Local analysis of matchings on sparse graphs

Asymptotic local law of matchings on random graphs

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Erdös-Renyi model

Let $n \in \mathbb{N}^*$, $p \in [0,1]$

Definition

$G_{n,p}$ is the random graph on $V = \{1, 2, ..., n\}$ such that independently on every (i,j), $\mathbb{P}((i,j) \in E) = p$.

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Let $\mathcal{W}$ be some law on $\mathbb{R}_+$ with finite expectation. (for example, uniform on $[0, 1]$).

Definition

$G_{n,p,\mathcal{W}}$ is the weighted random graph generated on $G_{n,p} = (V, E)$ by drawing $(W(e))_{e \in E}$ i.i.d of law $\mathcal{W}$.

Maximum matching, Minimum cost maximum matching

Let $G = (V, E, w)$ a weighted graph.

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- We say that a matching is of maximum cardinality of its size is maximal among all matchings.
- We say that it is a minimum cost maximum matching if the sum of the weight on its edge is minimum among matchings of maximum cardinality.

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Let $H \in \mathbb{N}$, we define $N_H(G') (= N_H(o))$ the $H-$neighboorhood of the root $o = (o_-, o_+)$ in G' by the set of edges at distance less than H from the root and the set of vertices at distance less than $H - 1$ from o_- or o_+.

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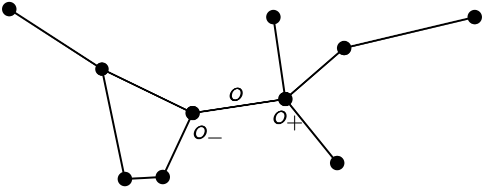

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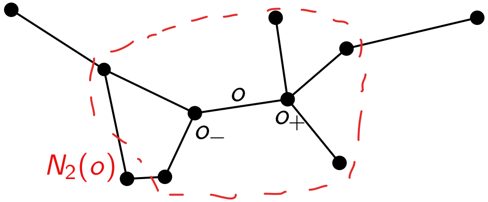

Local convergence (2)

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Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of rooted random graphs.

Definition

We say that G_n converges locally to G if for every $H \in \mathbb{N}$, the $H-$neighborhood of the root of G_n converges in law to the $H-$neighborhood of the root of G.

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Context for main theorems

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Assume that $\mathcal{W}$ is a continuous law with no atoms.
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For any n, let us root $G_{n,\frac{c}{n},\mathcal{W}}$ by picking o_n uniformly among the edges.
Let us call $\mathcal{T}_{\mathcal{W}}$ the weighted edge-rooted Galton-Watson tree of reproduction law $\mathcal{P}(c)$ with i.i.d weights on edges drawn with law $\mathcal{W}$.
It is known that $G_{n,\frac{c}{n},\mathcal{W}}$ converge locally to $\mathcal{T}_{\mathcal{W}}$.

Main theorems

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Theorem (2024, Liu et al.)

$((G_{(n,\frac{c}{n},\mathcal{W})}, o_n), (\mathcal{M}_n^{opt}, o_n))$ converge locally to $(\mathcal{T}_{\mathcal{W}}, \mathcal{M})$ where $\mathcal{M}$ is a matching on $\mathcal{T}_{\mathcal{W}}$ that can be described.

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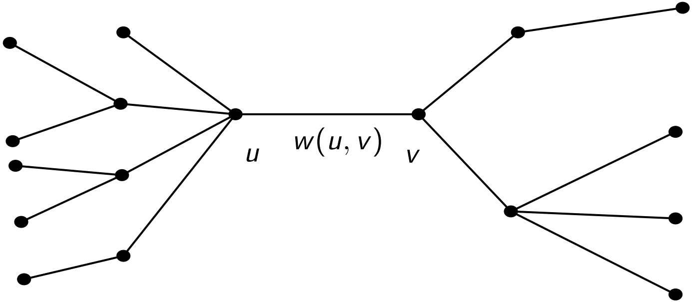
Dynamic programming heuristic



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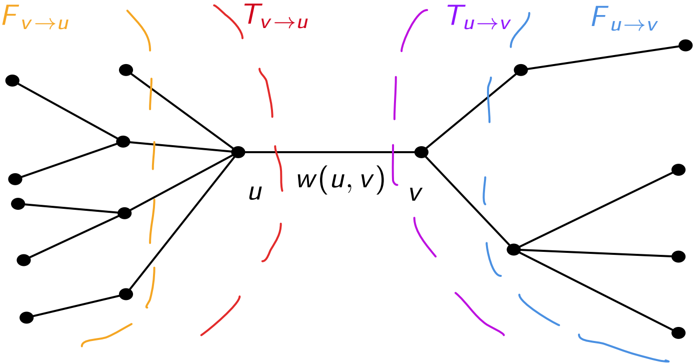

