# Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ 

Francesc Fité (UB) Xevi Guitart (UB)

## COUNT 2023

## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\mathcal{A}_{g, d}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / k\right.$ of dimension $g$ and $\left.[k: \mathbb{Q}]=d\right\} / \simeq$


The set $\mathcal{A}_{g, d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings.
- Very little is known:


## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A)=\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms


The set $\mathcal{A}_{g, d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings.
- Very little is known:


## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A)=\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- $\mathcal{A}_{g, d}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / k\right.$ of dimension $g$ and $\left.[k: \mathbb{Q}]=d\right\} / \simeq$
- There is even a stronger version for endomorphism rings.
- Very little is known:


## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A)=\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- $\mathcal{A}_{g, d}=\left\{\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A): A / k\right.$ of dimension $g$ and $\left.[k: \mathbb{Q}]=d\right\} / \simeq$


## Conjecture (Coleman)

The set $\mathcal{A}_{g, d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings. - Very little is known:


## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A)=\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- $\mathcal{A}_{g, d}=\left\{\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A): A / k\right.$ of dimension $g$ and $\left.[k: \mathbb{Q}]=d\right\} / \simeq$


## Conjecture (Coleman)

The set $\mathcal{A}_{g, d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings.


## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A)=\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- $\mathcal{A}_{g, d}=\left\{\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A): A / k\right.$ of dimension $g$ and $\left.[k: \mathbb{Q}]=d\right\} / \simeq$


## Conjecture (Coleman)

The set $\mathcal{A}_{g, d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings.
- Very little is known:
> - It is known for $\mathcal{A}_{1, d}$ (elliptic curves)

> It is known if we restrict to CM abelian varieties: $\mathcal{A}_{g, d}^{\mathrm{CM}}$ is finite (Orr-Skorobogatov 2018)
> - We are interested in $\mathcal{A}_{2,1}$ (abelian surfaces over ©)

## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A)=\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- $\mathcal{A}_{g, d}=\left\{\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A): A / k\right.$ of dimension $g$ and $\left.[k: \mathbb{Q}]=d\right\} / \simeq$


## Conjecture (Coleman)

The set $\mathcal{A}_{g, d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings.
- Very little is known:
- It is known for $\mathcal{A}_{1, d}$ (elliptic curves)
(Orr-Skorobogatov 2018)
- We are interested in $\mathcal{A}_{2,1}$ (abelian surfaces over $\mathbb{Q}$ )


## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A)=\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- $\mathcal{A}_{g, d}=\left\{\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A): A / k\right.$ of dimension $g$ and $\left.[k: \mathbb{Q}]=d\right\} / \simeq$


## Conjecture (Coleman)

The set $\mathcal{A}_{g, d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings.
- Very little is known:
- It is known for $\mathcal{A}_{1, d}$ (elliptic curves)
- It is known if we restrict to CM abelian varieties: $\mathcal{A}_{g, d}^{\mathrm{CM}}$ is finite (Orr-Skorobogatov 2018)
- We are interested in $\mathcal{A}_{2,1}$ (abelian surfaces over $\mathbb{Q}$ )


## A conjecture of Coleman

- $A$ abelian variety over a number field $k$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A)=\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- $\mathcal{A}_{g, d}=\left\{\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A): A / k\right.$ of dimension $g$ and $\left.[k: \mathbb{Q}]=d\right\} / \simeq$


## Conjecture (Coleman)

The set $\mathcal{A}_{g, d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings.
- Very little is known:
- It is known for $\mathcal{A}_{1, d}$ (elliptic curves)
- It is known if we restrict to CM abelian varieties: $\mathcal{A}_{g, d}^{\mathrm{CM}}$ is finite (Orr-Skorobogatov 2018)
- We are interested in $\mathcal{A}_{2,1}$ (abelian surfaces over $\mathbb{Q}$ )


## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e.c.,$[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}^{0}(E) \simeq$
- Constructing elliptic curves over $\mathbb{C}$ with CM by M - $I \subseteq M$ a fractional ideal
- If $E / k$ has CM by $M, \mathbb{Q}(j(E)) \subseteq k \Rightarrow \# \mathrm{Cl}(M) \leq[k: \mathbb{Q}]=d$.
- Heilbronn (1934): $\exists$ finitely many $\mathbb{Q}(\sqrt{-D})$ with $\# \mathrm{Cl}(M) \leq d$ - $\mathcal{A}_{1, d}$ is finite for all $\alpha$
- For $d \leq 100$ the set $\mathcal{A}_{1, d}$ is known explicitly (Watkins)
- $\mathcal{A}_{1,1}=\{\mathbb{Q}\} \cup\{\mathbb{Q}(\sqrt{-D}): D=3,4,7,8,11,19,43,67,163\}$


## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c.,$[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}>0\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by M - $I \subseteq M$ a fractional ideal
- If $E / k$ has CM by $M, \mathbb{Q}(j(E)) \subseteq k \Rightarrow \# \mathrm{Cl}(M) \leq[k: \mathbb{Q}]=d$.
- Heilbronn (1934): $\exists$ finitely many $\mathbb{Q}(\sqrt{-D})$ with $\# \mathrm{Cl}(M) \leq d$ - $\mathcal{A}_{1, d}$ is finite for all $d$
- For $d \leq 100$ the set $\mathcal{A}_{1, d}$ is known explicitly (Watkins)
- $\mathcal{A}_{1,1}=\{\mathbb{Q}\} \cup\{\mathbb{Q}(\sqrt{-D}): D=3,4,7,8,11,19,43,67,163\}$


## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e.c.,$[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}_{>0}\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
$\rightarrow I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / /$ is an elliptic curve with End $(E) \simeq O_{M}$
$\square$ $\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \stackrel{1: 1}{\longrightarrow}^{\text {I }} I \subseteq M$ fractional ideals $\} / I \sim \lambda I=\mathrm{Cl}(M)$


## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c.,$[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}>0\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
- $I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / I$ is an elliptic curve with $\operatorname{End}(E) \simeq \mathcal{O}_{M}$

$$
\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \stackrel{1: 1}{\stackrel{1}{\longleftrightarrow}}\{I \subseteq M \text { fractional ideals }\} / I \sim \lambda I=\mathrm{Cl}(M)
$$

## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c.,$[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}_{>0}\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
- $I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / I$ is an elliptic curve with $\operatorname{End}(E) \simeq \mathcal{O}_{M}$

Theory of Complex Multiplication
$\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \stackrel{1: 1}{\longleftrightarrow}\{I \subseteq M$ fractional ideals $\} / I \sim \lambda I=\operatorname{Cl}(M)$
If $E$ has CM by $\mathcal{O}_{M}$ then $j(E) \in \overline{\mathbb{Q}}$ and $[\mathbb{Q}(j(E)): \mathbb{Q}]=\# \mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O} \subseteq \mathcal{O}_{M}$ then $\# \mathrm{Cl}(M) \leq[\mathbb{Q}(j(E)): \mathbb{Q}]$

## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c. , $[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}>0\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
- $I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / I$ is an elliptic curve with $\operatorname{End}(E) \simeq \mathcal{O}_{M}$

Theory of Complex Multiplication
$\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \xrightarrow{1: 1} \stackrel{1}{\longrightarrow}\{I \subseteq M$ fractional ideals $\} / I \sim \lambda I=\mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O}_{M}$ then $j(E) \in \overline{\mathbb{Q}}$ and $[\mathbb{Q}(j(E)): \mathbb{Q}]=\# \mathrm{Cl}(M)$


## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c. , $[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}>0\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
- $I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / I$ is an elliptic curve with $\operatorname{End}(E) \simeq \mathcal{O}_{M}$


## Theory of Complex Multiplication

$\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \xrightarrow{1: 1} \underset{\longrightarrow}{ }\{I \subseteq M$ fractional ideals $\} / I \sim \lambda I=\mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O}_{M}$ then $j(E) \in \overline{\mathbb{Q}}$ and $[\mathbb{Q}(j(E)): \mathbb{Q}]=\# \mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O} \subseteq \mathcal{O}_{M}$ then $\# \mathrm{Cl}(M) \leq[\mathbb{Q}(j(E)): \mathbb{Q}]$

## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c. , $[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}>0\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
- $I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / I$ is an elliptic curve with $\operatorname{End}(E) \simeq \mathcal{O}_{M}$


## Theory of Complex Multiplication

$\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \xrightarrow{1: 1}\{I \subseteq M$ fractional ideals $\} / I \sim \lambda I=\mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O}_{M}$ then $j(E) \in \overline{\mathbb{Q}}$ and $[\mathbb{Q}(j(E)): \mathbb{Q}]=\# \mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O} \subseteq \mathcal{O}_{M}$ then $\# \mathrm{Cl}(M) \leq[\mathbb{Q}(j(E)): \mathbb{Q}]$

- If $E / k$ has CM by $M, \mathbb{Q}(j(E)) \subseteq k \Rightarrow \# \mathrm{Cl}(M) \leq[k: \mathbb{Q}]=d$.


## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c. , $[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}>0\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
- $I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / I$ is an elliptic curve with $\operatorname{End}(E) \simeq \mathcal{O}_{M}$


## Theory of Complex Multiplication

$\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \stackrel{1: 1}{\stackrel{1}{\longrightarrow}\{I \subseteq M \text { fractional ideals }\} / I \sim \lambda I=\mathrm{Cl}(M), ~(I)}$
If $E$ has CM by $\mathcal{O}_{M}$ then $j(E) \in \overline{\mathbb{Q}}$ and $[\mathbb{Q}(j(E)): \mathbb{Q}]=\# \mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O} \subseteq \mathcal{O}_{M}$ then $\# \mathrm{Cl}(M) \leq[\mathbb{Q}(j(E)): \mathbb{Q}]$

- If $E / k$ has CM by $M, \mathbb{Q}(j(E)) \subseteq k \Rightarrow \# \mathrm{Cl}(M) \leq[k: \mathbb{Q}]=d$.
- Heilbronn (1934): $\exists$ finitely many $\mathbb{Q}(\sqrt{-D})$ with $\# \mathrm{Cl}(M) \leq d$
- $\mathcal{A}_{1, d}$ is finite for all $d$


## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c. , $[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}>0\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
- $I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / I$ is an elliptic curve with $\operatorname{End}(E) \simeq \mathcal{O}_{M}$


## Theory of Complex Multiplication

$\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \stackrel{1: 1}{\stackrel{1}{\longrightarrow}\{I \subseteq M \text { fractional ideals }\} / I \sim \lambda I=\mathrm{Cl}(M), ~(I)}$
If $E$ has CM by $\mathcal{O}_{M}$ then $j(E) \in \overline{\mathbb{Q}}$ and $[\mathbb{Q}(j(E)): \mathbb{Q}]=\# \mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O} \subseteq \mathcal{O}_{M}$ then $\# \mathrm{Cl}(M) \leq[\mathbb{Q}(j(E)): \mathbb{Q}]$

- If $E / k$ has CM by $M, \mathbb{Q}(j(E)) \subseteq k \Rightarrow \# \mathrm{Cl}(M) \leq[k: \mathbb{Q}]=d$.
- Heilbronn (1934): $\exists$ finitely many $\mathbb{Q}(\sqrt{-D})$ with $\# \mathrm{Cl}(M) \leq d$
- $\mathcal{A}_{1, d}$ is finite for all $d$
- For $d \leq 100$ the set $\mathcal{A}_{1, d}$ is known explicitly (Watkins)


## The case $\mathcal{A}_{1, d}$ of elliptic curves

- $E / k$ e. c. , $[k: \mathbb{Q}]=d \rightsquigarrow \operatorname{End}_{\mathbb{Q}}^{0}(E) \simeq\left\{\begin{array}{l}\mathbb{Q} \\ M=\mathbb{Q}(\sqrt{-D}), D \in \mathbb{Q}>0\end{array}\right.$
- Constructing elliptic curves over $\mathbb{C}$ with CM by $M$
- $I \subseteq M$ a fractional ideal
- $I \subseteq \mathbb{C}$ is a lattice and $\mathbb{C} / I$ is an elliptic curve with $\operatorname{End}(E) \simeq \mathcal{O}_{M}$


## Theory of Complex Multiplication

$\left\{E / \mathbb{C}: \operatorname{End}(E) \simeq \mathcal{O}_{M}\right\} / \simeq \stackrel{1: 1}{\leftrightarrows}\{I \subseteq M$ fractional ideals $\} / I \sim \lambda I=\operatorname{Cl}(M)$
If $E$ has CM by $\mathcal{O}_{M}$ then $j(E) \in \overline{\mathbb{Q}}$ and $[\mathbb{Q}(j(E)): \mathbb{Q}]=\# \mathrm{Cl}(M)$
If $E$ has CM by $\mathcal{O} \subseteq \mathcal{O}_{M}$ then $\# \mathrm{Cl}(M) \leq[\mathbb{Q}(j(E)): \mathbb{Q}]$

