Constant Approximation for Private Interdependent Valuations

CIRM – *From matchings to markets*

Alon Eden
Hebrew Univ.

Michal Feldman
Tel Aviv Univ.

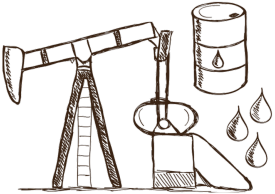
Kira Goldner
Boston Univ.

Simon Mauras
TAU $\rightarrow$ INRIA

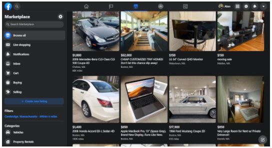
Divya Mohan
Tel Aviv Univ.

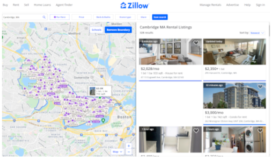
Motivation

Resale model



Mineral Rights





Online platforms

Ad auctions

Auctions with interdependence

- **Input:**

signal $s_1 \in \mathcal{S}$

$v_1 : \mathcal{S}^n \to \mathbb{R}_+$

signal $s_2 \in \mathcal{S}$

$v_2 : \mathcal{S}^n \to \mathbb{R}_+$

$\vdots$

signal $s_n \in \mathcal{S}$

$v_n : \mathcal{S}^n \to \mathbb{R}_+$

- **Output:** allocation rule.
  Distribution $x_1, x_2, \ldots, x_n \in [0, 1]$.

- **Objective:** social welfare.
  Maximize $SW = \sum_i x_i v_i(\mathbf{s})$.

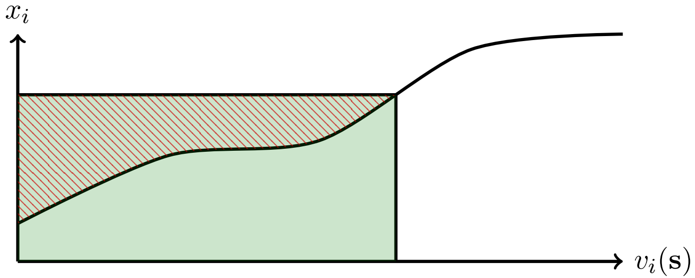
- **Constraint:** truthfulness.
  $x_i(v_i(\mathbf{s}), \mathbf{s}_{-i}, \mathbf{v}_{-i})$ monotone in $v_i(\mathbf{s})$.

- **Guarantee:** approximation ratio.
  Upper bound on OPT/SW.

Myerson's Lemma

Theorem. If $x_i(v_i(\mathbf{s}), \mathbf{s}_{-i}, \mathbf{v}_{-i})$ is **monotone** in $v_i(\mathbf{s})$, there exists a payment scheme which makes **truthful** reporting an optimal strategy.

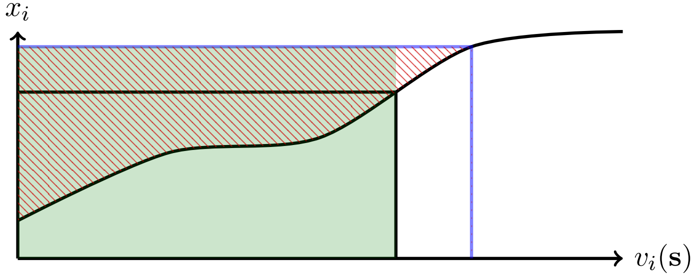
Proof. Set $p_i = \int_0^{v_i(\mathbf{s})} x_i(t, \mathbf{s}_{-i}, \mathbf{v}_{-i}) \mathrm{d}t$.



Assumption. Each agent i maximize her utility $u_i = x_i v_i - p_i$.

Theorem. If $x_i(v_i(\mathbf{s}), \mathbf{s}_{-i}, \mathbf{v}_{-i})$ is **monotone** in $v_i(\mathbf{s})$, there exists a payment scheme which makes **truthful** reporting an optimal strategy.

