Les mathématiques : le temps long

Pascal Hubert
Professeur à l'Université d'Aix-Marseille
Directeur du Cirm

William Thurston (1946-2012) mathématicien américain, médaille Fields en 1982

Très connu pour ses travaux sur les variétés de dimension 3 mais l'histoire que je raconte aujourd'hui concerne les surfaces.

66-67

ASTÉRISQUE

1991/1979

TRAVAUX DE THURSTON SUR LES SURFACES

Séminaire Orsay

(seconde édition)

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

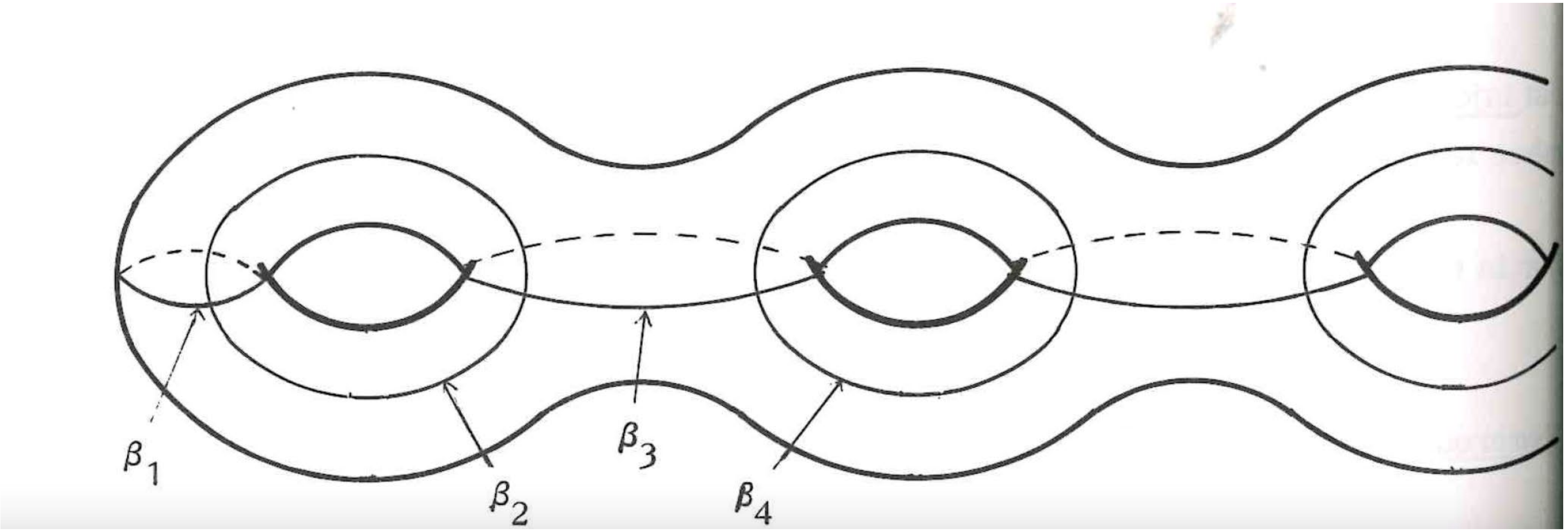
Auteurs : Fathi-Laudenbach-Poenaru
Contributions de Douady, Fried, Sullivan

Participation au séminaire d'Abikoff, Bers, Hubbard.

Meilleurs topologues français de l'époque, école d'Orsay.

Ce livre contient l'exposé de résultats de William Thurston en théorie des surfaces (feuilletages mesurés, compactification naturelle de l'espace de Teichmüller et classification des difféomorphismes). Notre démarche suit pour l'essentiel celle qui est indiquée dans le "research announcement" de Thurston, ainsi que dans les notes de ses cours de Princeton, écrites par M. Handel et W. Floyd.

Une partie de ce texte, notamment la classification des courbes et des feuilletages mesurés est une élaboration des exposés faits dans le Séminaire d'Orsay en 1976-1977. Mais nous n'avons pu fabriquer les textes démonstratifs pour le reste de la théorie que bien plus tard. Au printemps 1978, à Plans-sur-Bex, Thurston nous a expliqué comment regarder le projectifié de l'espace des feuilletages mesurés comme bord de l'espace de Teichmüller.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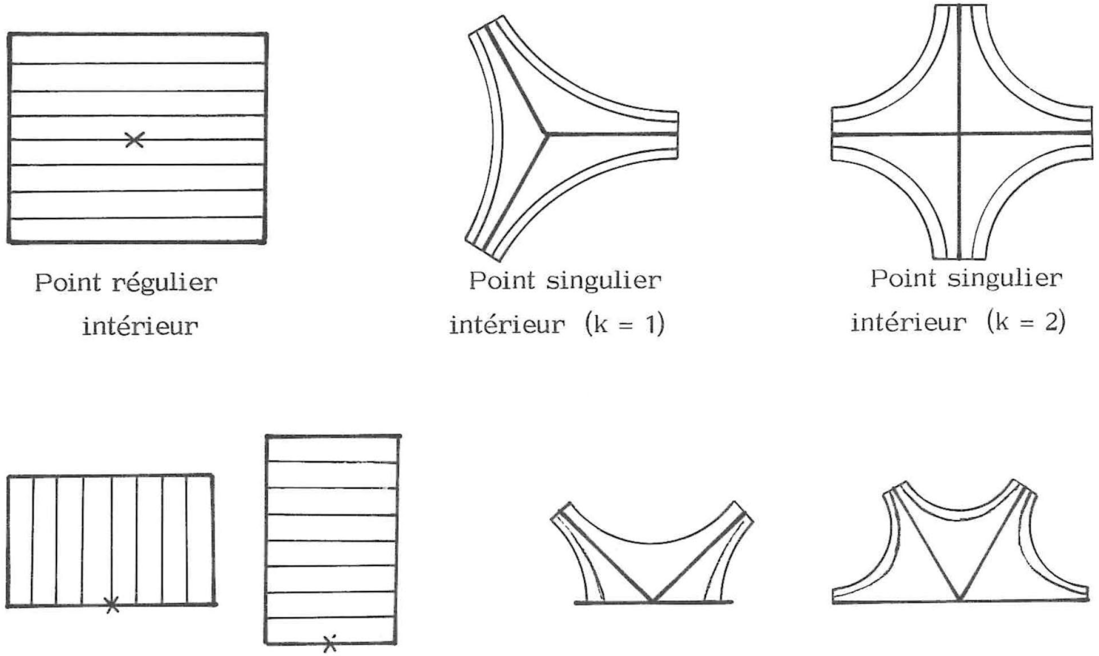

Point régulier intérieur
Point singulier intérieur (k = 1)
Point singulier intérieur (k = 2)
Points réguliers sur le bord
Points singuliers sur le bord

Figure 1

Remarque. Toutes les constructions précédentes conduisent à des pseudo-Anosov dont le facteur de dilatation est un entier quadratique. Les "membres du séminaire" ne savent pas construire d'exemple où il soit de degré plus élevé.

Page 250 de Fathi-Laudenbach-Poenaru

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

ON THE GEOMETRY AND DYNAMICS OF DIFFEOMORPHISMS OF SURFACES

WILLIAM P. THURSTON

PREFACE

This article was widely circulated as a preprint, about 12 years ago. At that time the *Bulletin* did not accept research announcements, and after a couple of attempts to publish it, I gave up, and the preprint did not find a home. I very soon saw that there were many ramifications of this theory, and I talked extensively about it in a number of places. One year I devoted my graduate course to this theory, and notes of Bill Floyd and Michael Handel from that course were circulated for a while. The participants in a seminar at Orsay in 1976–1977 went over this material, and wrote a volume [**FLP**] including some

The literature on this subject is now quite large, and I cannot even touch on all aspects of it here, such as algorithms, noncompact surfaces, handlebodies, measure theory, hyperbolic three-manifolds, etc.

There would be no simple stopping point if I began to incorporate the more recent developments in the original paper, so it is being published here in the original form.

Fin introduction de l’article de Thurston

Personne ne lit donc cet article !!!

Pourtant il y a … un chapitre supplémentaire

6. We will describe in detail an elementary construction for a large class of examples of diffeomorphisms in canonical form.

Ceci répond à la question de FLP

D’autres réponses à la question de FLP de nature différente :

Arnoux-Yoccoz en 1981, Veech 1982

La construction de Thurston est oubliée jusqu’en 2003 !

On organise au Cirm, 30 juin au 4 juillet 2003, la première conférence sur la dynamique dans les espaces de Teichmüller avec du très beau monde dont 4 lauréats de la médaille Fields Kontsevich, McMullen, Okounkov, Yoccoz

Hubbard fait un exposé mémorable après la bouillabaisse sur la construction de Thurston. C’était le seul à avoir lu le texte de Thurston. Les dernières années de sa carrière sont dédiées à rédiger les grands théorèmes de Thurston.

