo Aoudi and V. Robin /
M. Jonckheere, C. Ramirez and N. Soprano-Loto.

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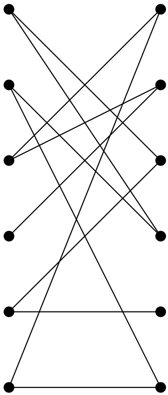
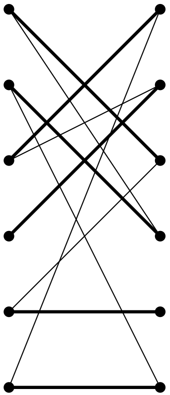
# Outline

# The marriage problem on bipartite graphs



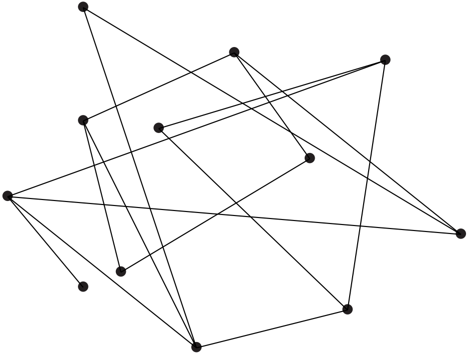
# The marriage problem on bipartite graphs



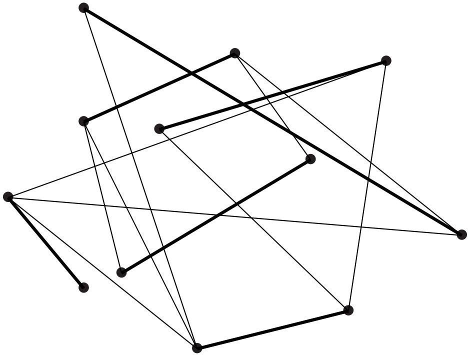
## Hall's Marriage Theorem (1935)

Let $G = (\mathcal{V}, \mathcal{E})$ be a bipartite graph and for all subsets $A \subset \mathcal{V}$, $\mathcal{E}(A)$ denote the neighborhood of the nodes of $A$. Then there exists a perfect matching iff for all $A \subset \mathcal{V}$, $|A| \leq |\mathcal{E}(A)|$.

# The marriage problem on general graphs



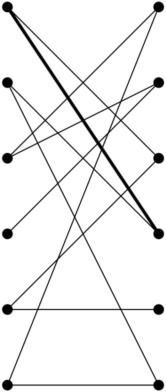
# The marriage problem on general graphs



## Tutte's Marriage Theorem (1947)

Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph. Then there exists a perfect matching iff for $o(G - U) \leq |U|$ for all $U \subset V$, where $o(G - U)$ is the number of connected components of odd sizes of the induced graph of $V \setminus U$ in $G$.

# A greedy matching algorithm



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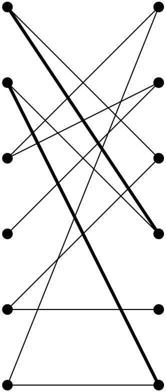

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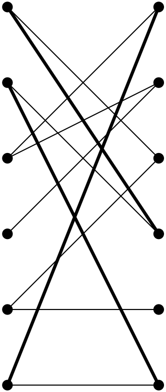

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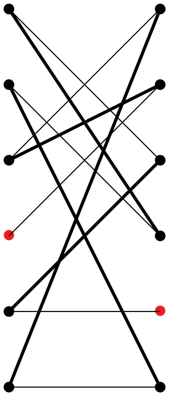


Greedy matching algorithms fail in general to construct a perfect matching.

# Matching algorithm on random graphs

We investigate the performance of greedy ("online") matching algorithms on large (random) graphs:

- On the **configuration model**;
- On **stochastic block models**.

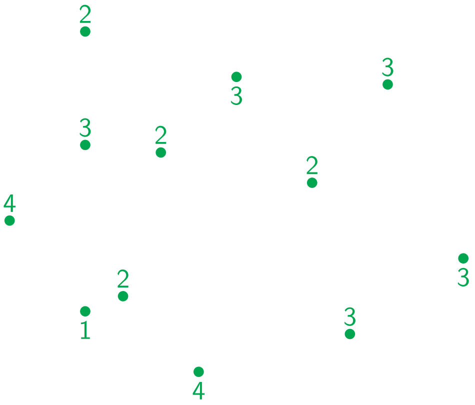
# Outline

# The configuration $\mathrm{CM}_n(\mathbf{d})$

- We fix the degree distribution: $D(i)$, $i \in [\![1, n]\!]$ are iid $\sim D$, a $[\![0, n-1]\!]$-valued r.v..
  **Exemples**:
  - deterministic $D$ ($\hookrightarrow$ regular graph): $\mathbb{P}[D = d] = 1$ for a certain $d \in [\![1, n-1]\!]$.
  - Poisson $\mathcal{P}(\lambda)$: for all $d$, $\mathbb{P}[D = d] = e^{-\lambda}\frac{\lambda^d}{d!}$ (asymptotic of Erdös-Rényi for $p_n \sim \frac{\lambda}{n}$.)
  - Power law: for all $d$, $\mathbb{P}[D > d] \sim d^{-\beta}$.
  - ...
- A (multi-) graph having such degree distribution can be constructed following a markov procedure.

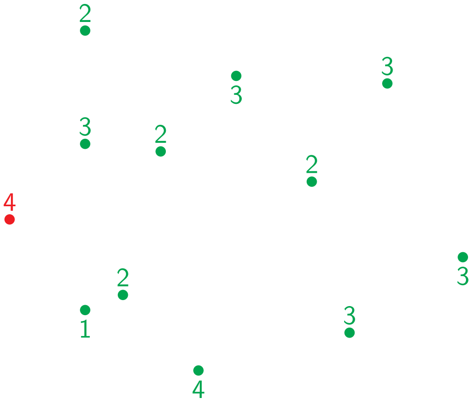
# Construction of the multi-graph

We start with $n$ nodes detached from the graph, having respective degrees $d_0(1), \ldots, d_0(n)$.



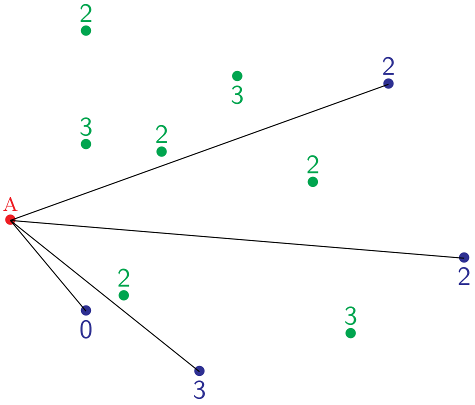
# Construction of the multi-graph

Bucket (node) $i$ is chosen uniformly at random, and attached to the graph.



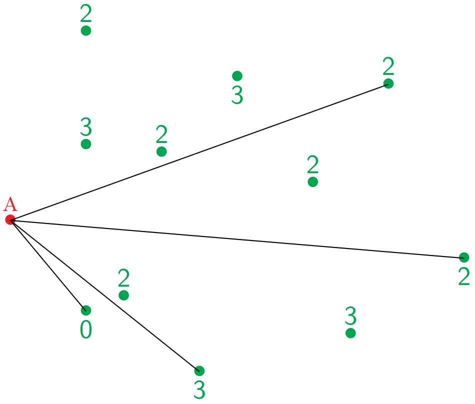
# Construction of the multi-graph

The $d_0(i)$ half-edges of $i$ are paired sequentially with uniformly drawn half-edges.