Proof. Set $p_i = \int_0^{v_i(\mathbf{s})} x_i(t, \mathbf{s}_{-i}, \mathbf{v}_{-i})\mathrm{d}t$.

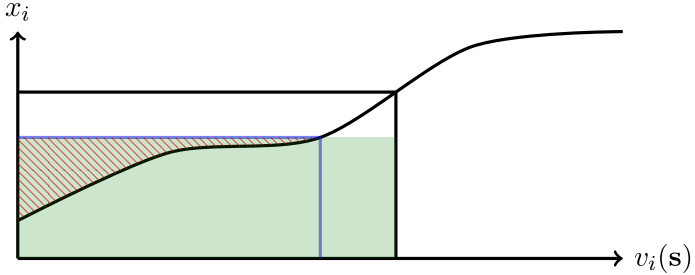


Assumption. Each agent i maximize her utility $u_i = x_i v_i - p_i$.

Myerson's Lemma

Theorem. If $x_i(v_i(\mathbf{s}), \mathbf{s}_{-i}, \mathbf{v}_{-i})$ is **monotone** in $v_i(\mathbf{s})$, there exists a payment scheme which makes **truthful** reporting an optimal strategy.

Proof. Set $p_i = \int_0^{v_i(\mathbf{s})} x_i(t, \mathbf{s}_{-i}, \mathbf{v}_{-i})\mathrm{d}t$.



Assumption. Each agent i maximize her utility $u_i = x_i v_i - p_i$.

Summary of existing results

Public valuations (v_i's are publicly known)

- **Independent values** ($v_i(\mathbf{s}) = s_i$ for all i)
  - $\rightarrow$ optimal welfare [Vickrey'61].
- **Single-crossing** ($\partial v_i / \partial s_i \geq \partial v_j / \partial s_i$ for all $i \neq j$)
  - $\rightarrow$ optimal welfare [Maskin'92].
- **Worst-case valuations**
  - $\rightarrow$ n lower-bound [EFFGK'19].
- **Submodular over signals**
  - $\rightarrow$ 2 lower-bound [EFFGK'19].
  - $\rightarrow$ 3.3 approximation [LSZ'22].

Private valuations (v_i's are reported by agents)

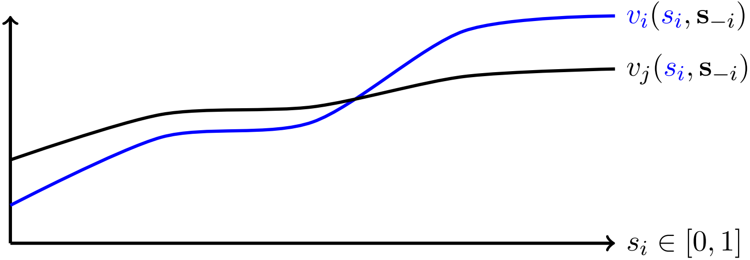
- **Submodular over signals**
  - $\rightarrow$ $\mathcal{O}(\log^2 n)$ approximation [EGZ'22].
  - $\rightarrow$ 5.55 **approximation [this work].**

Part I

Public Valuation Functions

Optimal welfare with single crossing

Assumption: $\partial v_i / \partial s_i \geq \partial v_j / \partial s_i$ for all $i \neq j$.



Algorithm (optimal). Define $x_i := \mathbb{1}[v_i(\mathbf{s}) > \max_{j \neq i} v_j(\mathbf{s})]$.

Lemma. Allocation $x_i(s_i, \mathbf{s}_{-i})$ is **monotone** in s_i.

Corollary. Optimal can be implemented **truthfully**.

Proof. Relies on each $v_i(s_i, \mathbf{s}_{-i})$ being monotone in s_i.