VEECH GROUPS WITHOUT PARABOLIC ELEMENTS

PASCAL HUBERT and ERWAN LANNEAU

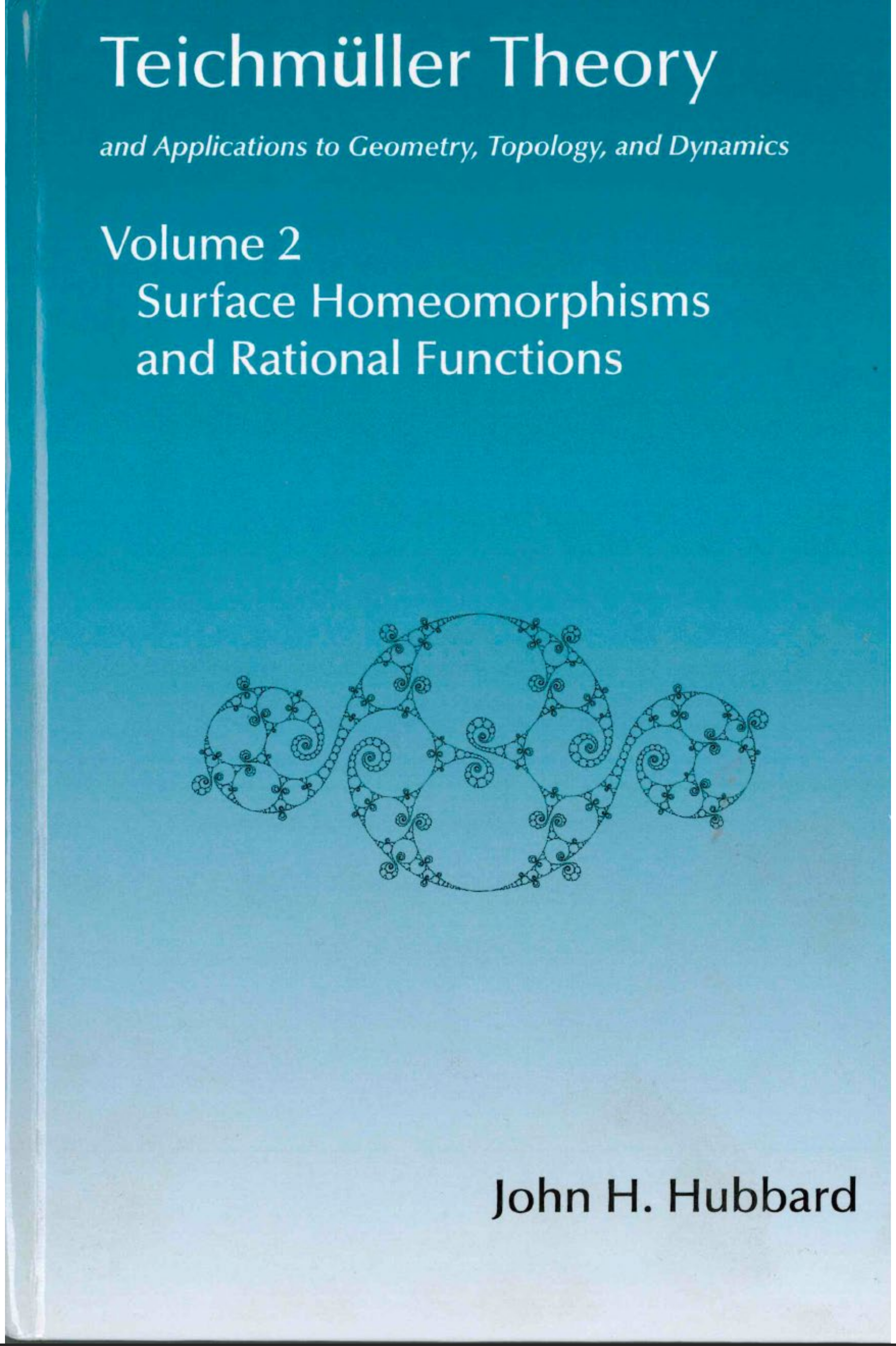
Abstract
We prove that a translation surface that has two transverse parabolic elements has a totally real trace field. As a corollary, nontrivial Veech groups that have no parabolic elements do exist.

The proof follows Veech's viewpoint on Thurston's construction of pseudo-Anosov diffeomorphisms.

1. Introduction

For a long time, it has been known that the ergodic properties of linear flows on a translation surface are strongly related to the behavior of its $SL_2(\mathbb{R})$-orbit in the moduli space of holomorphic one-forms (see [MT], [Z] for surveys of the literature on this subject). The $SL_2(\mathbb{R})$-orbit of a translation surface is called its *Teichmüller disc*. Its stabilizer under the action of $SL_2(\mathbb{R})$ is a Fuchsian group called the *Veech group*.

In 1989, Veech proved that a translation surface whose stabilizer is a lattice has



Cette construction est aujourd’hui si connue qu’elle est souvent appelée « construction bouillabaisse »

MIRZAKHANI, Maryam. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Inventiones mathematicae*, 2007, vol. 167, no 1, p. 179-222.

15/28 références ont + de 10 ans

Invent. math. 167, 179–222 (2007)
DOI: 10.1007/s00222-006-0013-2

Inventiones mathematicae

Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces

Maryam Mirzakhani

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544, USA (e-mail: mmirzakh@math.princeton.edu)

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Contents

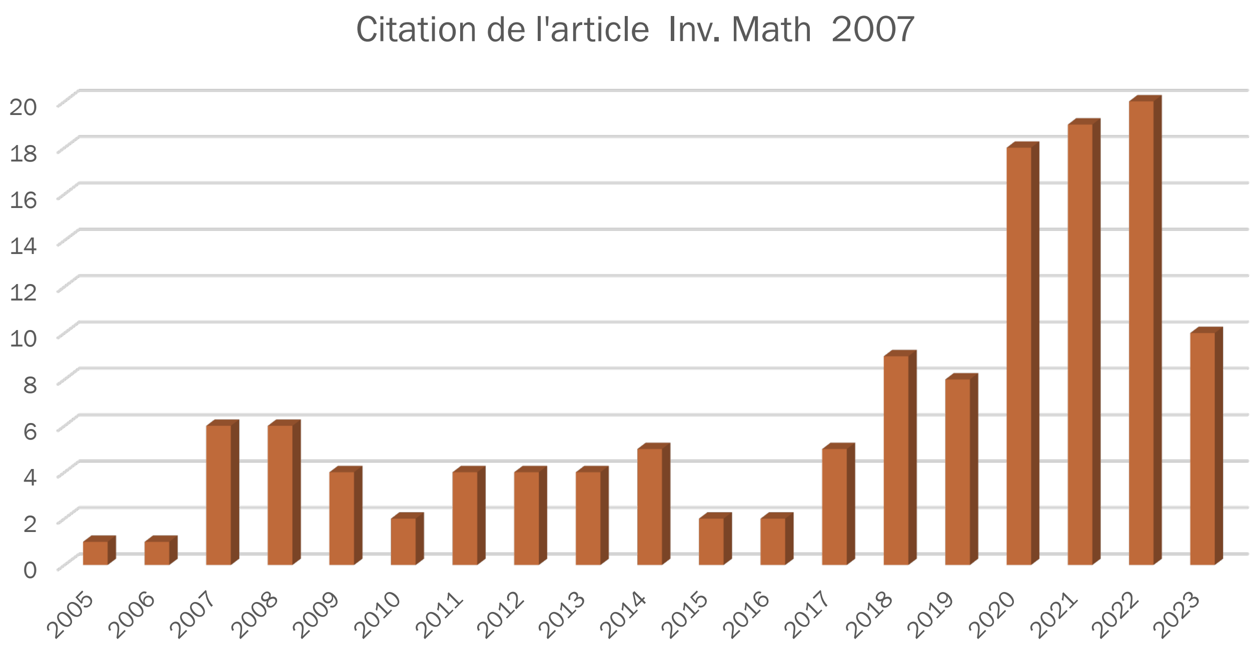
1. Introduction

In this paper we investigate the Weil-Petersson volume of the moduli space of curves with marked points. We develop a method for integrating geometric functions over the moduli space of curves, and obtain an effective recursive formula for the volume $V_{g,n}(L_1, \ldots, L_n)$ of the moduli space $\mathcal{M}_{g,n}(L_1, \ldots, L_n)$ of hyperbolic Riemann surfaces of genus g with n geodesic boundary components. We show that $V_{g,n}(L)$ is a polynomial whose coefficients are rational multiples of powers of π. The constant term of the polynomial $V_{g,n}(L)$ is the Weil-Petersson volume of the moduli space of closed surfaces of genus g with n marked points.