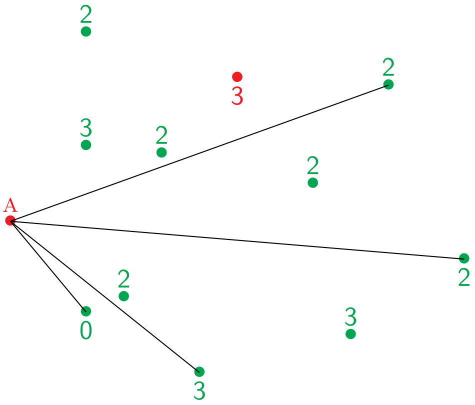
# Construction of the multi-graph

We repeat the same procedure on all nodes that are not yet fully attached to the graph.



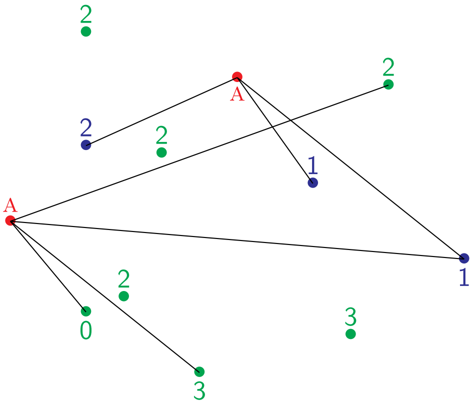
# Construction of the multi-graph

We draw uniformly at random a new node.



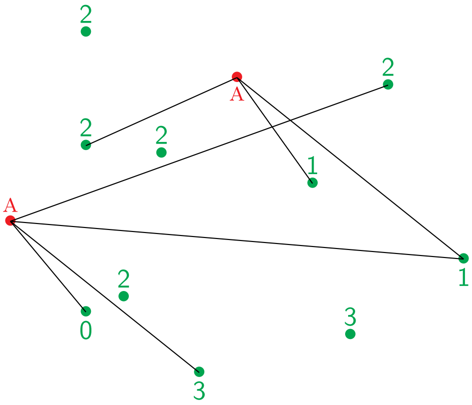
# Construction of the multi-graph

We draw its neighbors by pairing its half-edges uniformly at random.



# Construction of the multi-graph

We repeat the procedure...



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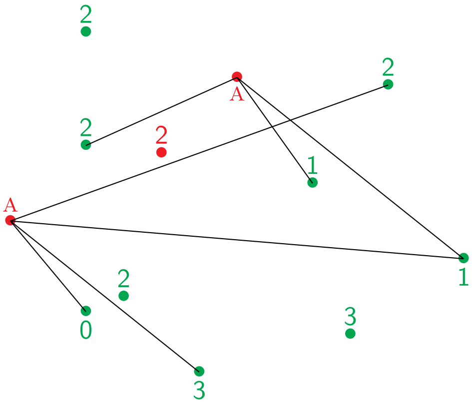

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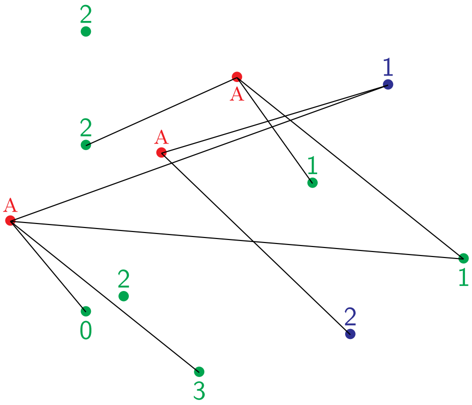

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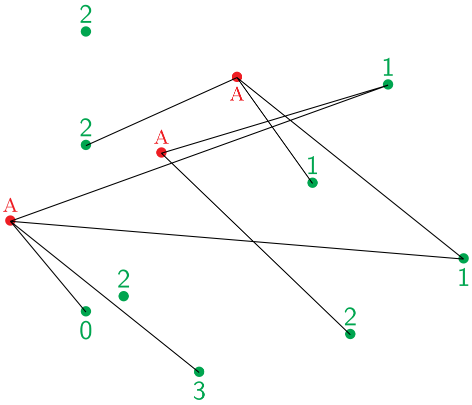

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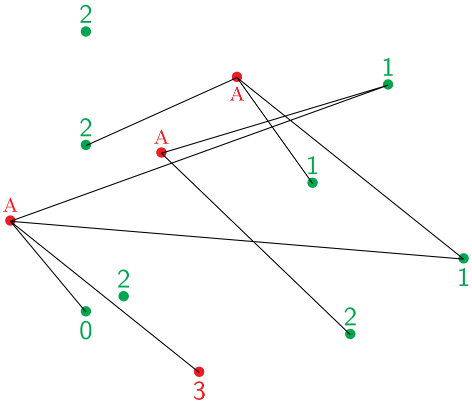

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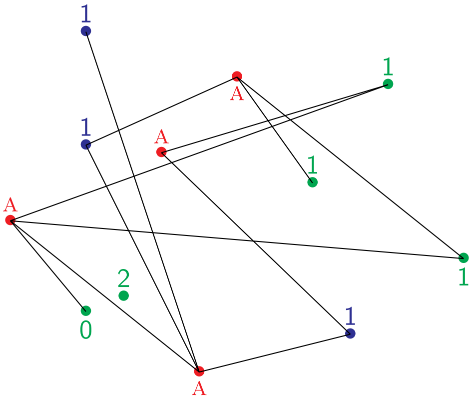

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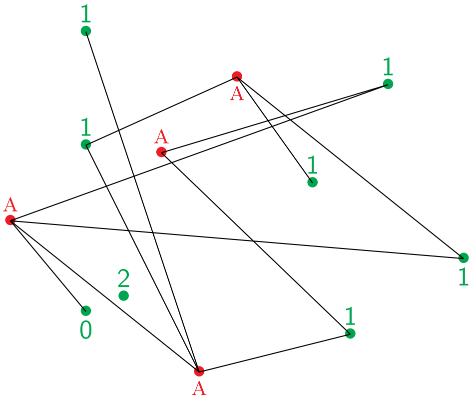

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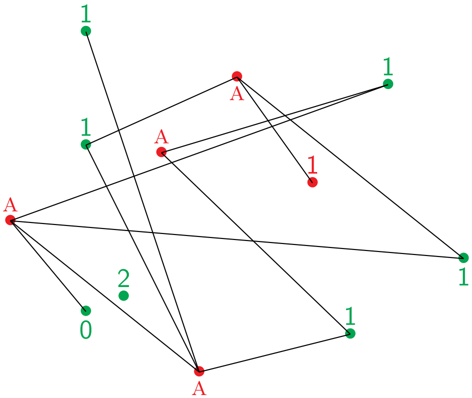

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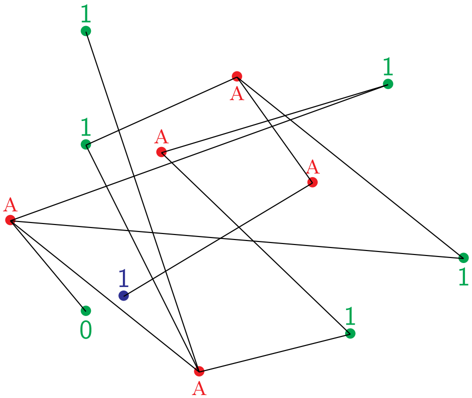