Hardness in the worst case

Intuition: when everyone can decrease other agents' values.

signal $s_1 \in \{0, 1\}$

$v_1(\mathbf{s}) = \varepsilon s_1 + \prod_{i \neq 1} s_i$

signal $s_2 \in \{0, 1\}$

$v_2(\mathbf{s}) = \varepsilon s_2 + \prod_{i \neq 2} s_i$

$\vdots$

signal $s_n \in \{0, 1\}$

$v_n(\mathbf{s}) = \varepsilon s_n + \prod_{i \neq n} s_i$

There exists an agent i such that

$$x_i(1_i, \mathbf{1}_{-i}) \leq 1/n.$$

By monotonicity, we have

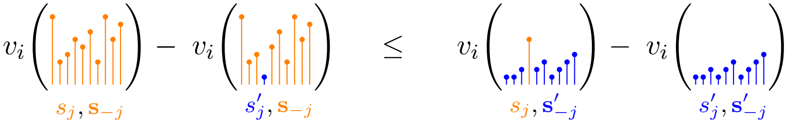
$$x_i(0_i, \mathbf{1}_{-i}) \leq 1/n.$$

Approximation ratio with $\mathbf{s} = (0_i, \mathbf{1}_{-i})$

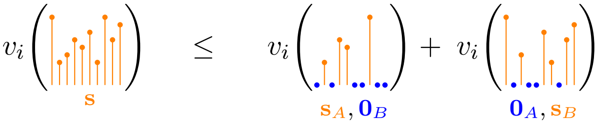
$$\frac{OPT}{SW} \geq \frac{1}{1/n + (n-1)\varepsilon} \approx n.$$

Submodular over signals (SOS)

Definition. For all signal vectors $\mathbf{s} \succeq \mathbf{s}'$, for all i and j, we require that



Lemma (sub-additive). For all signal vector $\mathbf{s}$, we have



With a random $A \subseteq [n]$, we have $v_i(\mathbf{s}) \leq 2\mathbb{E}_A[v_i(\mathbf{0}_A, \mathbf{s}_{-A})]$.

Constant approximation with SOS

Algorithm (from [EFFGK'19])

- Partition buyers $[n] = A \cup B$ at random.
- Winner is $i \in A$ who has the highest $v_i(s_i, \mathbf{s}_B, \mathbf{0}_{A\setminus\{i\}})$.
- Set $x_i = \mathbb{P}_A[i \text{ winner}]$.

Lemma. Allocation $x_i(s_i, \mathbf{s}_{-i})$ is **monotone** in s_i.

Definition. Let $i^\star := \operatorname{argmax}_i v_i(\mathbf{s})$ be the largest value buyer.

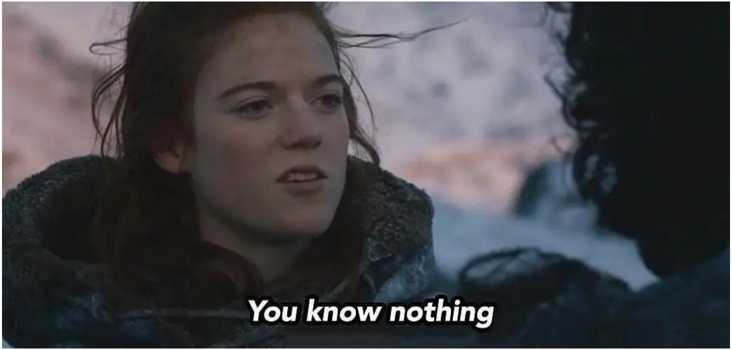
Lemma. $\mathbb{P}_A[i^\star \in A] = 1/2$.

Lemma. $\mathbb{E}_A[v_{i^\star}(s_{i^\star}, \mathbf{s}_B, \mathbf{0}_{A\setminus\{i^\star\}}) \,|\, i^\star \in A] \geq v_{i^\star}(\mathbf{s})/2$.

Theorem. The algorithm is a truthful 4-approximation.