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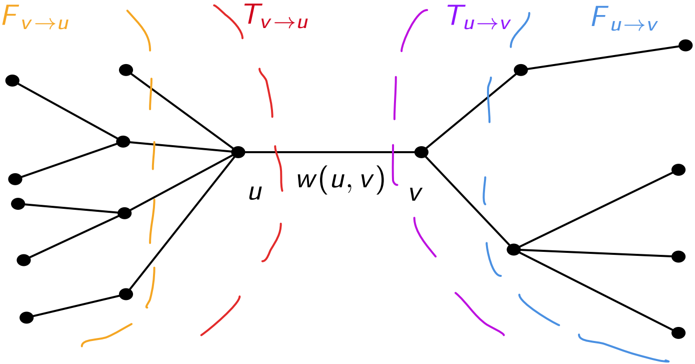

$Z(v, u) :=$
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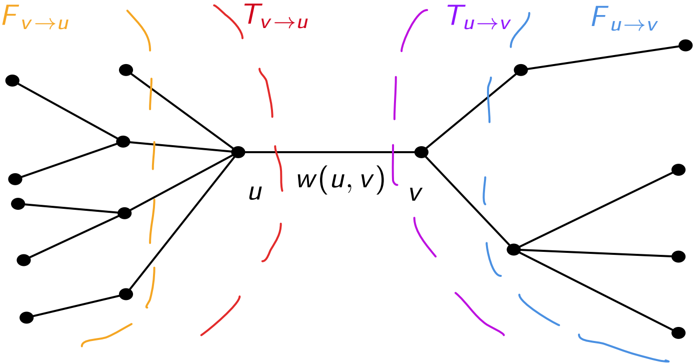

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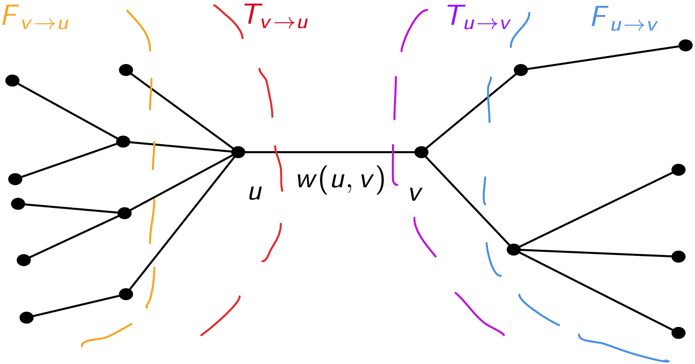

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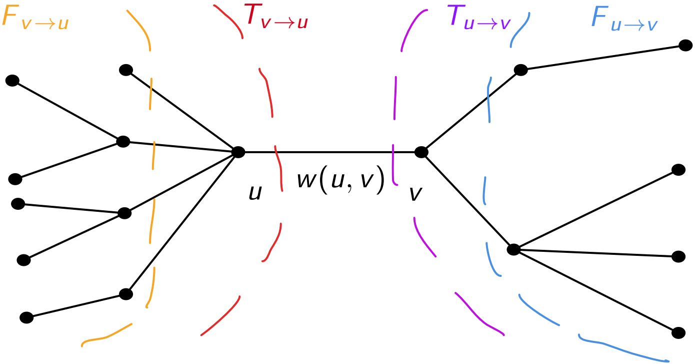

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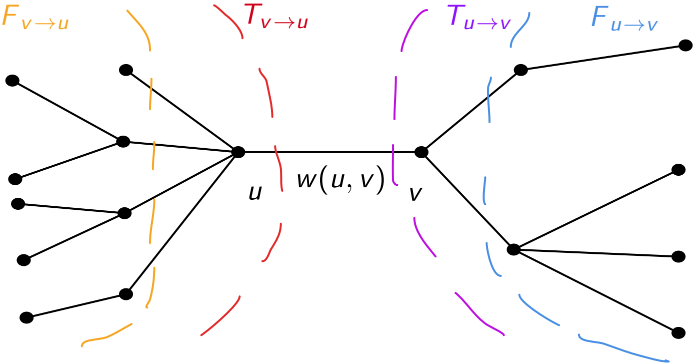

