Efficient and Strategy-proof Mechanism under General Constraints

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Matching under Constraints

Many real-life matching markets are often subject to constraints.

- Type-specific quotas
- Regional quotas
- Knapsack constraints
- Minimum quotas
- Proportional constraint

Desirable Properties

Three important properties in market design.

1. (one-sided) Pareto efficiency (PE)
2. Strategy-proofness (SP)
3. Individual rationality (IR)

How to design mechanism to satisfy the properties under constraints?

Two Models

1. Model with endowment
2. Model without endowment

Individual rationality

1. Model with endowment
   - Each student is assigned to a school that is at least as good as her endowment.
2. Model without endowment
   - Each student is assigned to a school that is at least as good as outside option.

Model with Endowment: Existing Results

Top Trading Cycles (TTC) satisfies PE, IR and SP under

- Capacity constraints
- Maximum and minimum quotas
- M-convex set

No mechanism satisfies PE, IR and SP under

- Knapsack constraints
- Generalized upper bound

Example: No PE, IR, and SP

Three students: i_1, i_2, i_3

$$s_3 \succ_{i_1} s_1 \succ_{i_1} s_2$$

$$s_3 \succ_{i_2} s_1 \succ_{i_2} s_2$$

$$s_2 \succ_{i_3} s_3$$

Three schools: s_1, s_2, s_3

$$\mathcal{F}_{s_1} = \mathcal{F}_{s_3} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}\} \text{ and } \mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_1, i_2\}\}$$

$\mu^E = \{(i_1, s_2), (i_2, s_2), (i_3, s_3)\}$ is endowment.

What constraint structure will be a key for the existence of PE, IR, and SP mechanism?

Model without Endowment

Serial Dictatorship (SD) satisfies PE, IR, and GSP under

- Knapsack constraints
- Generalized upper bound

Even if constraint is not general upper bound, SD can satisfy PE, IR, and GSP.

GUB is not Necessary

Two students: i_1, i_2

Two schools: s_1, s_2

$$\mathcal{F}_{s_1} = \{\emptyset, \{i_1\}, \{i_1, i_2\}\}, \mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}\}$$

$\mathcal{F}_{s_1}$ is not GUB, but PE, IR, and GSP mechanism exists.

However,

$$\mathcal{F}'_{s_1} = \{\emptyset, \{i_1, i_2\}\}, \mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}\}$$

No mechanism satisfies PE, IR, and GSP.

Example: No PE, GSP, and IR

$$\mathcal{F}'_{s_1} = \{\emptyset, \{i_1, i_2\}\}, \mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}\}$$

$$P^1 = (s_2 \succ_{i_1} \emptyset \succ_{i_1} s_1,\ s_1 \succ_{i_2} \emptyset \succ_{i_2} s_2)$$
$$P^2 = (s_2 \succ_{i_1} s_1 \succ_{i_1} \emptyset,\ s_1 \succ_{i_2} \emptyset \succ_{i_2} s_2)$$
$$P^3 = (s_2 \succ_{i_1} s_1 \succ_{i_1} \emptyset,\ s_1 \succ_{i_2} s_2 \succ_{i_2} \emptyset)$$
$$P^4 = (s_2 \succ_{i_1} s_1 \succ_{i_1} \emptyset,\ s_2 \succ_{i_2} s_1 \succ_{i_2} \emptyset)$$

Which constraint structure will be a key for the existence of PE, IR, and GSP mechanism?

Overview

1. Model with Endowment
   - Generalized matroid (g-matroid) is necessary and sufficient for PE, IR, and SP.
   - Generalized TTC based on Suzuki et al. (2018) satisfies PE, IR and GSP under g-matroid.

2. Model without Endowment
   - Accessibility is necessary for PE, IR and GSP.
   - Serial dictatorship (SD) mechanism satisfies PE, IR and GSP under ordered-accessibility.