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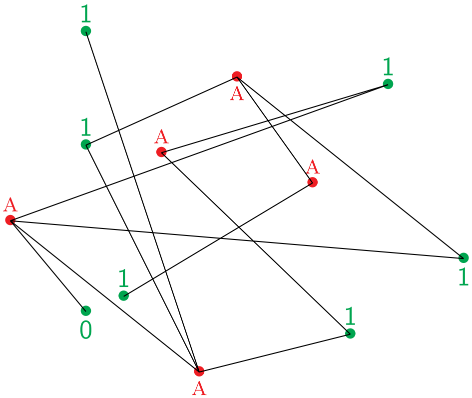

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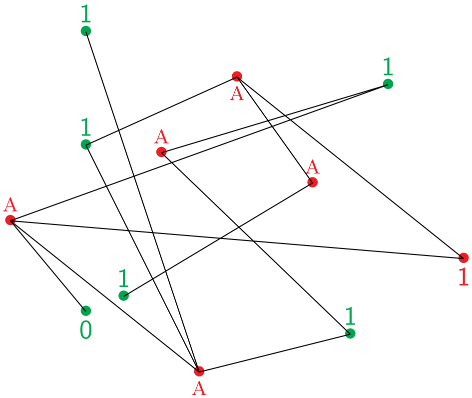

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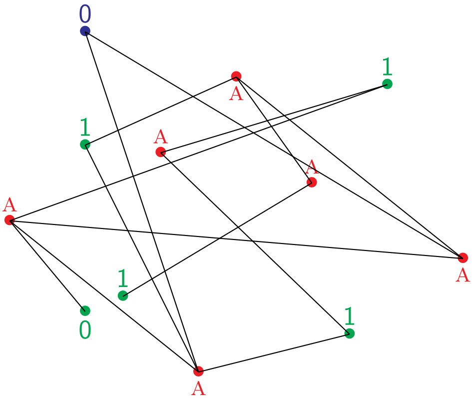

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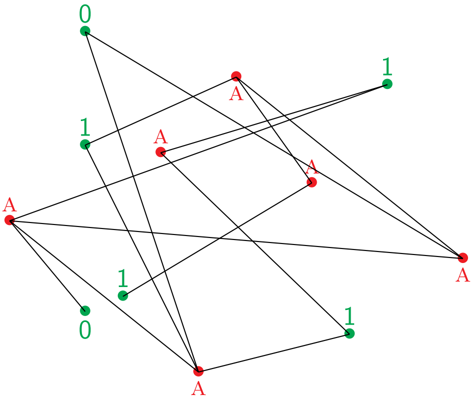

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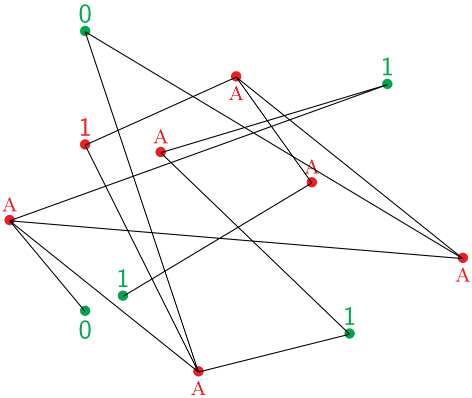

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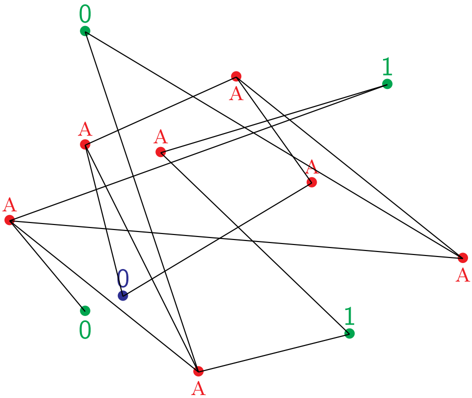

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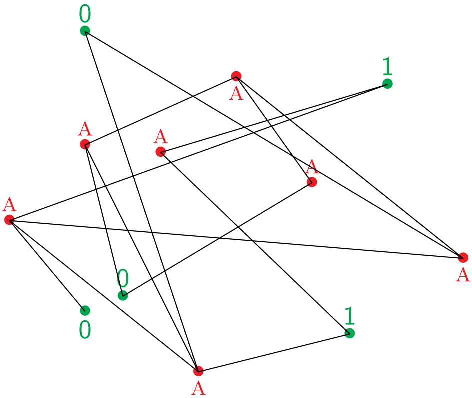

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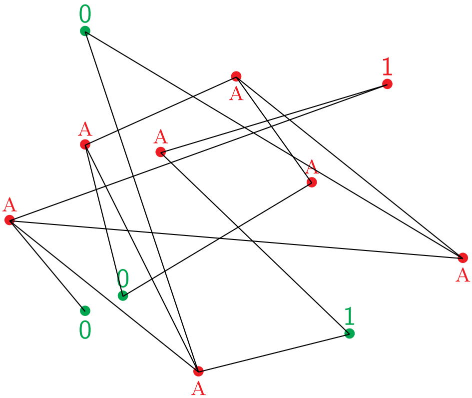

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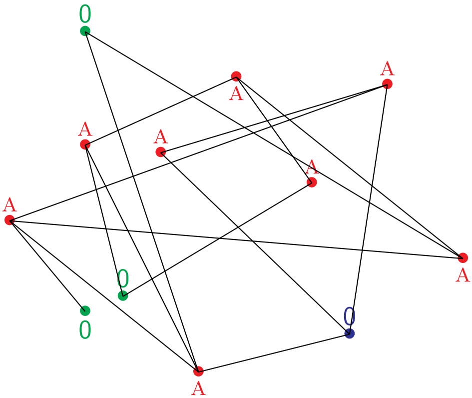
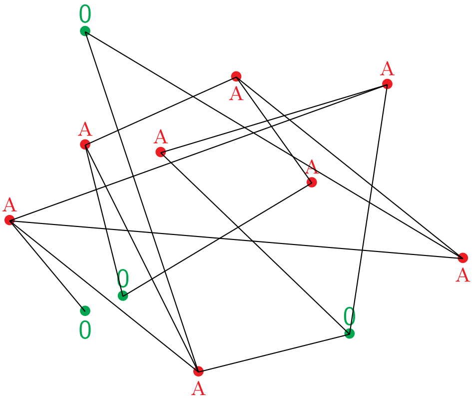


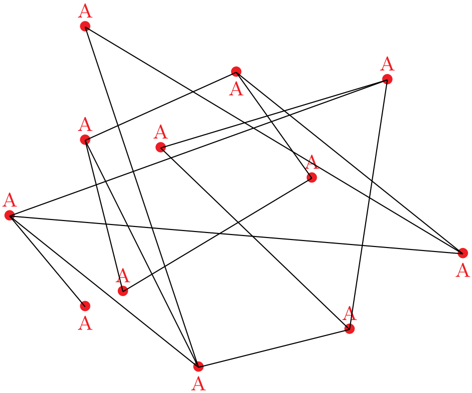
# Construction of the multi-graph

We repeat the procedure...until all unexplored nodes all have 0 available half-edge (time $T$).



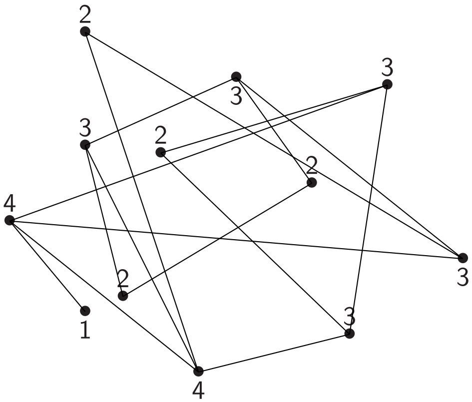
# Construction of the multi-graph

The multi-graph is now constructed.