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$(u,v) \in OPT$

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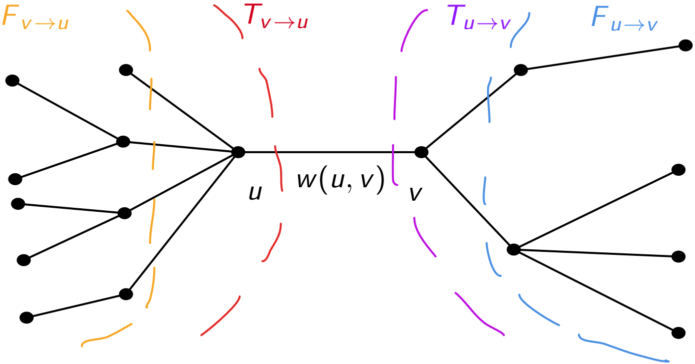

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$$(u,v) \in OPT \quad \Leftrightarrow OPT_{\text{forcing }(u,v)}(T) - OPT_{\text{without }(u,v)}(T) > 0$$

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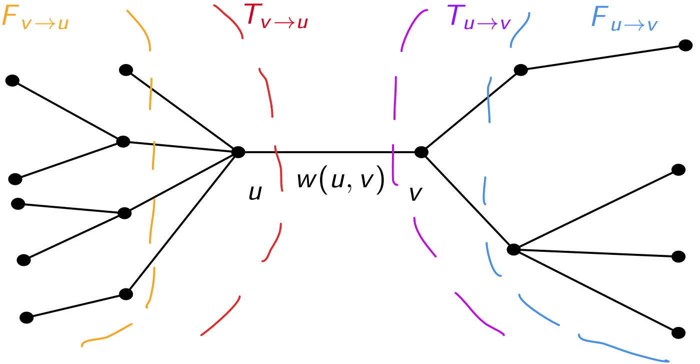

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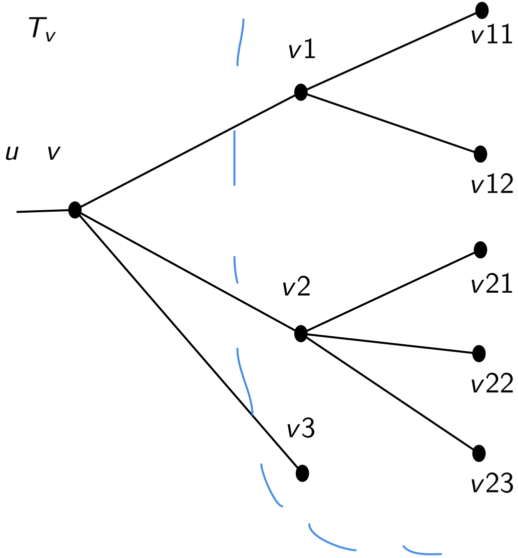
$$(u,v) \in OPT \Leftrightarrow OPT_{\text{forcing }(u,v)}(T) - OPT_{\text{without }(u,v)}(T) > 0$$
$$\Leftrightarrow Z(u,v) + Z(v,u) < w(u,v).$$

Recursive equation on heuristic



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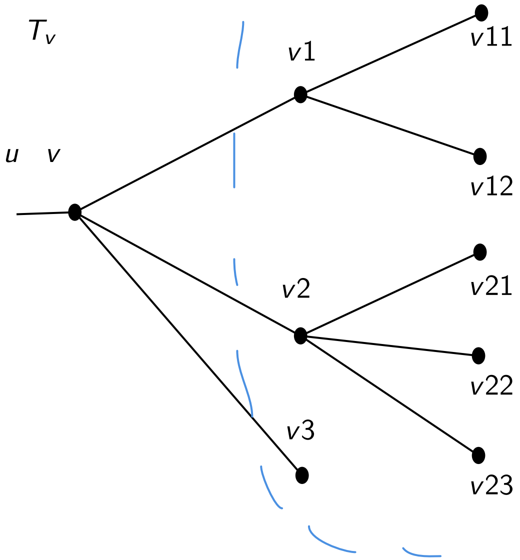