Model

$(I, S, (\succ_i)_{i \in I}, (\mathcal{F}_s)_{s \in S})$

- I: set of finite students
- S: set of finite schools
- $\succ_i$: strict preference over $S \cup \{\emptyset\}$
- $\mathcal{F}_s \subseteq 2^I$: constraint of school s
- μ^E: endowment
  - $\mu^E = \emptyset$ in model without endowment
  - Suppose $\mu^E \in \mathcal{F}$

Generalized Matroid

Definition

$\mathcal{F}$ is generalized matroid (g-matroid) if for any $X, Y \in \mathcal{F}$ and $x \in X \setminus Y$, either of the following holds.

1. $X \setminus \{x\} \in \mathcal{F}$ and $Y \cup \{x\} \in \mathcal{F}$.
2. There exists $y \in Y \setminus X$ such that $(X \setminus \{x\}) \cup \{y\} \in \mathcal{F}$ and $(Y \setminus \{y\}) \cup \{x\} \in \mathcal{F}$.

Necessary and Sufficient Constraint Structure

Theorem

- Generalized TTC satisfies PE, IR and SP (GSP) if $\mathcal{F}$ is g-matroid.
- Fix a set of students I with $|I| \geq 3$, a set of schools S with $|S| \geq 3$, and a school s^* with constraint $\mathcal{F}_{s^*}$. Suppose that $\mathcal{F}_{s^*}$ is not a g-matroid. Then, there must exist $((\mathcal{F}_s)_{s \in S}, \mu^E)$ with $s^* \in S$ and $\mathcal{F}_s = \{X \subseteq I: |X| \leq 1\}$ for all $s \in S \setminus \{s^*\}$ such that no mechanism satisfies PE, SP, and IR.

Model without Endowment

Necessity: Accessibility

Definition

Constraint $\mathcal{F}$ is accessible if for any $X \in \mathcal{F} \setminus \{\emptyset\}$, there exists $x \in X$ such that $X \setminus \{x\} \in \mathcal{F}$.

Theorem

Fix a set of students I with $|I| \geq 2$, a set of schools S with $|S| \geq 2$, and a school s^* with constraint $\mathcal{F}_{s^*}$. Suppose that $\mathcal{F}_{s^*}$ is not accessible. Then, there must exist $(\mathcal{F}_s)_{s \in S}$ with $s^* \in S$ and $\mathcal{F}_s = \{X \subseteq I : |X| \leq 1\}$ for all $s \in S \setminus \{s^*\}$ such that no mechanism satisfies PE, IR, and GSP.

Sufficiency: Ordered Accessibility

$\sigma \in \Sigma$: permutation of students

Definition

A constraint $\mathcal{F}$ is σ-accessible if for any non-empty matching $\mu \in \mathcal{F} \setminus \{\emptyset\}$ and $i \in \mathrm{argmax}\{\sigma^{-1}(i): i \in I, \mu(i) \neq \emptyset\}$

$$\mu \setminus \{(i, \mu(i))\} \in \mathcal{F}$$

This constraint appears in school choice in China (Huang, 2021), medical resource allocation under pandemic (Dur, Morrill, and Phan 2021), and dynamic matching (Bando and Kawasaki, 2021).

Theorem

If the constraint is σ-accessible for $\sigma \in \Sigma$, SD with σ satisfies PE, IR and GSP.

Relation to Existence of Stable Matchings

Choice function: $C: 2^I \to 2^I$ where $C(X) \subseteq X$ for all $X \in 2^I$

Feasibility constraint induced by C

$$\mathcal{F} = \{X \subseteq I: C(X) = X\}$$

- Path independent choice function C induces GUB $\mathcal{F}$.
- Unidirectional substitutes and complements conditions induces σ-accessible constraint.
- Non-accessible constraint is associated with stronger complementarities: there exists $X \in 2^I$ with $C(X) \neq \emptyset$ such that

$$C(C(X) \setminus \{i\} \subsetneq C(X) \setminus \{i\} \text{ for all } i \in C(X)$$

Remarks on Model without Endowment

1. Empty matching is feasible
   - If not, IR cannot be satisfied.
   - Dropping IR or assuming that outside option is worst for everyone, PE and GSP mechanism exists under general constraints.
2. Endowment is fixed to empty matching
   - Necessity in endowment scenario does not hold.