References

[B] Basmajian, A.: The orthogonal spectrum of a hyperbolic manifold. Am. J. Math. **115**, 1139–1159 (1993)
[BS] Birman, J.S., Series, C.: Geodesics with bounded intersection number on surfaces are sparsely distributed. Topology **24**, 217–225 (1985)
[BL] Bismut, J.-M., Labourie, F.: Symplectic geometry and the Verlinde formulas. In: Surveys in Differential Geometry: Differential Geometry Inspired by String Theory, vol. 5 of Surv. Differ. Geom., pp. 97–331. Boston, MA: Int. Press 1999
[Bus] Buser, P.: Geometry and Spectra of Compact Riemann Surfaces. Boston: Birkhäuser 1992
[CEG] Canary, R.D., Epstein, D.B.A., Green, P.: Notes on notes of Thurston. In: Analytical and Geometric Aspects of Hyperbolic Space, pp. 3–92. Cambridge: Cambridge University Press 1987
[D] Donaldson, S.: Gluing techniques in the cohomology of moduli spaces. In: Topological Methods in Modern Mathematics, pp. 137–170. Houston, TX: Publish or Perish 1993
[Gol] Goldman, W.: The symplectic nature of fundamental groups of surfaces. Adv. Math. **54**, 200–225 (1984)
[HP] Harer, J.L., Penner, R.C.: Combinatorics of Train Tracks. Annals of Math. Studies, vol. 125. Princeton, NJ: Princeton University Press 1992
[Har] Harris, J., Morrison, I.: Moduli of Curves. Graduate Texts in Mathematics, vol. 187. New York: Springer 1998
[HY] Hocking, J., Young, G.: Topology. New York: Dover Publication 1988
[IT] Imayoshi, Y., Taniguchi, M.: An Introduction to Teichmüller Spaces. Tokyo: Springer 1992
[JK] Jeffrey, L., Kirwan, F.: Intersection theory on moduli spaces of holomorphic bundles of arbitrary rank on a Riemann surface. Ann. Math. (2) **148**, 109–196 (1998)
[KMZ] Kaufmann, R., Manin, Y., Zagier, D.: Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves. Commun. Math. Phys. **181**, 736–787 (1996)
[Ki] Kirwan, F.: Momentum maps and reduction in algebraic geometry. Differ. Geom. Appl. **9**, 135–171 (1998)
[K] Kontsevich, M.: Intersection on the moduli space of curves and the matrix airy function. Commun. Math. Phys. **147**, 1–23 (1992)
[LM] Labourie, F., McShane, G.: Cross ratios and identities for higher Thurston theory. Preprint 2006
[MaZ] Manin, Y., Zograf, P.: Invertible cohomological field theories and Weil-Petersson volumes. Ann. Inst. Fourier **50**, 519–535 (2000)
[McD] McDuff, D.: Introduction to Symplectic Topology. Providence, RI: Am. Math. Soc. 1999
[M] McShane, G.: Simple geodesics and a series constant over Teichmüller space. Invent. Math. **132**, 607–632 (1998)
[Mirz1] Mirzakhani, M.: Growth of the number of simple closed geodesics on a hyperbolic surface. To appear in Ann. Math.
[Mirz2] Mirzakhani, M.: Weil-Petersson volumes and intersection theory on the moduli space of curves. To appear in J. Am. Math. Soc.
[NN] Nakanishi, T., Näätänen, M.: Areas of two-dimensional moduli spaces. Proc. Am. Math. Soc. **129**, 3241–3252 (2001)
[Pen] Penner, R.: Weil-Petersson volumes. J. Differ. Geom. **35**, 559–608 (1992)
[TWZ1] Tan, S.P., Wong, Y., Zhang, Y.: Necessary and sufficient conditions for McShane's identity and variations. Preprint 2004
[TWZ2] Tan, S.P., Wong, Y., Zhang, Y.: Generalizations of McShane's identity to hyperbolic cone-surfaces. J. Differ. Geom. **72**, 73–111 (2006)
[Wol1] Wolpert, S.: The Fenchel-Nielsen deformation. Ann. Math. **115**, 501–528 (1982)
[Wol2] Wolpert, S.: On the homology of the moduli space of stable curves. Ann. Math. (2) **118**, 491–523 (1983)
[Zo] Zograf, P.: The Weil-Petersson volume of the moduli space of punctured spheres. In: Mapping Class Groups and Moduli Spaces of Riemann Surfaces. Contemp. Math., vol. 150, pp. 367–372. Providence, RI: Am. Math. Soc. 1993

MIRZAKHANI, Maryam. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Inventiones mathematicae*, 2007, vol. 167, no 1, p. 179-222.



130 citations

MIRZAKHANI, Maryam. Weil-Petersson volumes and intersection theory on the moduli space of curves. *Journal of the American Mathematical Society*, 2007, vol. 20, no 1, p. 1-23.

20/34 références ont + de 10 ans

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WEIL-PETERSSON VOLUMES AND INTERSECTION THEORY ON THE MODULI SPACE OF CURVES

MARYAM MIRZAKHANI

1. Introduction

In this paper, we establish a relationship between the Weil-Petersson volume $V_{g,n}(b)$ of the moduli space $\mathcal{M}_{g,n}(b)$ of hyperbolic Riemann surfaces with geodesic boundary components of lengths $b_1, \ldots, b_n$, and the intersection numbers of tautological classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves. As a result, by using the recursive formula for $V_{g,n}(b)$ obtained in [22], we derive a new proof of the Virasoro constraints for a point. This result is equivalent to the Witten-Kontsevich formula [14].

Intersection theory of $\overline{\mathcal{M}}_{g,n}$. Let $\mathcal{M}_{g,n}$ be the moduli space of genus g curves with n distinct marked points and $\overline{\mathcal{M}}_{g,n}$ its Deligne-Mumford compactification. The space $\overline{\mathcal{M}}_{g,n}$ is a connected complex orbifold of dimension $3g-3+n$ [9]. These moduli spaces are endowed with natural cohomology classes. An example of such a class is the Chern class of a vector bundle on the moduli space. There are n tautological line bundles defined over $\overline{\mathcal{M}}_{g,n}$: for each marked point i, there exists a canonical line bundle $\mathcal{L}_i$ in the orbifold sense whose fiber at the point $(C, x_1, \ldots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space of C at x_i. The first Chern class of this bundle is denoted by $\psi_i = c_1(\mathcal{L}_i)$. Note that although the complex curve C may have nodes, x_i never coincides with the singular points.

For any set $\{d_1, \ldots, d_n\}$ of integers define the top intersection number of ψ classes by

$$\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i}.$$

Such products are well defined when the d_i's are nonnegative integers and $\sum_{i=1}^{n} d_i = 3g-3+n$. In other cases $\langle \tau_{d_1}, \ldots, \tau_{d_n} \rangle_g$ is defined to be zero. Since we are in the orbifold setting, these intersection numbers are rational numbers. See [15] and [9] for more details.

Introduce formal variables t_i, $i \geq 0$, and define F_g, the generating function of all top intersections of ψ classes in genus g, by

$$F_g(t_0, t_1, \ldots) = \sum \langle \prod \tau_{d_i} \rangle_g \prod t_r^{n_r}/n_r!$$