# Construction of the multi-graph

The multi-graph does have the prescribed degree distribution.



# Asymptotic simplicity

We assume that **d** is graphical.

## Proposition

Let

$$\begin{cases} M_n & = \text{Number of multiple edges in } \mathrm{CM}_n(\mathbf{d}); \\ S_n & = \text{Number of self-loops in } \mathrm{CM}_n(\mathbf{d}). \end{cases}$$

The sequence $\{(S_n, M_n)\}_{n\in\mathbb{N}^*}$ tends weakly to $(S, M)$, where $S$ and $M$ are independent r.v.'s of respective distributions $\mathcal{P}(\nu/2)$ and $\mathcal{P}(\nu^2/4)$, for

$$\nu = \frac{\mathbb{E}\left[D(D-1)\right]}{\mathbb{E}\left[D\right]}.$$

## Corollary

We have the convergence

$$\mathbb{P}\left[\mathrm{CM}_n(\mathbf{d}) \textit{ produces a (simple) graph }\right] \underset{n\to\infty}{\longrightarrow} \textit{exp}\left(-\frac{\nu}{2} - \frac{\nu^2}{4}\right).$$

# Uniform sampling

## Proposition

Let $\mathbf{d} \in [0, n-1]^n$, be a graphical degree vector. Then for all (simple) graph $\mathcal{G}$ of degree vector $\mathbf{d}$,

$$\mathbb{P}\left[\mathrm{CM}_n(\mathbf{d}) = \mathcal{G} \mid \mathrm{CM}_n(\mathbf{d}) \textit{ is simple}\right] = \frac{1}{\textit{Card}\{\mathcal{G} \in \mathscr{G}_n \ : \ \mathcal{G} \textit{ has degree vector } \mathbf{d}\}}.$$

In other words, the distribution of $\mathrm{CM}_n(\mathbf{d})$ is **uniform** within the (simple) graphs of degree distribution **d**.

# Measure-valued Markov representation

## Definition

For all $\mathbf{d} = (d(1), \dots, d(n)) \in [\![0, n-1]\!]^n$, we define the residual degree sequence of $\mathrm{CM}_n(\mathbf{d})$, by

$$\begin{cases} \mu_0^n &= \nu_{\mathbf{d}} := \sum_{i=1}^{n} \delta_{d(i)} \in \mathscr{M}(\mathbb{N}); \\ \mu_k^n &= \sum_{i \in \mathcal{U}_k} \delta_{d_k(i)},\ 0 < k \leq T, \end{cases}$$

where

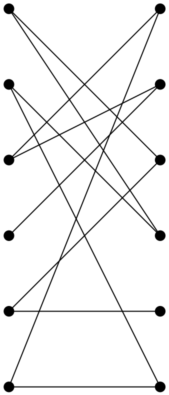
- For all $k \in [\![0, T]\!]$, $\mathcal{U}_k \subset [\![1, n]\!]$ is the set of nodes that are not yet attached to the graph after step $k$;
- For all $k \geq 0$ and all $i \in \mathcal{U}_k$, $d_k(i)$ is the number of available half-edges (*residual degree*) of node $i$, after step $k$.

## Proposition

The process $\{\mu_k^n\}_{0 \leq k \leq T}$ is Markov on $\mathscr{M}(\mathbb{N})$.

# Outline

# A marriage algorithm based on exploration



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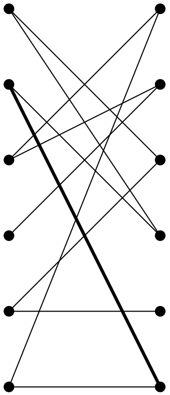
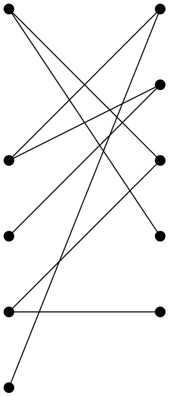


# A marriage algorithm based on exploration

We erase the married nodes from the graph, couple by couple.



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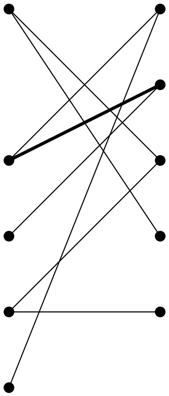

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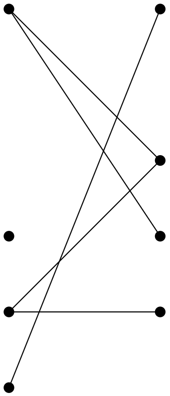

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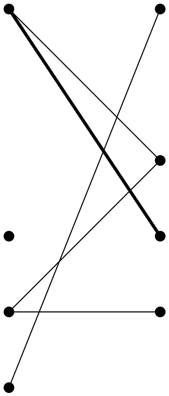

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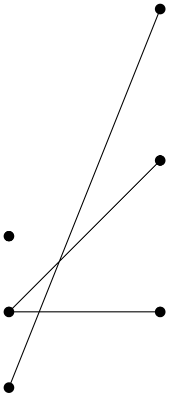

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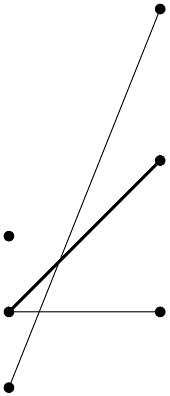

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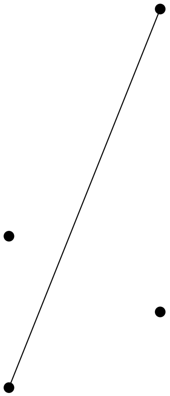

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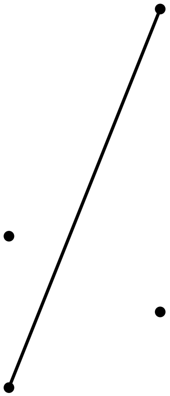
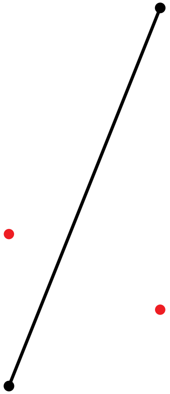


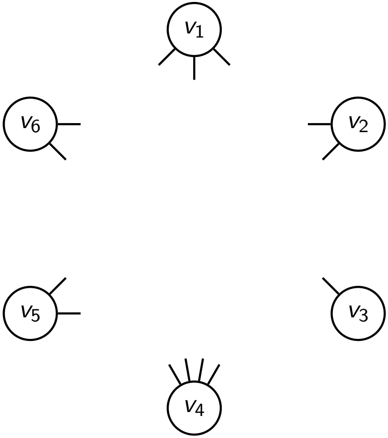
# A marriage algorithm based on exploration

Until there are no nodes left to match.