- If $E / k$ has CM by $M, \mathbb{Q}(j(E)) \subseteq k \Rightarrow \# \mathrm{Cl}(M) \leq[k: \mathbb{Q}]=d$.
- Heilbronn (1934): $\exists$ finitely many $\mathbb{Q}(\sqrt{-D})$ with $\# \mathrm{Cl}(M) \leq d$
- $\mathcal{A}_{1, d}$ is finite for all $d$
- For $d \leq 100$ the set $\mathcal{A}_{1, d}$ is known explicitly (Watkins)
- $\mathcal{A}_{1,1}=\{\mathbb{Q}\} \cup\{\mathbb{Q}(\sqrt{-D}): D=3,4,7,8,11,19,43,67,163\}$

The case $\mathcal{A}_{2,1}$ of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2\right\} / \simeq$
- $A / \mathbb{Q}$ an abelian surface
- geometrically simple if $A_{\bar{Q}}$ is simple
- geometrically split if $A_{\bar{Q}} \sim E_{1} \times E_{2}$
- $\operatorname{End}_{\frac{0}{\mathbb{Q}}}(A) \simeq\left\{\begin{array}{l}\mathbb{Q}, \mathbb{Q}(\sqrt{D}), C M \text { field }, B / \mathbb{Q} \text { division indef. quat. alg. } \\ \mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times M, M \times M^{\prime}, M_{2}(\mathbb{Q}), M_{2}(M), M=\mathbb{Q}(\sqrt{-D})\end{array}\right.$
- The case where $A$ is geometrically simple is open:
- There are 19 possibilities for the CM field (Murabayashi-Umegaki)
- Nothing is known for the real quadratic field or quaternion algebra
- $\mathcal{A}_{2,1}^{\text {split }}=\left\{\operatorname{End}_{\frac{0}{\mathbb{Q}}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2, A\right.$ geom. split $\}$
- $\mathcal{A}_{2,1}^{\text {split }}$ is finite (Shafarevich, 1996)


## The case $\mathcal{A}_{2,1}$ of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2\right\} / \simeq$
- $A / \mathbb{Q}$ an abelian surface
- geometrically simple if $A_{\overline{\mathbb{Q}}}$ is simple
- geometrically split if $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$

- The case where $A$ is geometrically simple is open:
- There are 19 possibilities for the CM field (Murabayashi-Umegaki)
- Nothing is known for the real quadratic field or quaternion algebra
- $\mathcal{A}_{2,1}^{\text {split }}=\left\{\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2, A\right.$ geom. split $\}$
- $A_{2.1}^{\text {split }}$ is finite (Shafarevich, 1996)


## The case $\mathcal{A}_{2,1}$ of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2\right\} / \simeq$
- $A / \mathbb{Q}$ an abelian surface
- geometrically simple if $A_{\overline{\mathbb{Q}}}$ is simple
- geometrically split if $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq\left\{\begin{array}{l}\mathbb{Q}, \mathbb{Q}(\sqrt{D}), \mathrm{CM} \text { field }, B / \mathbb{Q} \text { division indef. quat. alg. } \\ \mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times M, M \times M^{\prime}, \mathrm{M}_{2}(\mathbb{Q}), \mathrm{M}_{2}(M), M=\mathbb{Q}(\sqrt{-D})\end{array}\right.$
- The case where $A$ is geometrically simple is open:
- There are 19 possibilities for the CM field (Murabayashi-Umegaki) - Nothing is known for the real quadratic field or quaternion algebra - $\mathcal{A}_{2,1}^{\text {split }}=\left\{\operatorname{End}_{\frac{0}{0}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2, A\right.$ geom. split $\}$