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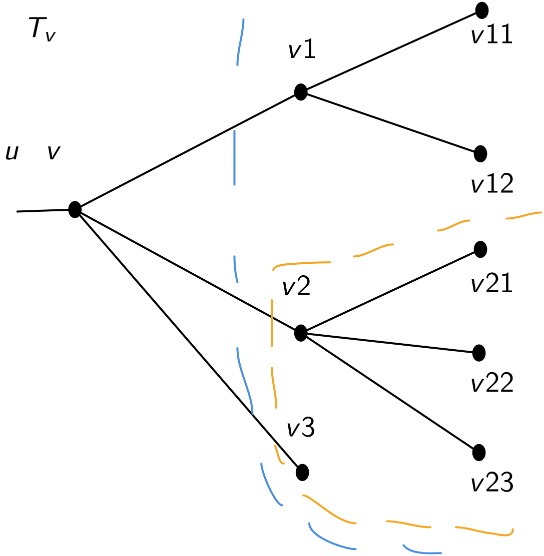
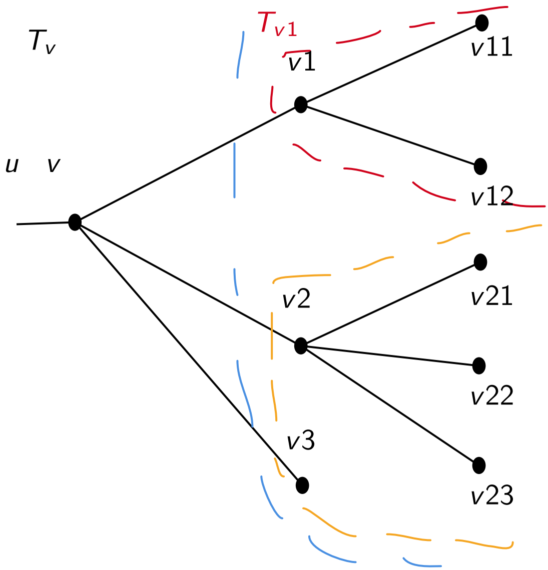
Recursive equation on heuristic



Assume that $OPT(T_v)$ picks v1, OPT(orange) is in common between $OPT(T_v)$ and $OPT(F_{u \to v})$.

Recursive equation on heuristic



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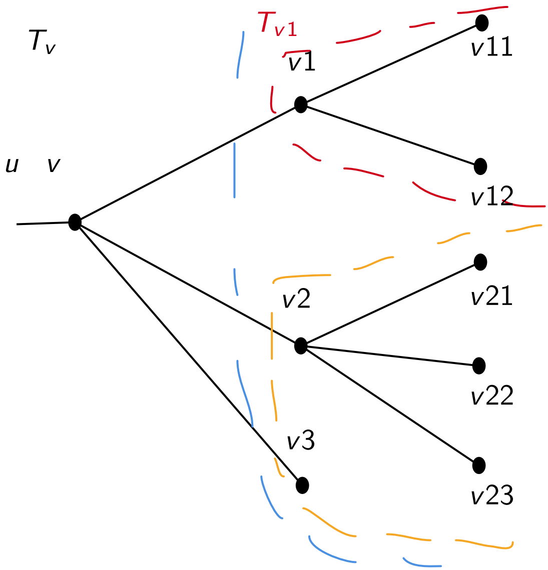
In red, $OPT(T_v)$ optimizes $OPT(T_{v11})$ and $OPT(T_{v12})$.

$OPT(F_{u \to v})$ optimizes $OPT(T_{v1})$.

Recursive equation on heuristic



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$OPT(F_{u\to v})$ *optimizes* $OPT(T_{v1})$.
Hence $Z(u,v) = OPT(T_v) - OPT(F_{u\to v})$
$= w(v, v1) - OPT(T_{v1}) + OPT(T_{v11})$
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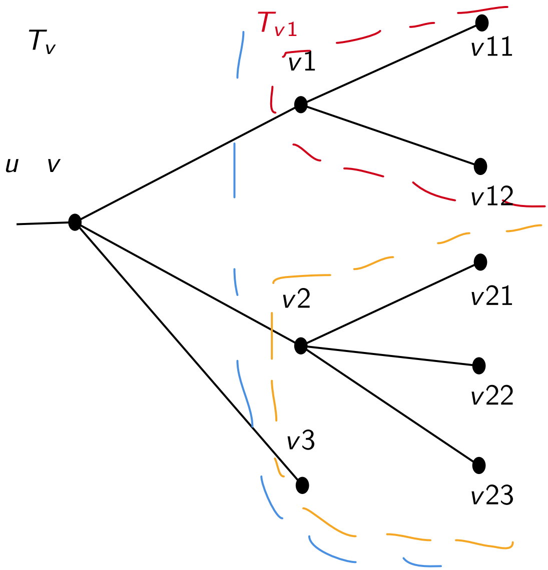
Recursive equation on heuristic