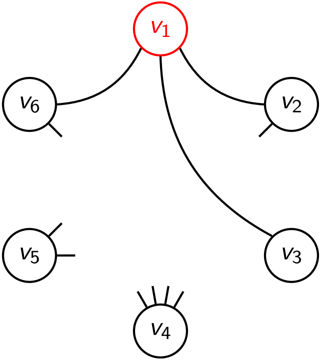
# Coupled construction with the Configuration Model

Figure: A running example: Initial state for $= (3, 2, 1, 4, 2, 2)$



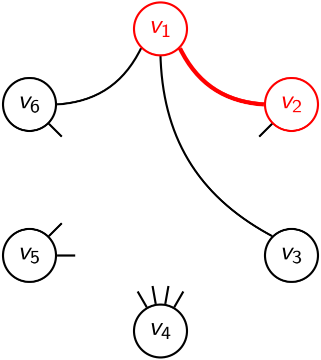
# Coupled construction with the Configuration Model

Figure: Chosing the neighbors of node $v_1$



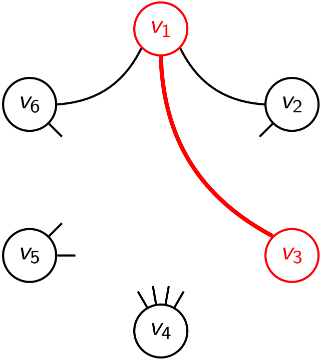
# Coupled construction with the Configuration Model

Figure: Chosing the match of $v_1$ in GREEDY



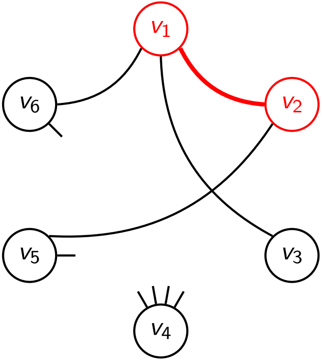
# Coupled construction with the Configuration Model

Figure: Chosing the match of $v_1$ in MINRES



# Coupled construction with the Configuration Model

Figure: Chosing the other nieghbors of the match of $v_1$ in GREEDY



# Measure-valued processes

## Definition

We let $\tilde{\mu}_t$ be the empirical degree distribution of all undetermined nodes at $t$ in the remaining graph $\tilde{G}_t$ of the first construction:

$$\tilde{\mu}_t = \sum_{\tilde{v} \in \tilde{U}_t^+} \delta_{d_t(\tilde{v})},$$

and $\mu_t$ be the empirical distribution representing the availabilities of all undetermined nodes at $t$ in the second construction:

$$\mu_t = \sum_{v \in U_t} \delta_{a_t(v)}. \tag{1}$$

# Markov property

## Proposition

The process $((\mu_n),\ n \in [\![0, T]\!])$ is Markov.

## Theorem

Let $\tilde{G}$ be a graph of size $n$, and let $\mathbf{d}$ be a graphical degree distribution. Fix the initial value

$$\tilde{\mu}_0 = \mu_0 = \nu_{\mathbf{d}} := \sum_{i=1}^{n} \delta_{d(i)}.$$

Let $G$ be the resulting multigraph of the second construction. Then for any $n \in [\![0, T]\!]$ and any measure $\nu$ we get that

$$\mathbb{P}\left[\mu_n = \nu \mid G = \tilde{G}\right] = \mathbb{P}\left[\tilde{\mu}_n = \nu\right].$$

# Hydrodynamic approximation

By applying Wormald's differential equation approximation procedure,

## Theorem

As $n \to \infty$, the sequences of Markov processes $\{\frac{1}{n}\mu^n\}$ tends weakly to the unique solution of the family of ODE's

$$\bar{\mu}_0 = \nu_{\mathbf{d}};$$

$$\frac{d\,\langle \bar{\mu}_t, \varphi \rangle}{dt} = -\Bigg( \langle \Phi_1(\bar{\mu}_t), \varphi \rangle + \langle \Phi_2(\bar{\mu}_t), \varphi \rangle + \frac{\langle \bar{\mu}_t, X \nabla \varphi \rangle}{\langle \bar{\mu}_t, X \rangle} \left( \langle \Phi_1(\bar{\mu}_t), X - \mathbf{1} \rangle + \langle \Phi_2(\bar{\mu}_t), X - \mathbf{1} \rangle \right) \Bigg),$$

for all $\varphi$ bounded or in $\{X, X^2\}$,, $\nabla$ the discrete gradient, and $\Phi_1$ and $\Phi_2$ characterizing the matching criterion $\Phi$.

## Corollary: asymptotic of the matching coverage

$$\mathbf{M}^n_\Phi(\nu) \xrightarrow{\mathbb{P}} \mathbf{M}_\Phi(\nu) := 1 - \bar{\mu}_1(\{0\}).$$

# Particular case: bipartite matching

Consider a bipartite graph of size $2n$, and append superscripts $^+$ and $^-$ to the characteristics of both sides.

## Theorem

As $n \to \infty$, the sequences of Markov processes $\{\frac{1}{n}\mu^n\}$ tends to the unique solution of the following family of systems of ODE's: for all $t \in [0,1]$,

$$\begin{cases} (\bar{\mu}_0^+, \bar{\mu}_0^-) & = (\nu_{\mathbf{d}^+}, \nu_{\mathbf{d}^-}); \\ \frac{\mathrm{d}\left\langle \bar{\mu}_t^+, \varphi \right\rangle}{\mathrm{d}\, t} & = -\left\langle \bar{\mu}_t^+, \varphi \right\rangle - \left\langle \bar{\mu}_t^+, F_X^\Phi(\bar{\mu}_s^-) \right\rangle \frac{\left\langle \bar{\mu}_t^+, X\Delta\varphi \right\rangle}{\left\langle \bar{\mu}_t^+, X \right\rangle} \\ \frac{\mathrm{d}\left\langle \bar{\mu}_t^-, \varphi \right\rangle}{\mathrm{d}\, t} & = -\left\langle \bar{\mu}_t^+, F_\varphi^\Phi(\bar{\mu}_t^-) \right\rangle - \left\langle \bar{\mu}_t^+, X \right\rangle \frac{\left\langle \bar{\mu}_t^-, X\Delta\varphi \right\rangle}{\left\langle \bar{\mu}_t^-, X \right\rangle}, \end{cases},$$

for $\varphi$ bounded or in $\{X, X^2\}$, where the mapptings $F_\cdot^\Phi$ characterize $\Phi$.

# Two local matching algorithms

1. For $\Phi = \textsc{Greedy}$, the match $J^-$ of $I^+$ is chosen **uniformly** at random among its neighbors. We obtain

$$F^{\mathrm{GREEDY}}(\mu)(k) = \frac{\langle \mu, X\tau_1\varphi \rangle}{\langle \mu, X \rangle} \mathbf{1}_{\mathbb{N}^*}(k), \quad k \in \mathbb{N}.$$

2. For $\Phi = \textsc{MinRes}$, the match $J^-$ of $I^+$ is the one having the **least availability**. We obtain that

$$F_{\varphi}^{\mathrm{MINRES}}(\mu)(k) \\ = \sum_{a \in \mathbb{N}^*} \varphi(a-1) \left( \frac{\langle \mu^-, X\mathbf{1}_{[a,+\infty)} \rangle^k - \langle \mu^-, X\mathbf{1}_{[a+1,+\infty)} \rangle^k}{\langle \mu^-, X \rangle^k} \right), \quad k \in \mathbb{N}.$$

# For $\Phi = \text{GREEDY}$ and 3-regular graphs

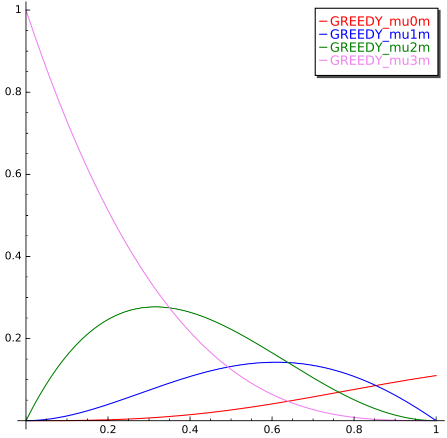



Figure: $\bar{\mu}_s^-(k)$ for $k = 0, \ldots, 3$ and $s \in [0, 1]$.