Part II

Private Valuation Functions



Hardness with single crossing!

Intuition: when everyone can decrease other agents' values.

signal $s_1 \in \{0, 1\}$
$v_1(\mathbf{s}) = \prod_i s_i$

signal $s_2 \in \{0, 1\}$
$v_2(\mathbf{s}) = \prod_i s_i$

$\vdots$

signal $s_n \in \{0, 1\}$
$v_n(\mathbf{s}) = \prod_i s_i$

There exists an agent i such that

$$x_i(\mathbf{1}, \mathbf{v}) \leq 1/n.$$

If i lies ($\tilde{s}_i = 0$ and $\tilde{v}_i(\mathbf{s}) = 1 + s_i$)

$$x_i(0_i, \mathbf{1}_{-i}, \tilde{v}_i, \mathbf{v}_{-i}) \leq 1/n.$$

Approx. ratio for $(0_i, \mathbf{1}_{-i}, \tilde{v}_i, \mathbf{v}_{-i})$

$$\frac{OPT}{SW} \geq \frac{1}{1/n} = n.$$

Warmup mechanism

Assumption: each valuation only depends on d signals.

Algorithm (from [EGZ'22])

- Buyer j is a **candidate** if $v_j(\mathbf{s}) > v_i(0_j, \mathbf{s}_{-i})$ for all $i \neq j$, with ties broken in favor of lower indices.
- Set $x_j = 1/(d+1)$ if j is a candidate, and $x_j = 0$ otherwise.

Lemma. Candidate $i^\star := \operatorname{argmax}_i v_i(\mathbf{s})$ is a candidate.

Lemma. There are at most $d+1$ candidates.

Proof. For each candidate $j \neq i^\star$, we must have

$$v_{i^\star}(\mathbf{s}) > v_j(\mathbf{s}) > v_{i^\star}(0_j, \mathbf{s}_{-j})$$

hence $v_{i^\star}(\mathbf{s})$ depends on j's signal.

Theorem. The algorithm is a truthful $(d+1)$-approximation.

Submodular over signals

Definition. For all signal vectors $\mathbf{s} \succeq \mathbf{s}'$, for all i and j, we require that

$$v_i\left(s_j, \mathbf{s}_{-j}\right) - v_i\left(s'_j, \mathbf{s}_{-j}\right) \leq v_i\left(s_j, \mathbf{s}'_{-j}\right) - v_i\left(s'_j, \mathbf{s}'_{-j}\right)$$

Lemma (self-bounding). For all signal vector $\mathbf{s}$, we have

$$v_i\left(\mathbf{s}\right) \geq \sum_{j \in [n]} v_i\left(s_j, \mathbf{s}_{-j}\right) - v_i\left(0_j, \mathbf{s}_{-j}\right)$$

There is at most one j such that $v_i(0_j, \mathbf{s}_{-j}) < v_i(\mathbf{s})/2$.

Attempt #1

Assumption: each valuation is submodular over signals.

Algorithm #1 (with parameter $C \geq 1$)

- $f(v) :=$ round down v to the nearest 2^k.
- Buyer j is a **candidate** if $f(v_j(\mathbf{s})) > f(v_i(0_j, \mathbf{s}_{-i}))$ for all $i \neq j$, with ties broken in favor of lower indices.
- Set $x_j = \mathbb{1}[i \text{ is a candidate}]/C$.

Lemma. Candidate $i^\star := \text{argmax}_i\ f(v_i(\mathbf{s}))$ is a candidate.

~~**Lemma.** There are at most a constant number C of candidates.~~

Does not work! If all $v_i(\mathbf{s}) = 2.001$ and $v_i(0_j, \mathbf{s}_{-j}) = 1.999$.

Attempt #2

Assumption: each valuation is submodular over signals.