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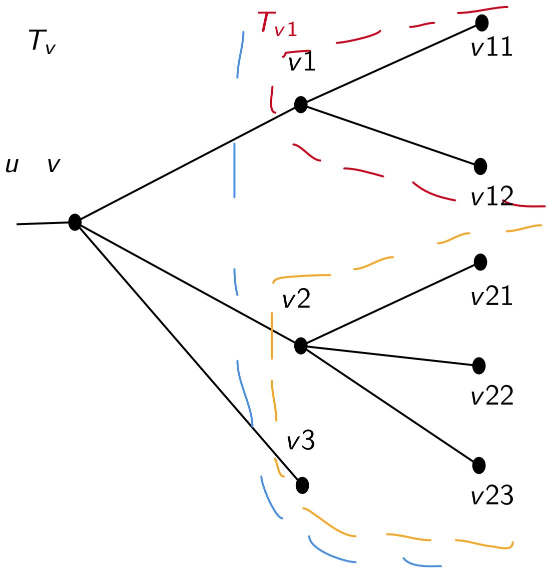

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$+ OPT(T_{v12})$
$= w(v, v1) - Z(v, v1)$
$v1$ *has to be maximal for this quantity hence :*

$$Z(u, v) = max_{w \sim v, w \neq u}(0, w(v, w) - Z(v, w))$$

Equation in law

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Assume there exists a law $\mathcal{L}$ such that for Z, Z_k i.i.d of law $\mathcal{L}$, w_k of law $\mathcal{W}$ and $K \sim \mathcal{P}(c)$ all mutually independent such that:

$$Z \stackrel{\mathcal{D}}{=} \max(0, \max_{1 \leq k \leq K} (w_k - Z_k)).$$

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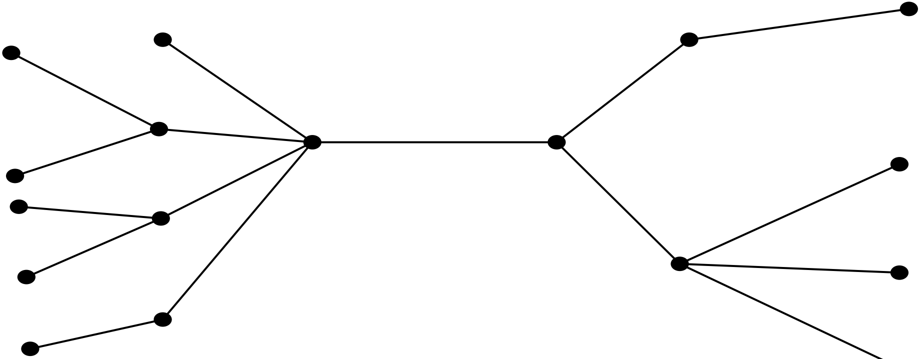
Fix $H \in \mathbb{N}$. We define Z in $N_H(o)$ of $T_{\mathcal{W}}$ in a way such that the law is compatible as H increases.

Existence of Z on T and Kolmogorov



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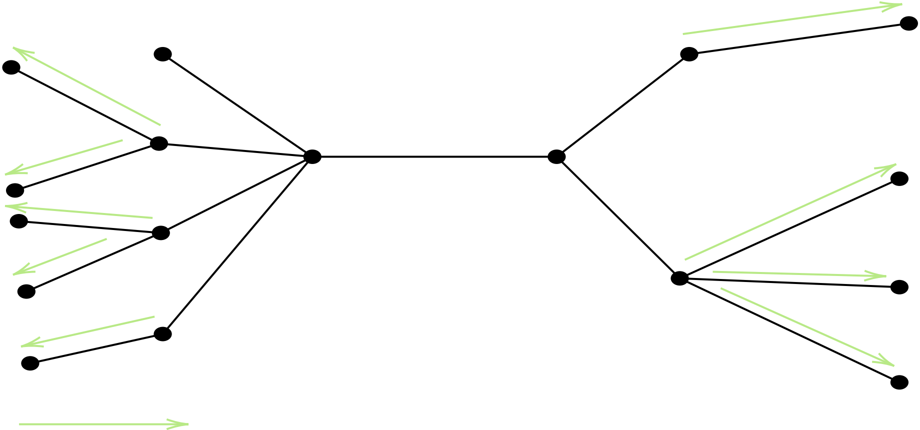
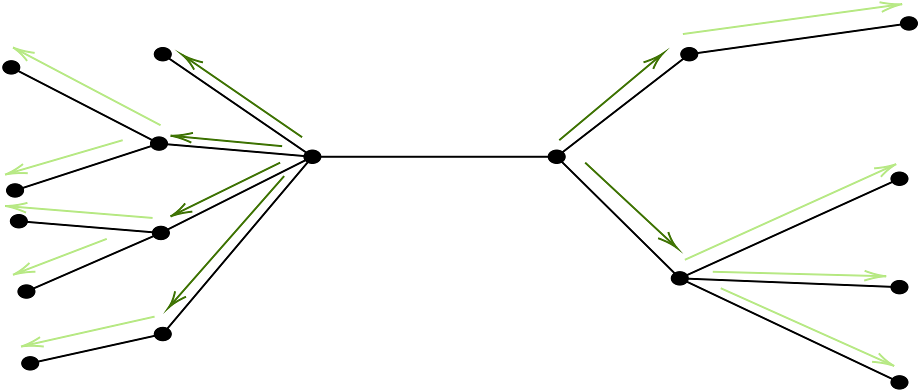