# Regular graphs

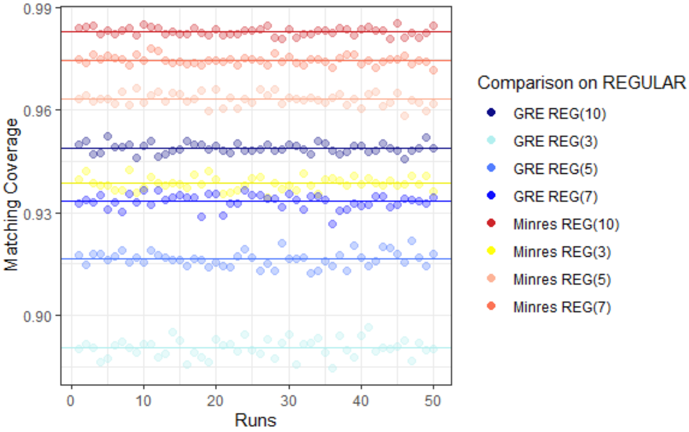



Figure: Performance on Regular Degree Distr.

# Poisson degree distributions

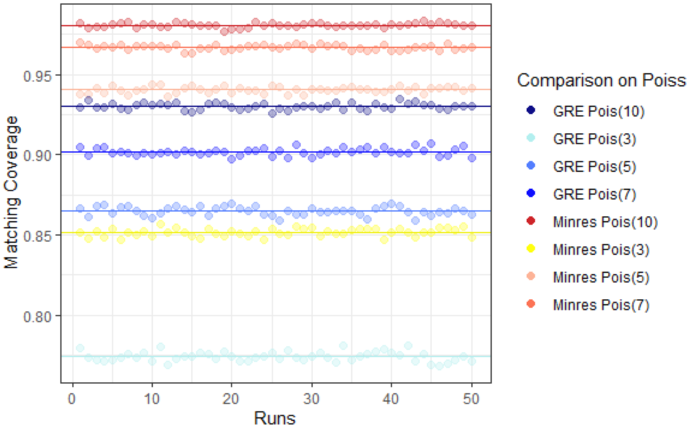



Figure: Performance on Poisson Degree Distribution

# The CM *vs* the graph of KVV90

We compare, for both algorithms, the matching coverage obtained on the CM to that obtained on the graph on which Karp et al. construct their worst case scenario.

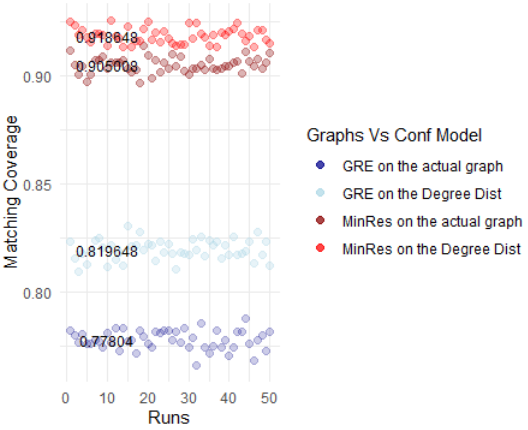



Figure: Exploration Vs Conf Model

# Work in progress

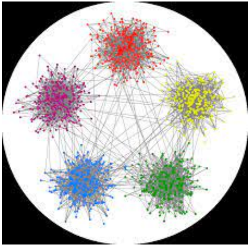
1. Proof of the convergence to the ODE for GREEDY and MINRES in the bipartite case: done.
2. Proof of the convergence to the ODE for GREEDY and MINRES in the general case: done.
3. Explicit forms of $\Phi$ for other local algorithms?
4. Optimality of MINRES? In what sense?


- Diallo Aoudi, M.H.A., Moyal, P. and Robin, V. (2022). "Markovian online matching algorithms on large bipartite random graphs", *Methodology and Computing in Applied Probability* **24**: 3195-3225.
- Diallo Aoudi, M.H.A., Moyal, P. and Robin, V. (2023+). "Hydrodynamic limits for local matching algorithms on large uniform random graphs", *Preprint*.

# Outline

1. Classical problem: the marriage problem on a graph
2. Reminder: the Configuration Model (CM)
3. Two local marriage algorithms on uniform graphs
4. Perfect matchings on the Stochastic Block Model

# A stochastic block model (SBM) with $p$ communities



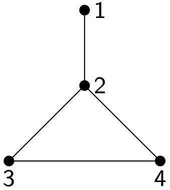
A random graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$:

- There are $p$ communities $\mathbf{C}^1, ..., \mathbf{C}^p$ forming a partition of $\mathbf{V}$.
- For any nodes $u_i \in \mathbf{C}^1$ and $u_j \in \mathbf{C}^j$, the edge $\{u_i, u_j\} \in \mathbf{E}$ with probability $P_{ij}$, independently of everything else.
- Set $G$, the *root* graph (with self-loop) on the set of nodes $[1, p]$, such that for all $i, j \in [1, p]$,

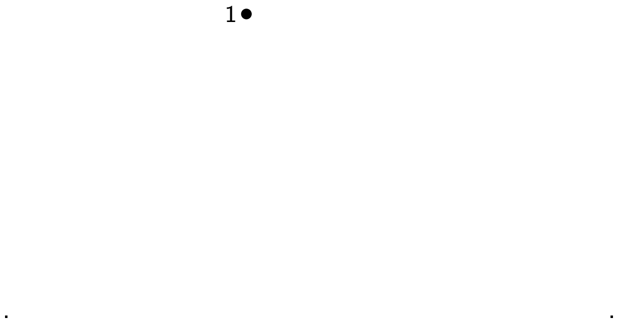
$$i \sim j \iff P_{ij} > 0.$$

# Online construction of a SBM

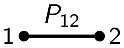
Start with a root graph



# Online construction of a SBM



# Online construction of a SBM



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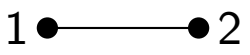

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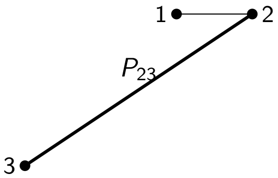

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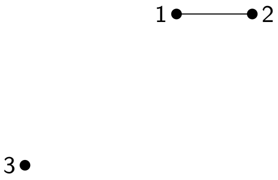

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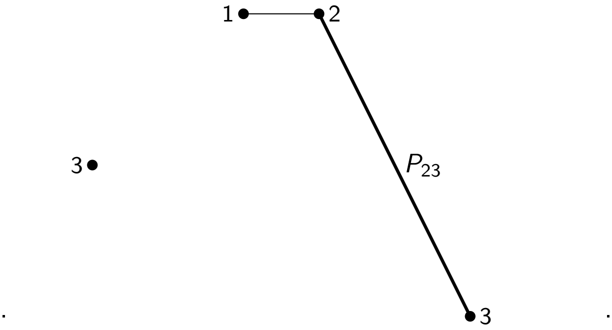

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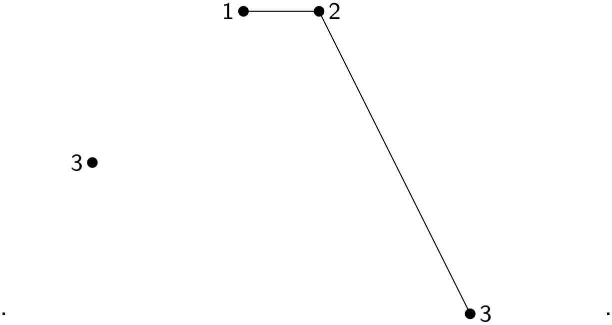