## The case $\mathcal{A}_{2,1}$ of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2\right\} / \simeq$
- $A / \mathbb{Q}$ an abelian surface
- geometrically simple if $A_{\overline{\mathbb{Q}}}$ is simple
- geometrically split if $A_{\bar{Q}} \sim E_{1} \times E_{2}$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq\left\{\begin{array}{l}\mathbb{Q}, \mathbb{Q}(\sqrt{D}), \mathrm{CM} \text { field }, B / \mathbb{Q} \text { division indef. quat. alg. } \\ \mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times M, M \times M^{\prime}, \mathrm{M}_{2}(\mathbb{Q}), \mathrm{M}_{2}(M), \quad M=\mathbb{Q}(\sqrt{-D})\end{array}\right.$
- The case where $A$ is geometrically simple is open:
- There are 19 possibilities for the CM field (Murabayashi-Umegaki)
- Nothing is known for the real quadratic field or quaternion algebra


## The case $\mathcal{A}_{2,1}$ of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2\right\} / \simeq$
- $A / \mathbb{Q}$ an abelian surface
- geometrically simple if $A_{\overline{\mathbb{Q}}}$ is simple
- geometrically split if $A_{\bar{Q}} \sim E_{1} \times E_{2}$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq\left\{\begin{array}{l}\mathbb{Q}, \mathbb{Q}(\sqrt{D}), \mathrm{CM} \text { field }, B / \mathbb{Q} \text { division indef. quat. alg. } \\ \mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times M, M \times M^{\prime}, \mathrm{M}_{2}(\mathbb{Q}), \mathrm{M}_{2}(M), \quad M=\mathbb{Q}(\sqrt{-D})\end{array}\right.$
- The case where $A$ is geometrically simple is open:
- There are 19 possibilities for the CM field (Murabayashi-Umegaki)
- Nothing is known for the real quadratic field or quaternion algebra
- $\mathcal{A}_{2,1}^{\text {split }}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2, A\right.$ geom. split $\}$


## The case $\mathcal{A}_{2,1}$ of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2\right\} / \simeq$
- $A / \mathbb{Q}$ an abelian surface
- geometrically simple if $A_{\overline{\mathbb{Q}}}$ is simple
- geometrically split if $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq\left\{\begin{array}{l}\mathbb{Q}, \mathbb{Q}(\sqrt{D}), \mathrm{CM} \text { field }, B / \mathbb{Q} \text { division indef. quat. alg. } \\ \mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times M, M \times M^{\prime}, \mathrm{M}_{2}(\mathbb{Q}), \mathrm{M}_{2}(M), M=\mathbb{Q}(\sqrt{-D})\end{array}\right.$
- The case where $A$ is geometrically simple is open:
- There are 19 possibilities for the CM field (Murabayashi-Umegaki)
- Nothing is known for the real quadratic field or quaternion algebra
- $\mathcal{A}_{2,1}^{\text {split }}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2, A\right.$ geom. split $\}$
- $\mathcal{A}_{2,1}^{\text {split }}$ is finite (Shafarevich, 1996)
$\square$


## The case $\mathcal{A}_{2,1}$ of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2\right\} / \simeq$
- $A / \mathbb{Q}$ an abelian surface
- geometrically simple if $A_{\bar{Q}}$ is simple
- geometrically split if $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq\left\{\begin{array}{l}\mathbb{Q}, \mathbb{Q}(\sqrt{D}), \mathrm{CM} \text { field }, B / \mathbb{Q} \text { division indef. quat. alg. } \\ \mathbb{Q} \times \mathbb{Q}, \mathbb{Q} \times M, M \times M^{\prime}, \mathrm{M}_{2}(\mathbb{Q}), \mathrm{M}_{2}(M), \quad M=\mathbb{Q}(\sqrt{-D})\end{array}\right.$
- The case where $A$ is geometrically simple is open:
- There are 19 possibilities for the CM field (Murabayashi-Umegaki)
- Nothing is known for the real quadratic field or quaternion algebra
- $\mathcal{A}_{2,1}^{\text {split }}=\left\{\operatorname{End}_{\mathbb{Q}}^{0}(A): A / \mathbb{Q}, \operatorname{dim}(A)=2, A\right.$ geom. split $\}$
- $\mathcal{A}_{2,1}^{\text {split }}$ is finite (Shafarevich, 1996)


## Our goal

To determine $\mathcal{A}_{2,1}^{\text {split }}$ explicitly

The case $\mathcal{A}_{2,1}^{\text {split }}$
If $A / \mathbb{Q}$ is a geometrically split abelian surface:

(0) $A_{\bar{Q}} \sim E^{2}$ and $E$ has CM by some $M$

## Main question <br> What imaginary quadratic fields $M$ do in fact appear?

The case $\mathcal{A}_{2,1}^{\text {split }}$
If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have $C M$ by some $M$
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$

Main question
What imaainary auadratic fields $M$ do in fact appear?