References

1. E. Arbarello, *Sketches of kdv*, Symposium in Honor of C. H. Clemens (Salt Lake City, UT, 2000), Contemp. Math., vol. 312, Amer. Math. Soc., 2002, pp. 9–69. MR1941578 (2004i:14039)
2. L. Bers, *Spaces of degenerating Riemann surfaces*, Discontinuous groups and Riemann surfaces, Annals of Math. Studies, vol. 76, Princeton University Press, 1974, pp. 43–55. MR0361051 (50:13497)
3. M. Boggi and M. Pikaart, *Galois covers of moduli of curves*, Compositio Math. **120** (2000), 171–191. MR1739177 (2002a:14025)
4. P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser Boston, 1992. MR1183224 (93g:58149)
5. R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity*, Nuclear Phys. B **384** (1991), 435–456. MR1083914 (92a:81171)
6. W. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. Math. **54** (1984), 200–225. MR0762512 (86i:32042)
7. ______, *Ergodic theory on moduli spaces*, Ann. of Math. **146** (1997), 475–507. MR1491446 (99a:58024)
8. V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian t^n-spaces*, Birkhäuser Boston, Inc., Boston, MA, 1994. MR1301331 (96e:58064)
9. J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, 1998. MR1631825 (99g:14031)
10. Y. Imayoshi and M. Taniguchi, *An introduction to Teichmüller spaces*, Springer-Verlag, 1992. MR1215481 (94b:32031)
11. C. Itzykson and J. Zuber, *Combinatorics of the modular group. II. The Kontsevich integrals*, Internat. J. Modern Phys. A **7** (1992), 5661–5705. MR1180858 (94m:32029)
12. R. Kaufmann, Y. Manin, and D. Zagier, *Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves*, Comm. Math. Phys. **181** (1996), 736–787. MR1414310 (98i:14029)
13. F. Kirwan, *Momentum maps and reduction in algebraic geometry*, Differential Geom. Appl. **9** (1998), 135–171. MR1636303 (99e:58072)
14. M. Kontsevich, *Intersection on the moduli space of curves and the matrix Airy function.*, Comm. Math. Phys. **147** (1992). MR1171758 (93e:32027)
15. E. Looijenga, *Intersection theory on Deligne-Mumford compactifications (after Witten and Kontsevich)*, Séminaire Bourbaki, 1992/93, Astérisque, volume 216, 1993, pp. 187–212. MR1246398 (95b:32033)
16. ______, *Smooth Deligne-Mumford compactification by means of Prym level structures*, J. Algebraic Geom. **3** (1994), 283–293. MR1257324 (94m:14029)
17. Y. Manin and P. Zograf, *Invertible cohomological field theories and Weil-Petersson volumes*, Ann. Inst. Fourier (Grenoble) **50** (2000), 519–535. MR1775360 (2001g:14046)
18. H. Masur, *The extension of the Weil-Petersson metric to the boundary of Teichmüller space*, Duke Math. J. **43** (1976), 623–635. MR0417456 (54:5506)
19. D. McDuff, *Introduction to symplectic topology*, Amer. Math. Soc., Providence, RI, 1999. MR1702941 (2000e:53099)
20. G. McShane, *Simple geodesics and a series constant over Teichmüller space*, Invent. Math. **132** (1998), 607–632. MR1625712 (99i:32028)
21. J. Milnor and J. Stasheff, *Characteristic classes*, Annals of Mathematics Studies. MR0440554 (55:13428)
22. M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Preprint, 2003.
23. T. Nakanishi and M. Näätänen, *Areas of two-dimensional moduli spaces*, Proc. Amer. Math. Soc. **129** (2001), 3241–3252. MR1844999 (2002e:32020)
24. A. Okounkov, *Random trees and moduli of curves*, Asymptotic combinatorics with applications to mathematical physics, Lecture Notes in Mathematics, vol. 1815, Springer-Verlag, 2003, pp. 89–126. MR2009837 (2004m:14049)
25. A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and matrix models, I*, Preprint.
26. R. Penner, *Weil-Petersson volumes*, J. Differential Geom. **35** (1992), 559–608. MR1163449 (93d:32029)
27. J. Weitsman, *Geometry of the intersection ring of the moduli space of flat connections and the conjectures of Newstead and Witten*, Topology **37** (1998). MR1480881 (99m:57030)
28. E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry, Lehigh Univ., Bethlehem, PA, 1991. MR1144529 (93e:32028)
29. ______, *Two dimensional gauge theories revisited*, J. Geom. Phys. **9** (1992), 303–368. MR1185834 (93m:58017)
30. S. Wolpert, *An elementary formula for the Fenchel-Nielsen twist*, Comment. Math. Helv. **56** (1981), 132–135. MR0615620 (82k:32053)
31. ______, *On the homology of the moduli space of stable curves*, Ann. of Math.(2) **118** (1983), 491–523. MR0727702 (86h:32036)
32. ______, *On obtaining a positive line bundle from the Weil-Petersson class*, Amer. J. Math. **107** (1985), 1485–1507. MR0815769 (87f:32058)
33. ______, *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985), 969–997. MR0796909 (87b:32040)
34. P. Zograf, *The Weil-Petersson volume of the moduli space of punctured spheres*, Mapping class groups and moduli spaces of Riemann surfaces, Contemp. Math., vol. 150, Amer. Math. Soc., 1993, pp. 367–372. MR1234274 (94g:32030)

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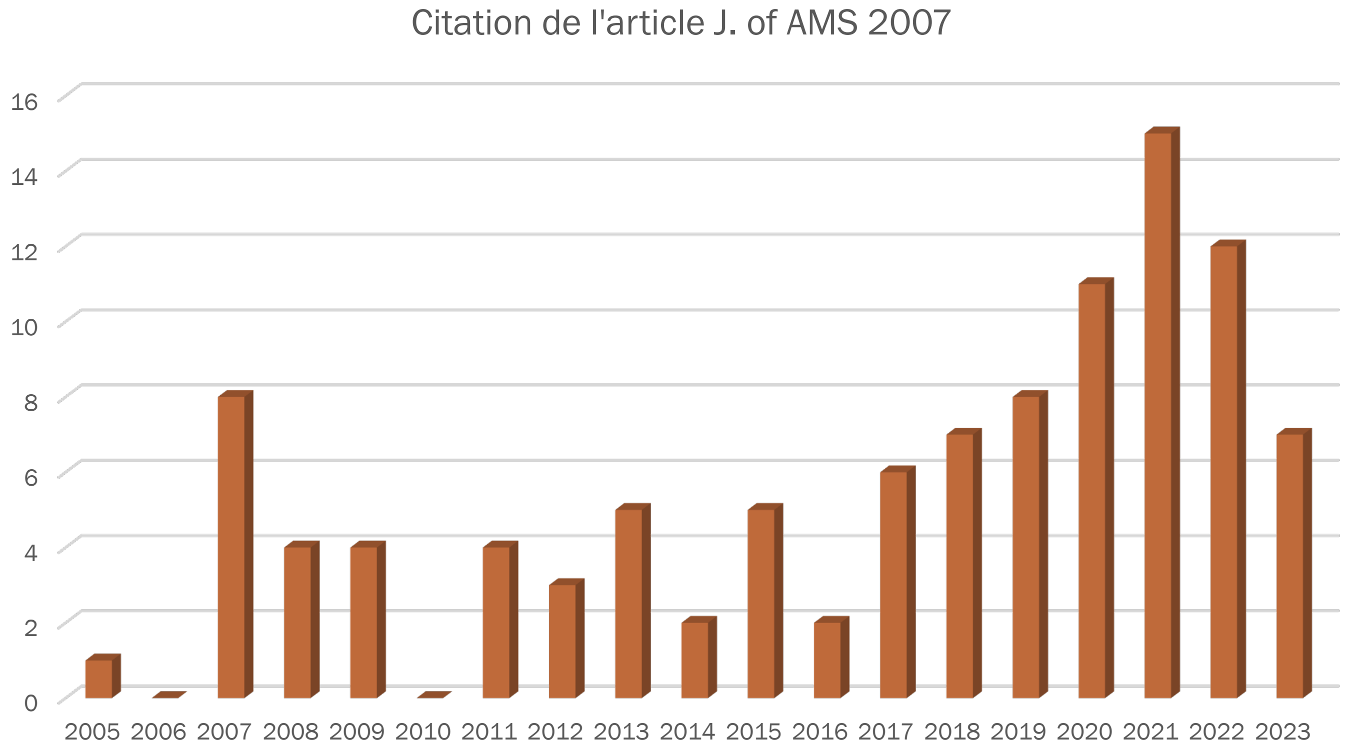
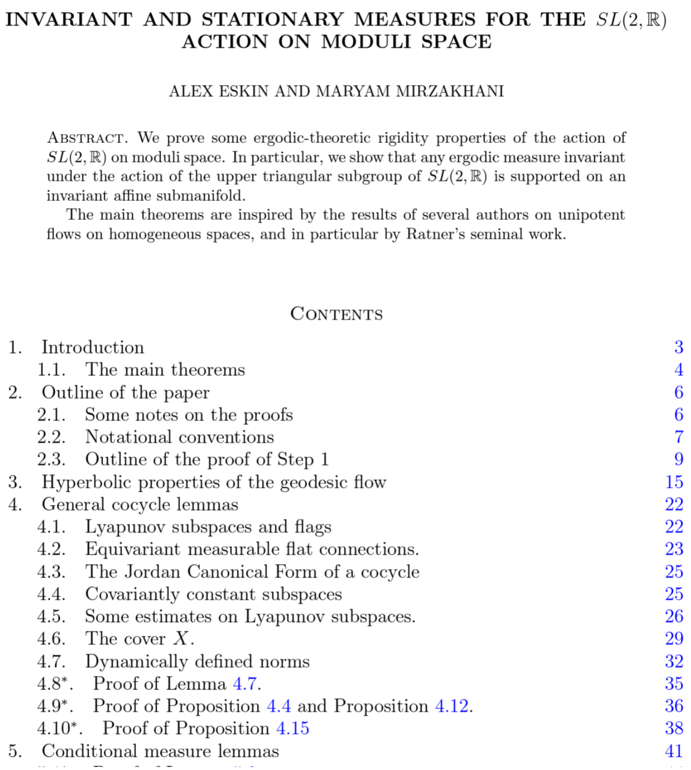
MIRZAKHANI, Maryam. Weil-Petersson volumes and intersection theory on the moduli space of curves. *Journal of the American Mathematical Society*, 2007, vol. 20, no 1, p. 1-23.



ESKIN, Alex et MIRZAKHANI, Maryam. Invariant and stationary measures for the action on moduli space. *Publications mathématiques de l'IHÉS*, 2018, vol. 127, p. 95-324.