Classes of Constraints

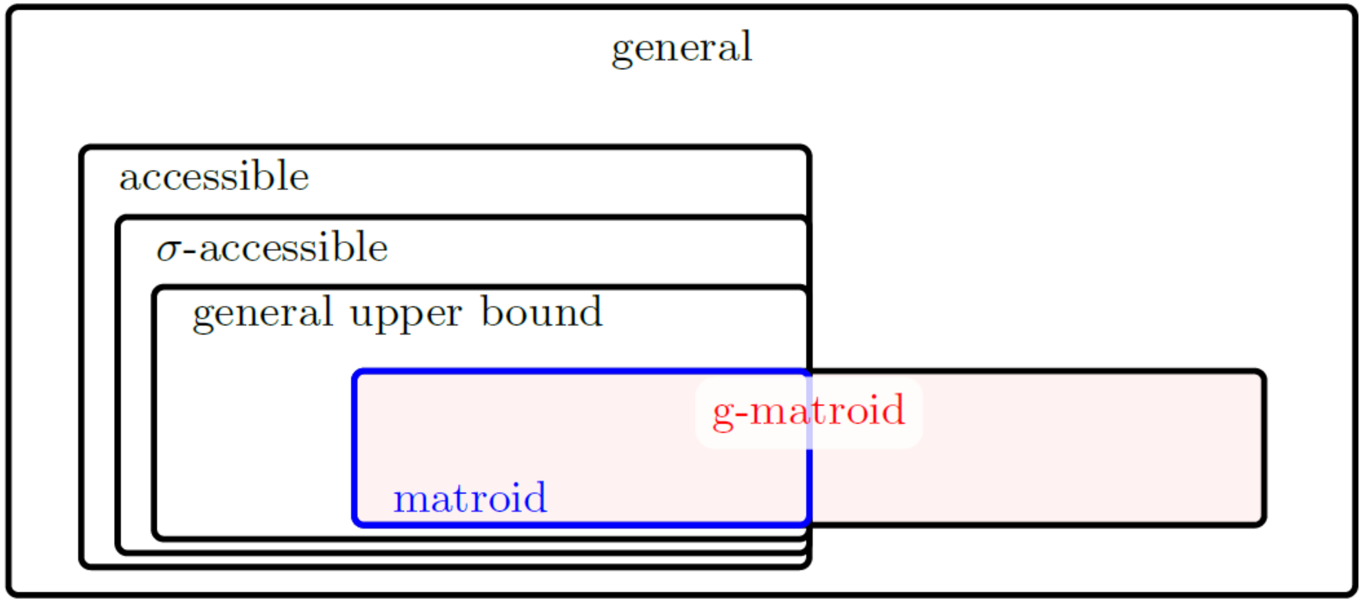

Figure 1: Classes of constraints we dealt with in this study.

Generalized TTC

1. PairをNodeにする必要があること
2. Cycleの選び方をFeasibilityとSPのために工夫する必要があること
   1. Shortest cycle
   2. Common Priorityを作りHigh priorityを優先する
3. 例でみせる
4. YRMH-IGYT, TTCC with PE and SP chain rule

Examples of Constraints

1. Overlapping types
   1. One-to-one
   2. One-to-all

Discussion

1. Necessity of PE, IR, and SP is open question.
2. Any two of PE, IR, and GSP can be achieved for general constraints

- Allocating Medical Resources During a Pandemic, Umut Dur, Thayer Morrill, William Phan, 2021
- Unidirectional substitutes and complements, Chao Huang, 2021