The case $\mathcal{A}_{2,1}^{\text {split }}$
If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{Q} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$

Main question
What imaginary quadratic fields $M$ do in fact appear?

The case $\mathcal{A}_{2,1}^{\text {split }}$
If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have $C M$ by some $M$
(3) $A_{\bar{Q}} \sim E^{2}$ and $E$ has CM by some $M$

Main question
What imaginary quadratic fields $M$ do in fact appear?

## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\bar{Q}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
$-\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
(3) $A_{\bar{Q}} \sim E^{2}$ and $E$ has CM by some $M$
What imaginary quadratic fields $M$ do in fact appear?


## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\bar{Q}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End} \frac{0}{\mathbb{Q}}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\bar{Q}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End} \frac{0}{\mathbb{Q}}(A) \simeq \mathbb{Q} \times M_{1}$ or $\left.\operatorname{End}{\underset{\widetilde{\mathbb{Q}}}{ }}_{0}^{( }\right) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- $A_{\bar{Q}} \sim E^{2}$ and $E$ has $C M$ by some $M$
What imaginary quadratic fields $M$ do in fact appear?


## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(-1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(3) $A_{\bar{Q}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $\mathrm{M}^{\prime} \mathrm{s}$


## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $M^{\prime}$ s
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$
- Key observation (Shafarevic):
$\exists d \in \mathbb{Z}_{>1}$ s.t. $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$ for some $K$ with $[K: \mathbb{Q}] \leq d$
- $E / K$ and $[K: \mathbb{Q}] \leq d \rightsquigarrow$ finitely many $M$ 's since $\mathcal{A}_{1, d}$ is finite.

What imaginary quadratic fields $M$ do in fact appear?

## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
$-\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $M^{\prime}$ s
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(M)$
- Key observation (Shafarevic):
$\exists d \in \mathbb{Z}_{>1}$ s.t. $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$ for some $K$ with $[K: \mathbb{Q}] \leq d$
- $E / K$ and $[K: \mathbb{Q}] \leq d \rightsquigarrow$ finitely many $M$ 's since $\mathcal{A}_{1, d}$ is finite.

What imaginary quadratic fields $M$ do in fact appear?

## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $M^{\prime}$ s
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(M)$
- Key observation (Shafarevic): $\exists d \in \mathbb{Z}_{>1}$ s.t. $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$ for some $K$ with $[K: \mathbb{Q}] \leq d$

What imaginary quadratic fields $M$ do in fact appear?

## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $M^{\prime}$ s
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(M)$
- Key observation (Shafarevic): $\exists d \in \mathbb{Z}_{>1}$ s.t. $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$ for some $K$ with $[K: \mathbb{Q}] \leq d$ $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ acts on $\operatorname{End}\left(A_{\overline{\mathbb{D}}}\right) \simeq \mathbb{Z}^{8}$ with kernel $\operatorname{Gal}(\overline{\mathbb{Q}} / K)$

What imaginary quadratic fields $M$ do in fact appear?

## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $M^{\prime}$ s
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(M)$
- Key observation (Shafarevic): $\exists d \in \mathbb{Z}_{>1}$ s.t. $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$ for some $K$ with $[K: \mathbb{Q}] \leq d$ $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ acts on $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \simeq \mathbb{Z}^{8}$ with kernel $\operatorname{Gal}(\overline{\mathbb{Q}} / K)$ $\operatorname{Gal}(K / \mathbb{Q}) \hookrightarrow \mathrm{GL}_{8}(\mathbb{Z})$

What imaginary quadratic fields $M$ do in fact appear?

## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $M^{\prime}$ s
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(M)$
- Key observation (Shafarevic): $\exists d \in \mathbb{Z}_{>1}$ s.t. $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$ for some $K$ with $[K: \mathbb{Q}] \leq d$ $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ acts on $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \simeq \mathbb{Z}^{8}$ with kernel $\operatorname{Gal}(\overline{\mathbb{Q}} / K)$ $\operatorname{Gal}(K / \mathbb{Q}) \hookrightarrow \operatorname{GL}_{8}(\mathbb{Z})$ and $\operatorname{sogal}(K / \mathbb{Q}) \hookrightarrow \mathrm{GL}_{8}(\mathbb{Z} / 3 \mathbb{Z})$

What imaginary quadratic fields $M$ do in fact appear?

## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $M^{\prime}$ s
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has CM by some $M$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(M)$
- Key observation (Shafarevic):
$\exists d \in \mathbb{Z}_{>1}$ s.t. $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$ for some $K$ with $[K: \mathbb{Q}] \leq d$ $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ acts on $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \simeq \mathbb{Z}^{8}$ with kernel $\operatorname{Gal}(\overline{\mathbb{Q}} / K)$ $\operatorname{Gal}(K / \mathbb{Q}) \hookrightarrow \mathrm{GL}_{8}(\mathbb{Z})$ and $\operatorname{so} \operatorname{Gal}(K / \mathbb{Q}) \hookrightarrow \mathrm{GL}_{8}(\mathbb{Z} / 3 \mathbb{Z})$
- $E / K$ and $[K: \mathbb{Q}] \leq d \rightsquigarrow$ finitely many $M$ 's since $\mathcal{A}_{1, d}$ is finite.

What imaginary quadratic fields $M$ do in fact appear?

## The case $\mathcal{A}_{2,1}^{\text {split }}$

If $A / \mathbb{Q}$ is a geometrically split abelian surface:
(1) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM

- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$, and $E_{1}$ or $E_{2}$ have CM by some $M$
- $\operatorname{End}_{\overline{\mathbb{Q}}}^{0}(A) \simeq \mathbb{Q} \times M_{1}$ or $\operatorname{End}_{\mathbb{\mathbb { Q }}}^{0}(A) \simeq M_{1} \times M_{2}$ with $M_{i}=\mathbb{Q}\left(\sqrt{-D_{i}}\right)$
- Fite-Kedlaya-Rotger-Sutherland: $E_{i} \sim C_{i}$ for some $C_{i} / \mathbb{Q}$
- Thus $\# \mathrm{Cl}(M)=1 \rightsquigarrow$ finitely many $M^{\prime}$ s
(3) $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by some $M$
- $\operatorname{End}_{\mathbb{Q}}^{0}(A) \simeq \mathrm{M}_{2}(M)$
- Key observation (Shafarevic):
$\exists d \in \mathbb{Z}_{>1}$ s.t. $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$ for some $K$ with $[K: \mathbb{Q}] \leq d$ $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ acts on $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right) \simeq \mathbb{Z}^{8}$ with kernel $\operatorname{Gal}(\overline{\mathbb{Q}} / K)$ $\operatorname{Gal}(K / \mathbb{Q}) \hookrightarrow \mathrm{GL}_{8}(\mathbb{Z})$ and $\operatorname{so} \operatorname{Gal}(K / \mathbb{Q}) \hookrightarrow \mathrm{GL}_{8}(\mathbb{Z} / 3 \mathbb{Z})$
- $E / K$ and $[K: \mathbb{Q}] \leq d \rightsquigarrow$ finitely many $M$ 's since $\mathcal{A}_{1, d}$ is finite.


## Main question

What imaginary quadratic fields $M$ do in fact appear?

## Main Theorem

Theorem (Fité-G.)
The set $\mathcal{A}_{21}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:

$\square$ In particular, the set $A_{2}^{\text {split }}$ has cardinality 92 .
(1f $A_{\bar{\infty}} \sim E_{1} \times E_{2}$ or $A_{\bar{\infty}} \sim E_{1}^{2}$ with $E_{i}$ non-CM
(2) If $A_{\overline{0}} \sim E_{1} \times E_{2}$ and $E_{i}$ can have CM
(3) Here $A_{\bar{\infty}} \sim E^{2}$ with $E$ with CM by $M$ : here is where the work is

## Main Theorem

Theorem (Fité-G.)
The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:
© $\mathbb{Q} \times \mathbb{Q}, \mathrm{M}_{2}(\mathbb{Q})$;
 and $M$ distinct from