INVARIANT AND STATIONARY MEASURES FOR THE $SL(2,\mathbb{R})$ ACTION ON MODULI SPACE

ALEX ESKIN AND MARYAM MIRZAKHANI

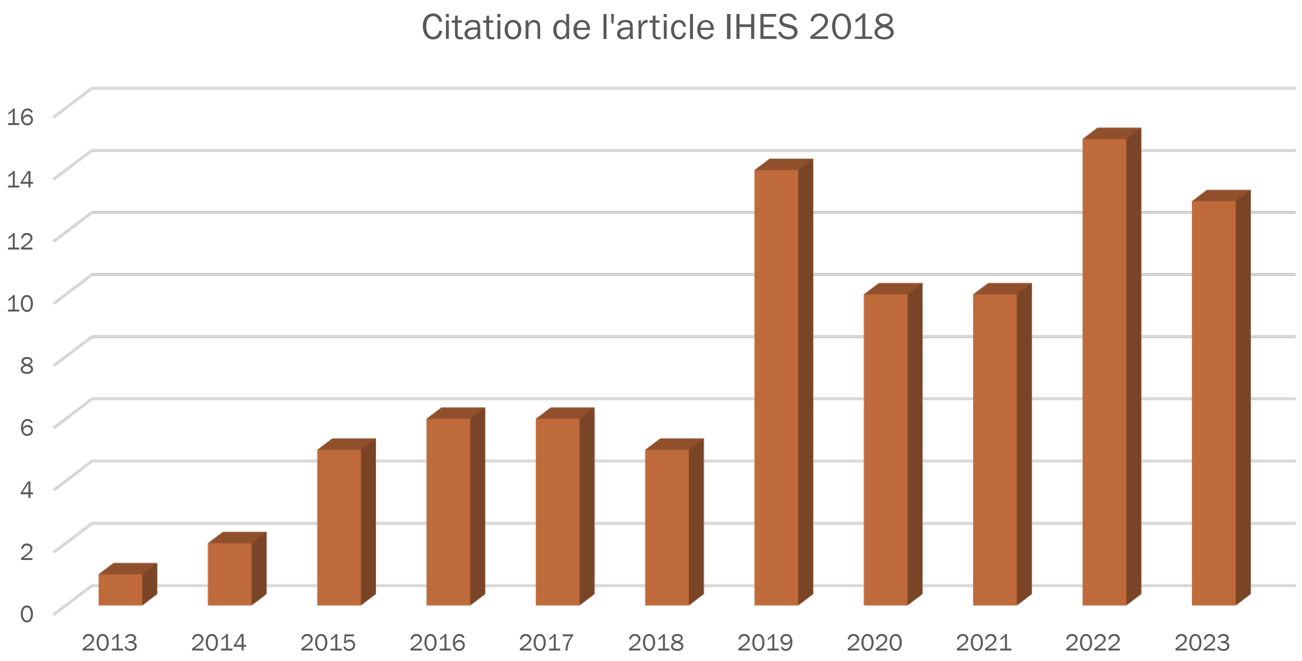
ABSTRACT. We prove some ergodic-theoretic rigidity properties of the action of $SL(2,\mathbb{R})$ on moduli space. In particular, we show that any ergodic measure invariant under the action of the upper triangular subgroup of $SL(2,\mathbb{R})$ is supported on an invariant affine submanifold.

The main theorems are inspired by the results of several authors on unipotent flows on homogeneous spaces, and in particular by Ratner's seminal work.

CONTENTS

72/112 références ont + de 10 ans
8 références datent d'avant 1980, le plus ancien article cité a été publié en 1947

ESKIN, Alex et MIRZAKHANI, Maryam. Invariant and stationary measures for the action on moduli space. *Publications mathématiques de l'IHÉS*, 2018, vol. 127, p. 95-324.



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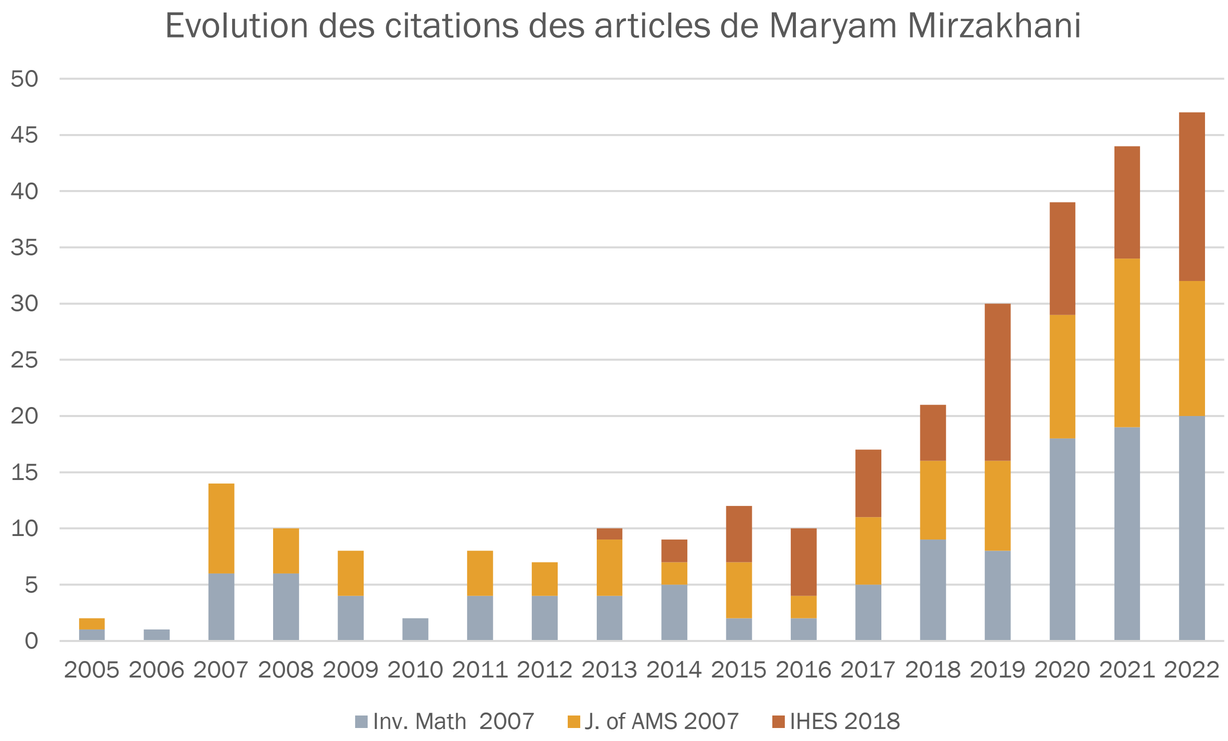

Année	Inv. Math 2007	J. of AMS 2007	IHES 2018
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2006	1	0	
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2008	6	4	
2009	4	4	
2010	2	0	
2011	4	4	
2012	4	3	
2013	4	5	1
2014	5	2	2
2015	2	5	5
2016	2	2	6
2017	5	6	6
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2019	8	8	14
2020	18	11	10
2021	19	15	10
2022	20	12	15
2023	10	7	13
Total	130	104	87

Gauss measures for transformations on the space of interval exchange maps

By William A. Veech

In memory of Rufus Bowen

1. Introduction

This paper brings a certain completeness to the study, initiated in [9], [10], and Rauzy [6], [7], of a class of transformations on the set of interval exchange maps. The main result is Theorem 1.1 which asserts that these transformations possess "Gauss measures," ergodic conservative invariant measures whose densities are rational functions. The measures are in general infinite.

The reduction ([9], [10]) yields as one consequence of Theorem 1.1 the "Keane Conjecture" [3], which has also been affirmed recently by H. Masur [4]. The constructions used to prove Theorem 1.1 yield a large class of examples of pseudo-Anosov maps (Thurston [8]), indeed, by reasoning in [1], all such maps whose stable/unstable foliations are orientable (Section 8).

We recall first the definition of "interval exchange map." Fix $m > 1$, and let $\mathfrak{S}_m^0$ be the set of irreducible permutations $j \to \pi j$, $1 \leq j \leq m$. To say $\pi \in \mathfrak{S}_m^0$ is to say $\pi\{1,\ldots,k\} = \{1,\ldots,k\}$ implies $k = m$. Λ_m is the positive cone in $\mathbf{R}^m$, and to $\lambda \in \Lambda_m$ one associates the several objects $\beta_0(\lambda) = 0$, $\beta_j(\lambda) = \Sigma_{i=1}^{j}\lambda_i$, $1 \leq j \leq m$, $|\lambda| = \beta_m(\lambda)$, and $I_j^\lambda = [\beta_{j-1}(\lambda), \beta_j(\lambda)) \subseteq [0, |\lambda|) = I^\lambda$. Also, define $\lambda_j^\pi = \lambda_{\pi^{-1}j}$, $1 \leq j \leq m$. The (λ, π) *interval exchange*, $T = T_{\lambda,\pi}$, is the map of I^λ defined by

$$Tx = x + \beta_{\pi i-1}(\lambda^\pi) - \beta_{i-1}(\lambda)(x \in I_i^\lambda).$$

Next, define $\tau(\lambda, \pi) = \mathrm{Max}(\beta_{m-1}(\lambda), \beta_{m-1}(\lambda^\pi))$, and let $J(\lambda, \pi) =$

VEECH, William A. Gauss measures for transformations on the space of interval exchange maps. *Annals of Mathematics*, 1982, vol. 115, no 2, p. 201-242.