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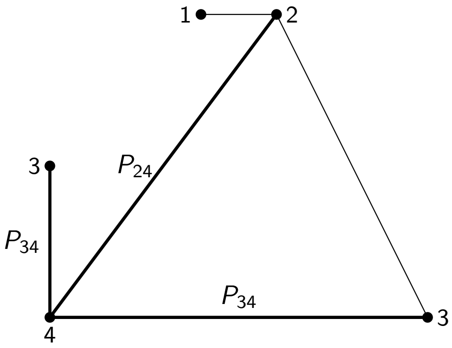

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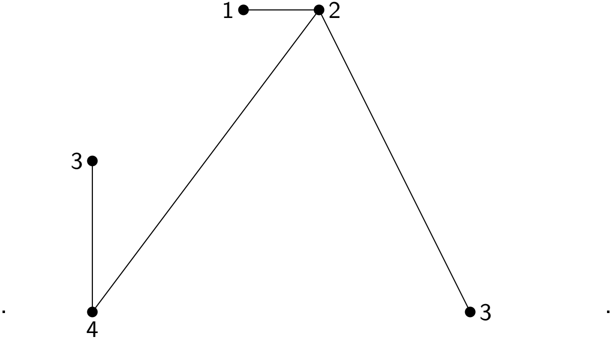

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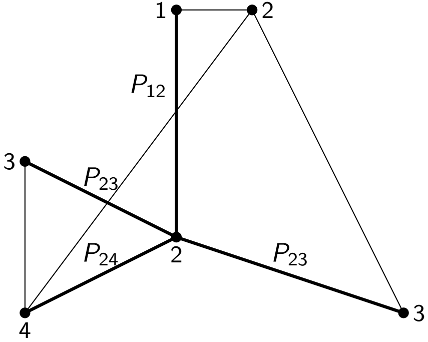
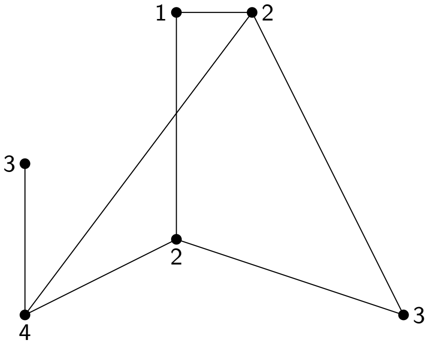


# Online construction of a SBM



# Joint construction of an online matching on the SBM

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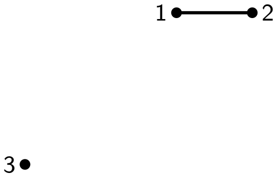
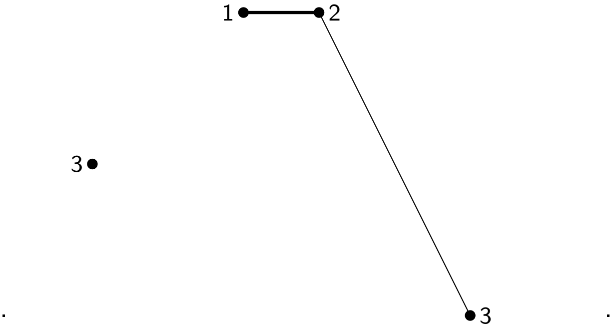
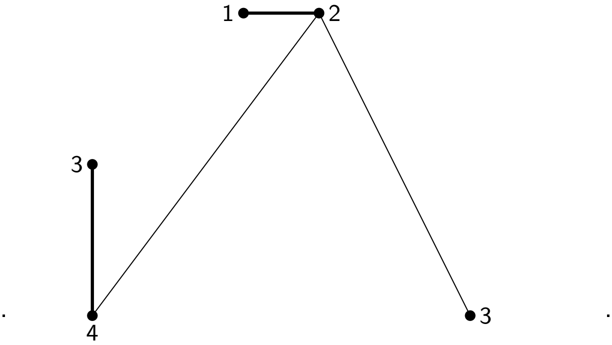


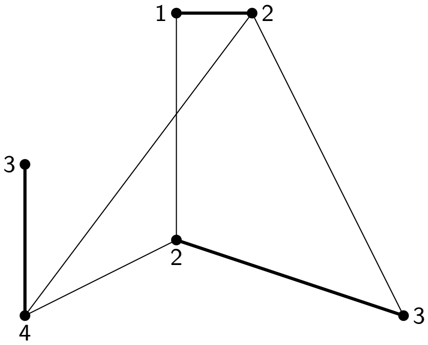
# Joint construction of an online matching on the SBM



# Joint construction of an online matching on the SBM



# Joint construction of an online matching on the SBM



# Online construction of a SBM $\mathbf{G}_n$

- Start with a connectivity matrix $P$ inducing a root graph $G$, and a probability measure $\mu$ on $[1, p]$.
- At first, set $\mathbf{G}_0 = \emptyset$
- At any step $n$, we are given a graph $\mathbf{G}_{n-1} = (\mathbf{V}_{n-1}, \mathbf{E}_{n-1})$ of communities $\mathbf{C}^1, ..., \mathbf{C}^p$,

Then,

1. Create a new node $v_n$ by setting $\mathbf{V}_n = \mathbf{V}_{n-1} \cup \{v_n\}$.
   Draw $i \in [1, p]$ from distribution $\mu$: $v_n$ is of community $\mathbf{C}^i$.
2. For any node $u \in \mathbf{G}_{n-1}$ of community $\mathbf{C}^j$, we set
   $\mathbf{E}_n = \mathbf{E}_{n-1} \cup \{\{u, v_n\}\}$ with probability $P_{ij}$.

$\Rightarrow$ $\mathbf{G}_n = (\mathbf{V}_n, \mathbf{E}_n)$ is a SBM with communities $\mathbf{C}^1, ..., \mathbf{C}^p$.

# Online construction of a SBM $\mathbf{G}_n$ and joint construction of a matching on $\mathbf{G}_n$

- Start with a connectivity matrix $P$ inducing a root graph $G$, and a probability measure $\mu$ on $[\![1, p]\!]$.
- At first, set $\mathbf{G}_0 = \emptyset$ and $\mathbf{M}_0 = \emptyset$.
- At any step $n$, we are given a graph $\mathbf{G}_{n-1} = (\mathbf{V}_{n-1}, \mathbf{E}_{n-1})$ of communities $\mathbf{C}^1, ..., \mathbf{C}^p$, and set for any $i$, $\mathbf{X}^i_{n-1}$ as the set of unmatched nodes of $\mathbf{C}^i$ in $\mathbf{G}_{n-1}$. Then,
  1. Create a new node $v_n$ by setting $\mathbf{V}_n = \mathbf{V}_{n-1} \cup \{v_n\}$. Draw $i \in [\![1, p]\!]$ from distribution $\mu$: $v_n$ is of community $\mathbf{C}^i$.
  2. For any node $u \in \mathbf{G}_{n-1}$ of community $\mathbf{C}^j$, we set $\mathbf{E}_n = \mathbf{E}_{n-1} \cup \{\{u, v_n\}\}$ with probability $P_{ij}$.

  $\Rightarrow$ $\mathbf{G}_n = (\mathbf{V}_n, \mathbf{E}_n)$ is a SBM with communities $\mathbf{C}^1, ..., \mathbf{C}^p$.