Algorithm #2 (with parameter $C \geq 1$)

- Draw $r \in [0, 1]$ uniformly at random.
- $f_r(v) :=$ round down v to the nearest 2^{k+r}.
- Buyer j is a **candidate** if $f_r(v_j(\mathbf{s})) > f_r(v_i(0_j, \mathbf{s}_{-i}))$ for all $i \neq j$, with ties broken in favor of lower indices.
- Set $x_j = \mathbb{P}_r[i \text{ is a candidate}]/C$.

Lemma. Candidate $i_r^\star := \operatorname{argmax}_i f_r(v_i(\mathbf{s}))$ is a candidate.

~~**Lemma.** There are at most a constant number C of candidates.~~

Does not work! Low indices having low values $\rightarrow \sqrt{n}$ candidates.

Attempt #3

Assumption: each valuation is submodular over signals.

Algorithm #3 (with parameter $C = 4$)

- Draw $r \in [0, 1]$ and permutation π uniformly at random.
- $f_r(v)$:= round down v to the nearest 2^{k+r}.
- Buyer j is a **candidate** if $f_r(v_j(\mathbf{s})) > f_r(v_i(0_j, \mathbf{s}_{-i}))$ for all $i \neq j$, with ties broken using π.
- Set $x_j = \mathbb{P}_{r,\pi}[i \text{ is a candidate}]/C$.

Lemma. Candidate $i^\star_{r,\pi} := \operatorname{argmax}_i \left(f_r(v_i(\mathbf{s})), \pi_i\right)$ is a candidate.

Lemma. There are at most $C = 4$ candidates in expectation.

Theorem. The algorithm is a truthful $(8 \ln 2)$-approximation.

Proof. Calculation gives $\mathbb{E}_{r,\pi}[v_{i^\star_{r,\pi}}(\mathbf{s})] \geq \max_i v_i(\mathbf{s})/(2 \ln 2)$.

Extension #1

Extension: non-monotone valuation functions

- Buyer 1 knows the color: $s_1 \in \{\bullet, \bullet, \bullet, \bullet, ...\}$
- Buyer 2 knows the shape: $s_2 \in \{\bigcirc, \triangle, \square, \lozenge, ...\}$
- ...

Remark: with unordered signal spaces

- $v_j(0_i, \mathbf{s}_{-i})$ is undefined.
- Submodular over signals is undefined

Idea: replace zeros by infimums

- Replace $v_i(0_j, \mathbf{s}_{-j})$ by $\underline{v}_i(\mathbf{s}_{-j}) := \inf\{v_i(\tilde{s}_j, \mathbf{s}_{-j}) \mid \tilde{s}_j \in \mathcal{S}_j\}$
- Self-bounding is well defined: $v_i(\mathbf{s}) \geq \sum_{j \in [n]} (v_i(\mathbf{s}) - \underline{v}_i(\mathbf{s}_{-j}))$.

Extension #2

Definition. Valuation v_i is d-self-bounding (with $d \geq 1$) if

$$d \cdot v_i(\mathbf{s}) \geq \sum_{j \in [n]} (v_i(\mathbf{s}) - \underline{v}_i(\mathbf{s}_{-i}))$$

Hierarchy of valution functions:

- monotone SOS is 1-self-bounding
- non-monotone SOS is 2-self-bounding
- ...
- any function is n-self-bounding.

Lemma. There are at most $C = 2 + 2d$ candidates in expectation.

Theorem. There is a truthful $\mathcal{O}(d)$ approximation (oblivious to d).

Proof. Set $x_i = \mathbb{P}_{r,\pi}[i \text{ is a candidate}]/(4 + 4 \max_{j \neq i} d_j)$.

Conclusion

Results:

- Constant approximation with private SOS valuations.

Extensions:

- Non-monotone valuation functions.
- Parametrized valuations: d-self-bounding $\rightarrow \mathcal{O}(d)$-approximation.
- Allocating multiple (identical) items.

Future works:

- Allocating multiple (non-identical) items.
- Interdependence with other optimization problems.

Thank you!