Année	Nb de citations	Année	Nb de citations
1981	1	2003	3
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1997	1	2019	18
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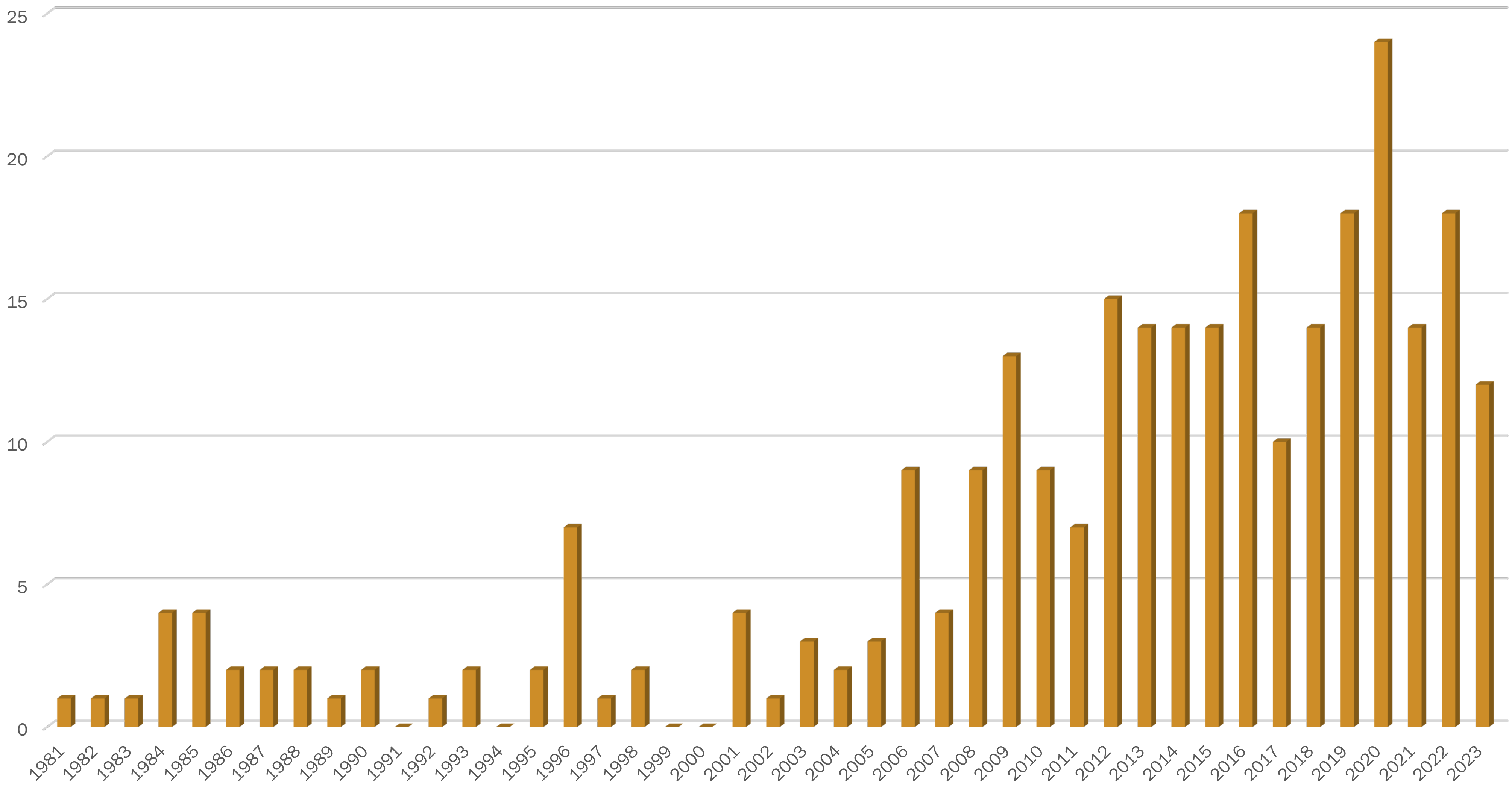
Nb de citations article de Veech (1982)



Invent. math. 97, 553–583 (1989)

Inventiones mathematicae

VEECH, William A. Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Inventiones mathematicae*, 1989, vol. 97, p. 553-583.

Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*

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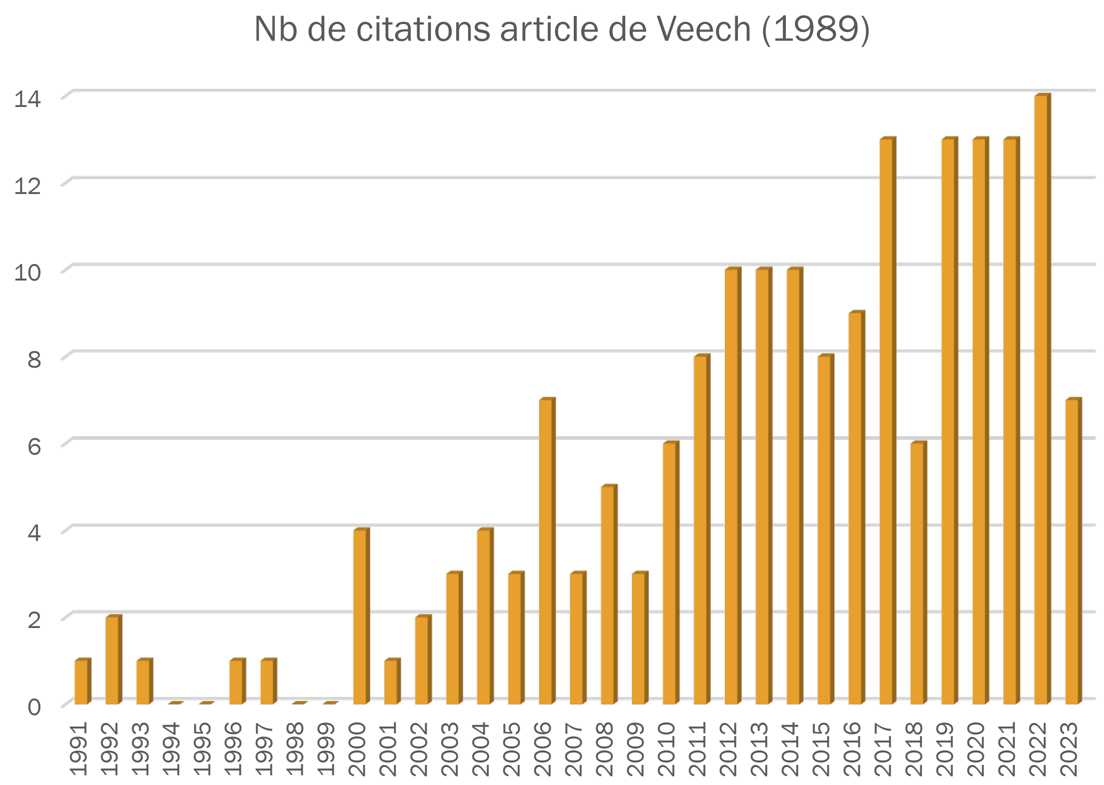
Contents

Summary. There exists a Teichmüller disc Δ_n containing the Riemann surface of $y^2 + x^n = 1$, in the genus $\left[\frac{n-1}{2}\right]$ Teichmüller space, such that the stabilizer of Δ_n in the mapping class group has a fundamental domain of finite (Poincaré) volume in Δ_n. Application is given to an asymptotic formula for the length spectrum of the billiard in isosceles triangles with angles $\left(\pi/n, \pi/n, \frac{n-2}{n}\pi\right)$ and to the uniform distribution of infinite billiard trajectories in the same triangles.



Année	Nb de citation	Cité par
1991	1	VEECH, W. A. Erratum: Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. Invent. Math. 97, 553-583 (1989). Inventiones mathematicae, 1991, vol. 104, p. 447-448.
1992	2	VEECH, William A. The billiard in a regular polygon. Geometric & Functional Analysis GAFA, 1992, vol. 2, no 3, p. 341-379. MASUR, Howard. Hausdorff dimension of the set of nonergodic foliations of a quadratic differential. 1992.
1993	1	HOLT, Fred. Periodic reflecting paths in right triangles. Geometriae Dedicata, 1993, vol. 46, no 1, p. 73-90.
1994	0	
1995	0	
1996	1	GUTKIN, Eugene. Billiards in polygons: survey of recent results. Journal of statistical physics, 1996, vol. 83, p. 7-26.
1997	1	VOROBETS, Ya B. Billiards in rational polygons: Periodic trajectories, symmetries, and d-stability. Mathematical Notes, 1997, vol. 62, p. 56-63.
1998	0	
1999	0	
2000	4	GUTKIN, Eugene et JUDGE, Chris. Affine mappings of translation surfaces: geometry and arithmetic. 2000. HUBERT, Pascal et SCHMIDT, Thomas A. Veech groups and polygonal coverings. Journal of Geometry and Physics, 2000, vol. 35, no 1, p. 75-91. ARNOUX, Pierre et HUBERT, Pascal. Fractions continues sur les surfaces de Veech. Journal d'Analyse Mathématique, 2000, vol. 81, p. 35-64.