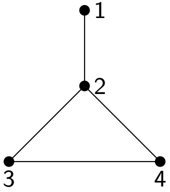
  3. Choose a match $u_n$ for $v_n$ of community $\mathbf{C}^j$, if any, in $\mathbf{V}_{n-1}$, applying a matching criterion

$$\phi_i : \begin{cases} \mathbb{N}^p & \longmapsto [\![1, p]\!]; \\ (|\mathbf{X}^1|, \cdots, |\mathbf{U}p|) & \longrightarrow j, \end{cases}$$

and set $\mathbf{M}_n = \mathbf{M}_{n-1} \cup \{\{u_n, v_n\}\}$.

# General stochastic matching model

Fix a simple connected graph $G = (\mathcal{V}, \mathcal{E})$,




- Items of the various classes in $\mathcal{V}$ arrive one by one; their class $i$ is drawn following $\mu$ on $\mathcal{V}$.
- Any incoming item is matched, if possible, with a compatible item present in the system. Otherwise it is stored in a buffer;
- If several possible matches are possible, the incoming item follows a given matching policy $\phi$ ($\phi =$ *First Come, First Matched*, *Match the Longest*, *Random*, Priorities, etc.)

# Stability problem

The **stability region** of $(G, \phi)$, denoted $\textsc{Stab}(G, \phi)$, is the set of probability measures $\mu$ on $\mathcal{V}$ such that the natural Markov chain of $(G, \mu, \phi)$ is positive recurrent.

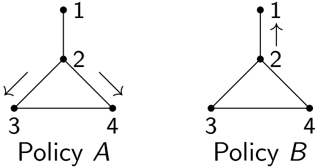
## Natural necessary condition on $\mu$

$\textsc{Stab}(G, \phi)$ is included in the set

$$\textsc{NCond}(G) := \left\{ \mu : \ \mu(\mathcal{I}) < \mu(\mathcal{E}(\mathcal{I})) \text{ for all independent sets } \mathcal{I} \right\}.$$

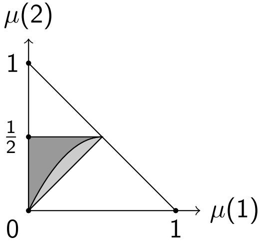
$\phi$ is said maximal if $\textsc{Stab}(G, \phi) = \textsc{NCond}(G)$.

# Dependence on the matching policy: example of the 'Paw graph'




## Stability regions

For $\mu(3) = \mu(4)$,



# Literature review: stability results

- A bipartite graph is not stabilizable, and 'Match the Longest' is maximal (*Mairesse and Moyal 16*, following *Bušić et al. 13*),
- 'Class-uniform' is never maximal, and there always exist a non-maximal priority-type matching policy (*Moyal and Perry 17*).
- 'First Come, First Matched' is maximal (*Adan et al. 17*, *Moyal et al. 21*) and the stationary distribution has a product form.
- Same results for graphs with self-loops (*Begeot et al. 21*).
- For 'Max-Weight' policies (defined in *Nazari and Stolyar 19*, and including 'Match the Longest'): Bounds on the speed of convergence and moments of the stationary distribution using Lyapunov techniques. (*Jonckheere et al. 23*).

# Comeback to the online SBM

Consider the following matching criterion: At $n$,

1. If the incoming item $v_n$ is of community $\mathbf{C}^i$, set

$$\phi_i\left(x(1), \cdots, x(p)\right) = \text{uniform in } \operatorname{Argmax}\left\{x(i) \,:\, x(i) > 0 \text{ and } i \sim j\right\}.$$

2. Then,
   - If $v_n$ indeed shares an edge with some node $u_n \in \mathbf{C}_j$ add the edge $\{u_n, v_n\}$ to $\mathbf{M}_{n-1}$.
   - Else, leave $\mathbf{M}_n = \mathbf{M}_{n-1}$.

# Comeback to the online SBM

Observe that:

- The matching $\mathbf{M}_n$ is perfect if and only if $(|\mathbf{X}_n(1)|, \cdots, |\mathbf{X}_n(p)|) = \mathbf{0}$.
- The process

$$(X_n)_{n\in\mathbb{N}} := ((|\mathbf{X}_n(1)|, \cdots, |\mathbf{X}_n(p)|))_{n\in\mathbb{N}}$$

  is an irreducible Markov DTMC.

$\Rightarrow$ The matching is perfect infinitely often if and only if the DTMC is positive recurrent.

## A simple observation

If $P_{ij} = 1$ for all $i \sim j$, the process $(X_n)_{n \in \mathbb{N}}$ is a stochastic upper-bound to the DTMC of the stochastic matching model on the root graph $G$ of the SBM, of matching policy 'Match the Longest'.

## A simple observation

If $P_{ij} = 1$ for all $i \sim j$, the process $(X_n)_{n\in\mathbb{N}}$ is a stochastic upper-bound to the DTMC of the stochastic matching model on the root graph $G$ of the SBM, of matching policy 'Match the Longest'.

$$\Downarrow$$

## Proposition

If $\mu$ is not an element of $\textsc{Ncond}(G)$, then there exists $c > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}\left[\|X_n\| \geq \sqrt{n}\right] \geq c.$$

# Results

Let

$$\eta := \eta(G, \mu) = \min\Big\{\mu(\mathcal{E}(\mathcal{I})) - \mu(\mathcal{I}) : \mathcal{I} \in \mathbb{I}\Big\}.$$

## Theorem (Jonckheere, M. and Soprano-Loto (23+))

Let the Lyapunov function $q : \mathbf{x} \longmapsto \sum_i x(i)^2$. Then for all $\mathbf{x} \in \mathbb{X} \subset \mathbb{N}^p$,

$$q(\mathbf{x}) \leq -2\eta\|\mathbf{x}\| - 2\langle \mathbf{x}, \mu\rangle + K.$$

## Corollaries

1. The CTMC is positive recurrent **if and only if** $\mu \in \textsc{Ncond}(G)$;
2. If $\eta > 0$, then the matching is a.s. perfect infinitely often;
3. If $\eta < 0$, then the matching is a.s. perfect finitely often;
4. Bounds on the expected number of unmatched nodes, to the large-graph limits.

# Comeback to the original problem

- Consider a similar construction on a SBM $\mathbf{G}_n$ of size $n$, that is constructed beforehand.
- For all $k \in [\![1, n]\!]$, let $\tilde{\mathbf{M}}_k$ be the matching constructed by applying the matching criterion $\phi_.$ on the sub-graph induced by the first $k$ nodes of the exploration of $\mathbf{G}_n$, and the corresponding state

$$\tilde{X}_k = (|\tilde{\mathbf{X}}_k(1)|, \cdots, |\tilde{\mathbf{X}}_k(p)|).$$

## Proposition

The processes $(\tilde{X}_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$ of the two constructions are equal in distribution.

# Back to Tutte's condition

## Tutte's Marriage Theorem (1947)

Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph. Then there exists a perfect matching iff for $o(G - U) \leq |U|$ for all $U \subset V$, where $o(G - U)$ is the number of connected components of odd sizes of the induced graph of $V \setminus U$ in $G$.

## Corollary of our results

Tutte's condition is satisfied a.s. infinitely often by the graphs $(\mathbf{G}_n)_{n \in \mathbb{N}}$.

# Work in progress

1. Bounds for the **return times** to the full matching?
2. **Sparse case**: what is the situation whenever the $P_{ij}$'s scale with $n$?


- M. Jonckheere, P. Moyal, C. Ramirez and N. Soprano-Loto. Generalized max-weight policies in stochastic matching. *Stochastic Systems* 13(1), 40-58, 2023.
- N. Soprano-Loto, M. Jonckheere and P. Moyal. Online matching for the multiclass stochastic block model. *ArXiv math.PR* / 2303.15374, 2023+.