Z *drawn i.i.d with law* $\mathcal{L}$

Existence of Z on T and Kolmogorov



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Z *first generation constructed by recursion*

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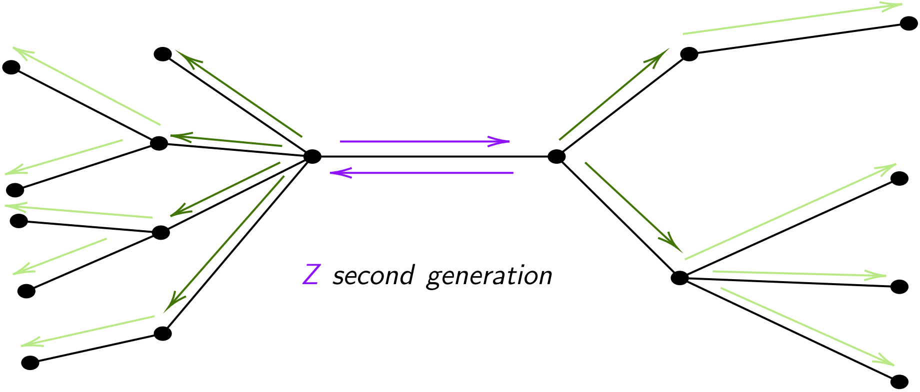

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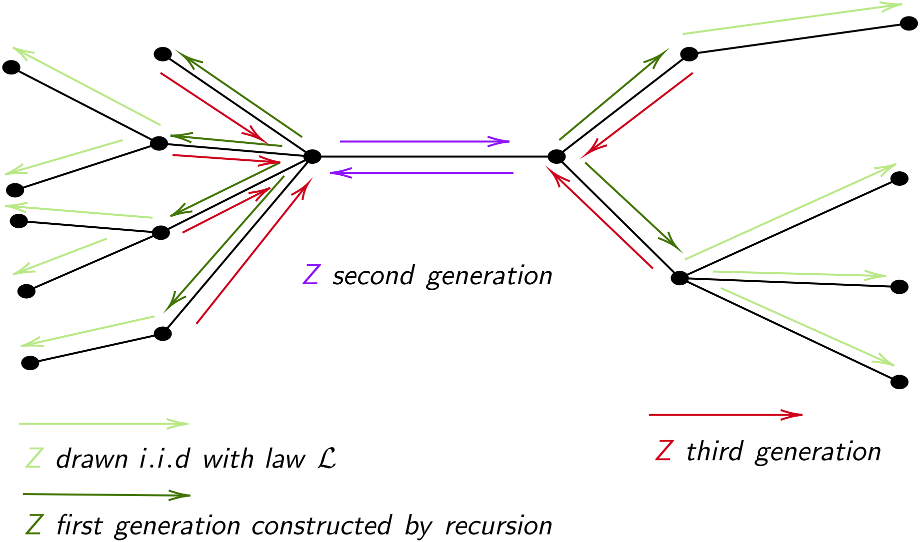