Année	Nb de citations
2001	1
2002	2
2003	3
2004	4
2005	3
2006	7
2007	3
2008	5
2009	3
2010	6
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STRICT ERGODICITY IN ZERO DIMENSIONAL DYNAMICAL SYSTEMS AND THE KRONECKER-WEYL THEOREM MOD 2[1]

BY
WILLIAM A. VEECH

In memory of L. Frank Lovett

1. **Introduction.** Our principal results[2] are number-theoretic. Let $X=\{x \mid 0 \leqq x<1\}$ be the compact group of real numbers modulo 1, and let $\theta \in X$ be irrational. The numbers $j\theta, j=0, \pm 1, \ldots$, (here and henceforth to be reduced modulo 1) comprise a dense subgroup of X. For each interval $I \subseteq X$ and $n>0$ define $S_n=S_n(\theta, I)$ to be the number of integers j, $1 \leqq j \leqq n$, such that $j\theta \in I$. By the Kronecker-Weyl theorem [12] $\lim_{n\to\infty} S_n/n=\nu(I)$, where ν is Lebesgue measure on X.

We will be interested in the behavior of the sequence $\{x_n\}$ of parities of $\{S_n\}$. That is x_n is 0 or 1 as S_n is even or odd. Our first result concerns the existence of the limit

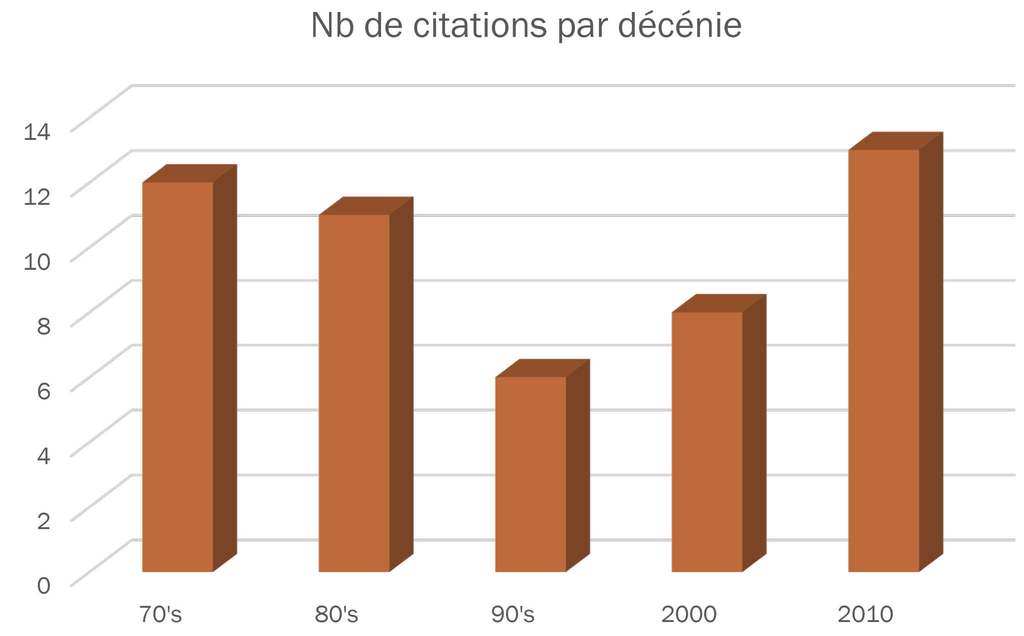
$$\mu_\theta(I) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n. \tag{1}$$

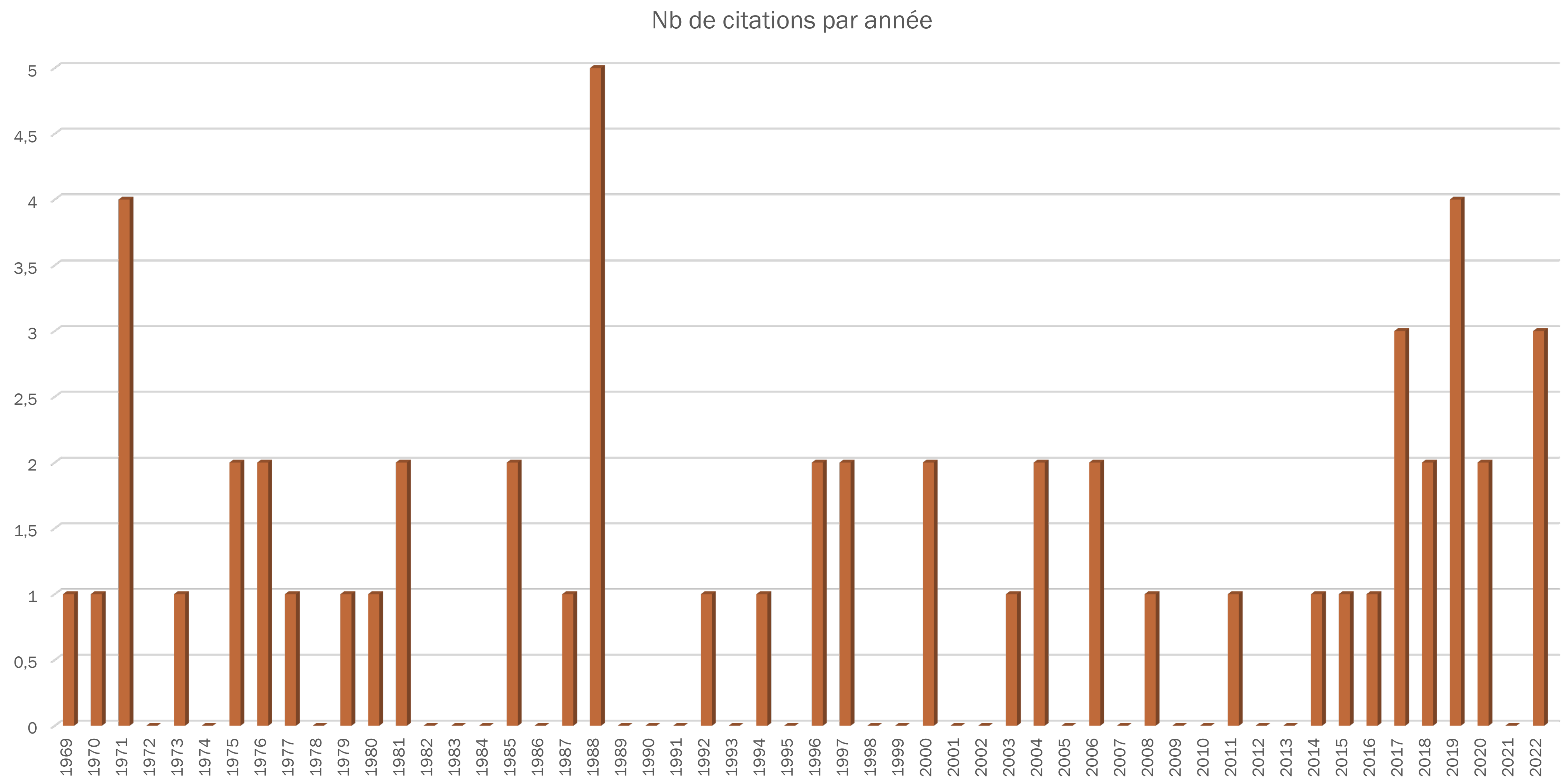
It is

THEOREM 1. *A necessary and sufficient condition for $\mu_\theta(I)$ to exist for every interval $I \subseteq X$ is that θ have bounded partial quotients.*

We will draw freely upon the language and results of continued fraction theory, as it is developed in [2] or [5]. Recall that θ has *bounded partial quotients* if and only if there exists a constant $c>0$ such that $|q\theta-p|>c/q$ for all integers p and q with $q>0$. In terms of $\|\cdot\|$, the closest integer function, the condition is $\|q\theta\|>c/q$, $q>0$. We note that $\|\cdot\|$ defines a group invariant metric on X.

VEECH, William A. Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem $mod 2$. *Transactions of the American Mathematical Society*, 1969, vol. 140, p. 1-33.

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0,5
1
1,5
2
2,5
3
3,5
4
4,5
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1969 1970 1971 1972 1973 1974 1975 1976 1977 1978 1979 1980 1981 1982 1983 1984 1985 1986 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016 2017 2018 2019 2020 2021 2022

Bull. Soc. math. France,
123, 1995, p. 257–270.

COMPLEXITÉ DE SUITES DÉFINIES PAR DES BILLARDS RATIONNELS

PAR

PASCAL HUBERT (*)

RÉSUMÉ. — Soit P un polygone rationnel convexe, $k_1\pi/r, \dots, k_q\pi/r$ les angles entre deux côtés consécutifs où $k_1, \dots, k_q, r$ sont premiers dans leur ensemble. Nous considérons le problème de billard dans ce polygone et codons les trajectoires suivant les côtés qu'elles rencontrent. Nous montrons que, si la suite ainsi obtenue n'est pas périodique, sa complexité est donnée par la formule $p(n) = n(q-2)r + 2r$. Cette expression de la complexité est valable pour n assez grand et est indépendante des conditions initiales du problème.

ABSTRACT. — Let P be a convex rational polygon, $k_1\pi/r, \dots, k_q\pi/r$ the interior angles ($k_1, \dots, k_q, r$ are coprime). Let us consider the billiard problem in this polygon. We code the trajectories according to the sides they meet. When the sequence so obtained is not periodic, we show that the complexity of this sequence is equal to $p(n) = n(q-2)r + 2r$. This formula is true for n large enough and does not depend on the initial conditions.

0. Introduction

On connaît peu de résultats concernant les propriétés ergodiques ou celles liées au codage du flot du billard dans un polygone quelconque. Trouver un exemple de flot de billard dans un polygone qui soit ergodique reste actuellement un problème ouvert. Toutefois, KERCKHOFF, MASUR et SMILLIE ont montré qu'il y a un G_δ dense de polygones pour lesquels le flot est ergodique (*cf.* [KMS]). Par ailleurs, KATOK démontre que ce sont des systèmes d'entropie nulle (*cf.* [K]). Certains cas particuliers sont tout de même bien connus, tant au niveau ergodique que du point de vue du

BIBLIOGRAPHIE

[A] ARNOUX (P.). — *Ergodicité générique des billards polygonaux, d'après Kerckhoff, Masur, Smillie*, Séminaire Bourbaki, t. **696**, 1988, p. 203–221.

[AMST] ARNOUX (P.), MAUDUIT (C.), SHIOKAWA (I.) and TAMURA (J.). — *Complexity of sequences defined by billiards in the cube*, Bull Soc. Math. France, t. **122**, 1994, p. 1–12.

[AR] ARNOUX (P.) et RAUZY (G.). — *Représentation géométrique de suites de complexité* $2n + 1$, Bull. Soc. Math. France, t. **119**, n° 2, 1991, p. 199–215.

[BKM] BOLDRIGINI (C.), KEANE (M.) and MARCHETTI (F.). — *Billiards in polygons*, Annals of Probability, t. **6**, 1978, p. 532–540.

[CFS] CORNFELD (I.P.), FORMIN (S.V.) and SINAI (Ya.G.). — *Ergodic theory*, Springer-Verlag, 1982.

[HM] HEDLUND (G.A.) and MORSE (M.). — *Symbolic Dynamics II, Sturmian trajectories*, Amer. J. Math., t. **62**, 1940, p. 1–42.

[K] KATOK (A.). — *The growth rate for the number of singular and periodic orbits for a polygonal billiard*, Commun. Math. Phys., t. **111**, 1987, p. 151–160.

[KMS] KERCKHOFF (S.), MASUR (H.) and SMILLIE (J.). — *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math., t. **124**, 1986, p. 293–311.

[NST] NISHIOKA (N.), SHIOKAWA (I.) and TAMURA (J.). — *Arithmetical properties of a certain power series*, J. Number Theory, t. **42**, 1992, p. 61–87.

[ZK] ZEMLYAKOF (A.N.) and KATOK (A.B.). — *Topological transivity of billiards in polygons*, Math. Notes, t. **18**, n° 2, 1976, p. 760–764.

06/12/2023

VEECH GROUPS WITHOUT PARABOLIC ELEMENTS

PASCAL HUBERT and ERWAN LANNEAU

Abstract

We prove that a translation surface that has two transverse parabolic elements has a totally real trace field. As a corollary, nontrivial Veech groups that have no parabolic elements do exist.

The proof follows Veech's viewpoint on Thurston's construction of pseudo-Anosov diffeomorphisms.

1. Introduction

For a long time, it has been known that the ergodic properties of linear flows on a translation surface are strongly related to the behavior of its $SL_2(\mathbb{R})$-orbit in the moduli space of holomorphic one-forms (see [MT], [Z] for surveys of the literature on this subject). The $SL_2(\mathbb{R})$-orbit of a translation surface is called its *Teichmüller disc*. Its stabilizer under the action of $SL_2(\mathbb{R})$ is a Fuchsian group called the *Veech group*.

In 1989, Veech proved that a translation surface whose stabilizer is a lattice has

[AY] P. ARNOUX and J.-C. YOCCOZ, *Construction de difféomorphismes pseudo-Anosov*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 75–78. MR 0610152 336, 337, 343

[C] K. CALTA, *Veech surfaces and complete periodicity in genus two*, J. Amer. Math. Soc. **17** (2004), 871–908. MR 2083470 335

[DFF] A. DOUADY, A. FATHI, D. FRIED, F. LAUDENBACH, V. POÉNARU, and M. SHUB, *Travaux de Thurston sur les surfaces*, Astérisque **66–67**, Soc. Math. France, Montrouge, 1979. MR 0568308 336

[Vo] Y. B. VOROBETS, *Planar structures and billiards in rational polygons: The Veech alternative*, Russian Math. Surveys **51** (1996), 779–817. MR 1436653 335

[W] C. C. WARD, *Calculation of Fuchsian groups associated to billiards in a rational triangle*, Ergodic Theory Dynam. Systems **18** (1998), 1019–1042. MR 1645350 335

[Z] A. ZORICH, "Flat surfaces" in *Frontiers in Number Theory, Physics and Geometry, Vol. 1: On Random Matrices, Zeta Functions and Dynamical Systems (Les Houches, France, 2003)*, Springer, Berlin, 2006, 439–586. 335

[FK] H. M. FARKAS and I. KRA, *Riemann Surfaces*, 2nd ed., Grad. Texts in Math. **71**, Springer, New York, 1992. MR 1139765 344

[GJ] E. GUTKIN and C. JUDGE, *Affine mappings of translation surfaces: Geometry and arithmetic*, Duke Math. J. **103** (2000) 191–213. MR 1760625 335, 339, 340

[H] J. HUBBARD, "Homeomorphisms of surface" in Dynamique dans l'espace de Teichmüller et applications aux billards rationnel (Marseille, 2003), lecture notes. 336

[HL] P. HUBERT and S. LELIÈVRE, *Prime arithmetic Teichmüller discs in $\mathscr{H}(2)$*, Israel J. Math. **151** (2006), 281–321. 341

[HS1] P. HUBERT and T. A. SCHMIDT, *Infinitely generated Veech groups*, Duke Math. J. **123** (2004), 49–69. MR 2060022 335, 336

[HS2] ———, *Geometry of infinitely generated Veech groups*, Conform. Geom. Dyn. **10** (2006), 1–20. MR 2192855 335

[K] S. KATOK, *Fuchsian Groups*, Chicago Lectures in Math., Univ. of Chicago Press, Chicago, 1992. MR 1177168 337

[KS] R. KENYON and J. SMILLIE, *Billiards on rational-angled triangles*, Comment. Math. Helv. **75** (2000), 65–108. MR 1760496 335, 337, 339, 340

[Le] C. J. LEININGER, *On groups generated by two positive multi-twists: Teichmüller curves and Lehmer's number*, Geom. Topol. **8** (2004), 1301–1359. MR 2119298 336

[MT] H. MASUR and S. TABACHNIKOV, "Rational billiards and flat structures" in *Handbook of Dynamical Systems, Vol. 1A*, North-Holland, Amsterdam, 2002, 1015–1089. MR 1928530 335, 337

[Mc1] C. T. MCMULLEN, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. **16** (2003), 857–885. MR 1992827 335, 337, 339, 340

[Mc2] ———, *Teichmüller geodesics of infinite complexity*, Acta Math. **191** (2003), 191–223. MR 2051398 335, 337, 339, 340

[Mc3] ———, *Teichmüller curves in genus two: Discriminant and spin*, Math. Ann. **333** (2005), 87–130. MR 2169830 337

[Mc4] ———, *Teichmüller curves in genus two: The decagon and beyond*, J. Reine Angew. Math. **582** (2005), 173–199. MR 2139715 337

[Mc5] ———, *Teichmüller curves in genus two: Torsion divisors and ratios of sines*, to appear in Invent. Math. 337

[Mö] M. MÖLLER, *Variations of Hodge structures of a Teichmüller curve*, J. Amer. Math. Soc. **19** (2006), 327–344. MR 2188128 337

[P] N. PURZITSKY, *A cutting and pasting of noncompact polygons with applications to Fuchsian groups*, Acta Math. **143** (1979), 233–250. MR 0549777 344

[T] W. P. THURSTON, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 417–431. MR 0956596 335, 336, 337, 340

[Ve1] W. A. VEECH, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2) **115** (1982), 201–242. MR 0644019 336

[Ve2] ———, *Teichmüller curves in modular space, Eisenstein series, and an application to triangular billiards*, Invent. Math. **97** (1989), 553–583. MR 1005006 335, 336, 337, 338, 339, 340, 341

[Ve3] ———, *The billiard in a regular polygon*, Geom. Funct. Anal. **2** (1992), 341–379. MR 1177316 335