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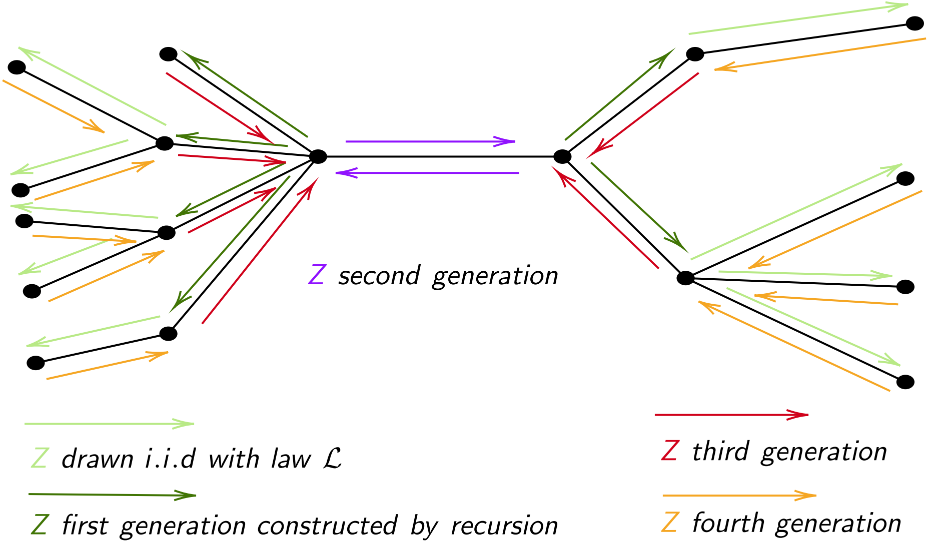

Description of $\mathcal{M}$

We now define $\mathcal{M}$ as the set of (u, v) such that

$$Z(u, v) + Z(v, u) < W(u, v).$$

This defines the limit law in Theorem 1.

Description of $\mathcal{M}^{mc}$

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Replace $\mathcal{W}$ by $\mathcal{W}_\epsilon$ the law of $1 - \epsilon W$ for $W \sim \mathcal{W}$ and $\epsilon > 0$.

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Replace $\mathcal{W}$ by $\mathcal{W}_\epsilon$ the law of $1 - \epsilon W$ for $W \sim \mathcal{W}$ and $\epsilon > 0$.
As ϵ goes to 0 and through renormalization methods, it is possible to show that we need to replace Z by:
$\Phi = (\Phi_0, \Phi_1)$ when $c \leq e$,
$\Phi = (\Phi_0, \Phi_{\frac{1}{2}}, \Phi_1)$ when $c > e$. with:

$$\Phi_0(u,v) = \max(0, \max_{\substack{w \sim v \\ w \neq u}}(-w(v,w) - \Phi_1(v,w))),$$

$$\Phi_1(u,v) = \max_{\substack{w \sim v \\ w \neq u}}(-w(v,w) - \Phi_0(v,w)),$$

$$\Phi_{\frac{1}{2}}(u,v) = \max_{\substack{w \sim v \\ w \neq u}}(-w(v,w) - \Phi_{\frac{1}{2}}(v,w)).$$

Description of $\mathcal{M}^{mc}(2)$

We then define $\mathcal{M}^{mc}$ as the set of (u, v) such that :

$$\Phi_0(u, v) + \Phi_1(v, u) < -w(u, v)$$

or

$$\Phi_1(u, v) + \Phi_0(v, u) < -w(u, v)$$

or

$$\Phi_{\frac{1}{2}}(u, v) + \Phi_{\frac{1}{2}}(v, u) < -w(u, v).$$

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Asymptotic costs

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Asymptotic cost of minimum cost maximum matching on Erdös-Renyi Graph

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{e\in\mathcal{M}_n^{\text{opt mc}}} w(e)\right] = \frac{c}{2}\mathbb{E}\left[w(o)\mathbb{1}_{o\in\mathcal{M}^{mc}}\right]$$

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We can also recover the known Karp-Sipser formula of maximum matching size

Corollary

$$\lim_{n\to\infty} \frac{1}{n}\left(\textit{Size of maximum matching on } G_{n,\frac{c}{n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{e\in\mathcal{M}_n^{opt\ mc}} 1\right] = \frac{c}{2}\mathbb{E}\left[\mathbb{1}_{o\in\mathcal{M}^{mc}}\right]$$

Conditioning on local geometry

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For either $\mathcal{M}_n = \mathcal{M}_n^{opt}$ or $\mathcal{M}_n^{\text{opt mc}}$, we can also compute any conditioning on local property such as:

$$\lim_{n\to\infty} \mathbb{P}\left(e \text{ is matched by } \mathcal{M}_n | w(e) \leq x\right)$$

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for some fixed x, or

$$\lim_{n\to\infty} \mathbb{P}\left(e \text{ is matched by } \mathcal{M}_n | \text{the degrees of } e \text{ are 3 and 4}\right)$$

Conditioning on local geometry

For either $\mathcal{M}_n = \mathcal{M}_n^{opt}$ or $\mathcal{M}_n^{\text{opt mc}}$, we can also compute any conditioning on local property such as:

$$\lim_{n\to\infty} \mathbb{P}\left(e \text{ is matched by } \mathcal{M}_n | w(e) \leq x\right)$$

for some fixed x, or

$$\lim_{n\to\infty} \mathbb{P}\left(e \text{ is matched by } \mathcal{M}_n | \text{the degrees of } e \text{ are 3 and 4}\right)$$

In general, fix some $H \in \mathbb{N}$, if $\mathcal{G}$ is a collection of rooted weighted graphs and $\mathcal{G}_H$ the collection of their $H-$neighborhood of their roots, we can compute

$$\lim_{n\to\infty} \mathbb{P}\left(e \text{ is matched by } \mathcal{M}_n | N_H(e_n) \in \mathcal{G}_H\right).$$

Conditioning on local geometry (2)

For the second limit, the limit will be the probability that $o \in \mathcal{M}$ with the following configuration (with independent weights drawn on edges) :

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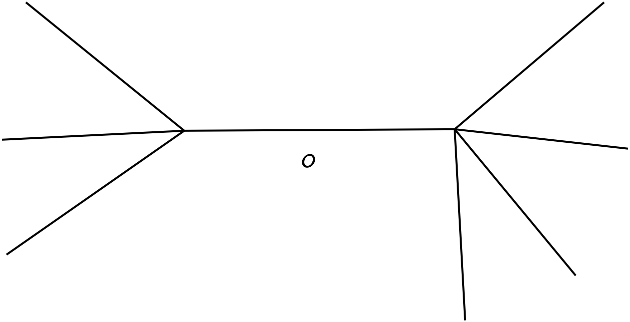

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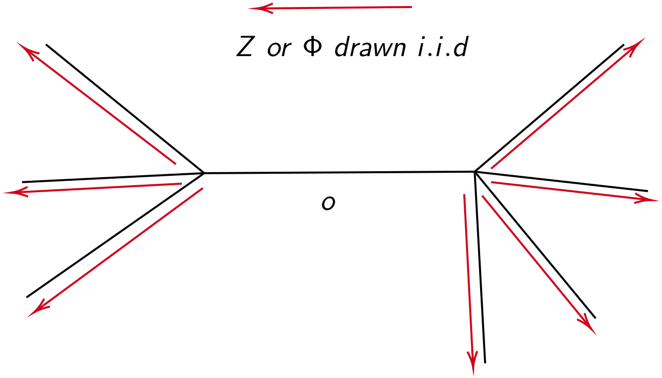

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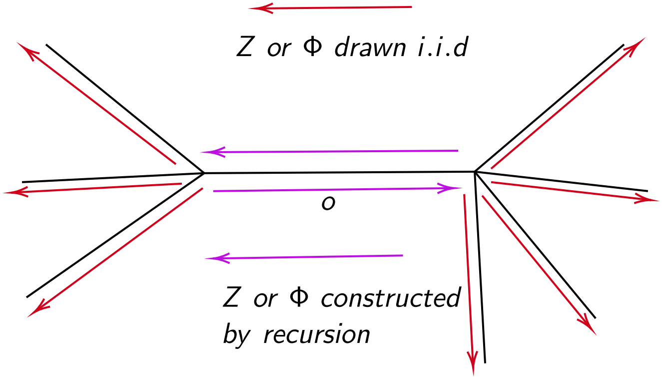

Surprising statistics

Surprisingly, there are some quantities that do not depend on the law $\mathcal{W}$. It is possible to compute

$$\lim_{n\to\infty} \mathbb{P}\left(e = (u,v) \text{ is matched by } \mathcal{M}_n^{\text{opt mc}} | deg(v) = k\right)$$

for any $k \in \mathbb{N}$ and any law $\mathcal{W}$ and show it is equal to :

$$\frac{1 - (1 + \frac{\ln(\beta)}{c})^k}{c},$$

where

$$\beta = \underline{\gamma} + \overline{\gamma} + \underline{\gamma}\overline{\gamma} - 1$$

with $\underline{\gamma}$ (resp. $\overline{\gamma}$) the smallest (resp. biggest) solution to $e^{-ce^{-cx}} = x$.