$\square$
(1f $A_{\bar{\infty}} \sim E_{1} \times E_{2}$ or $A_{\bar{\infty}} \sim E_{1}^{2}$ with $E_{i}$ non-CM
(2) If $A_{\text {© }} \sim E_{1} \times E_{2}$ and $E_{i}$ can have CM
(3) Here $A_{\bar{\infty}} \sim E^{2}$ with $E$ with CM by $M$ : here is where the work is

## Main Theorem

## Theorem (Fité-G.)

The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:
(1) $\mathbb{Q} \times \mathbb{Q}, \mathrm{M}_{2}(\mathbb{Q})$;
(2) $\mathbb{Q} \times M_{1}, M_{1} \times M_{2}$, with $M_{i}$ quadratic imag. fields of $\# \mathrm{Cl}\left(M_{i}\right)=1$;

$\square$
(If $A_{\bar{\infty}} \sim E_{1} \times E_{2}$ or $A_{\bar{\sigma}} \sim E_{1}^{2}$ with $E_{i}$ non-CM
(2) If $A_{\overline{0}} \sim E_{1} \times E_{2}$ and $E_{i}$ can have CM
(0) Here $A_{\overline{0}} \sim E^{2}$ with $E$ with $C M$ by $M$ : here is where the work is

## Main Theorem

## Theorem (Fité-G.)

The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:
(1) $\mathbb{Q} \times \mathbb{Q}, \mathrm{M}_{2}(\mathbb{Q})$;
(2) $\mathbb{Q} \times M_{1}, M_{1} \times M_{2}$, with $M_{i}$ quadratic imag. fields of $\# \mathrm{Cl}\left(M_{i}\right)=1$;
(3) $\mathrm{M}_{2}(M)$ with $M$ quadratic imaginary field, $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $M$ distinct from

$$
\mathbb{Q}(\sqrt{-195}), \mathbb{Q}(\sqrt{-312}), \mathbb{Q}(\sqrt{-340}), \mathbb{Q}(\sqrt{-555}), \mathbb{Q}(\sqrt{-715}), \mathbb{Q}(\sqrt{-760})
$$

(1f $A_{\bar{Q}} \sim E_{1} \times E_{2}$ or $A_{\bar{Q}} \sim E_{1}^{2}$ with $E_{i}$ non-CM
(2) If $A_{0} \sim E_{1} \times E_{2}$ and $E_{i}$ can have CM
(0) Here $A_{\bar{\infty}} \sim E^{2}$ with $E$ with CM by $M$ : here is where the work is

## Main Theorem

## Theorem (Fité-G.)

The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:
(1) $\mathbb{Q} \times \mathbb{Q}, \mathrm{M}_{2}(\mathbb{Q})$;
(2) $\mathbb{Q} \times M_{1}, M_{1} \times M_{2}$, with $M_{i}$ quadratic imag. fields of $\# \mathrm{Cl}\left(M_{i}\right)=1$;
(3) $\mathrm{M}_{2}(M)$ with $M$ quadratic imaginary field, $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $M$ distinct from

$$
\mathbb{Q}(\sqrt{-195}), \mathbb{Q}(\sqrt{-312}), \mathbb{Q}(\sqrt{-340}), \mathbb{Q}(\sqrt{-555}), \mathbb{Q}(\sqrt{-715}), \mathbb{Q}(\sqrt{-760})
$$

In particular, the set $\mathcal{A}_{2,1}^{\text {split }}$ has cardinality 92 .


## Main Theorem

## Theorem (Fité-G.)

The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:
(1) $\mathbb{Q} \times \mathbb{Q}, \mathrm{M}_{2}(\mathbb{Q})$;
(2) $\mathbb{Q} \times M_{1}, M_{1} \times M_{2}$, with $M_{i}$ quadratic imag. fields of $\# \mathrm{Cl}\left(M_{i}\right)=1$;
(3) $\mathrm{M}_{2}(M)$ with $M$ quadratic imaginary field, $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $M$ distinct from

$$
\mathbb{Q}(\sqrt{-195}), \mathbb{Q}(\sqrt{-312}), \mathbb{Q}(\sqrt{-340}), \mathbb{Q}(\sqrt{-555}), \mathbb{Q}(\sqrt{-715}), \mathbb{Q}(\sqrt{-760})
$$

In particular, the set $\mathcal{A}_{2,1}^{\text {split }}$ has cardinality 92 .
(1) If $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM
(2) If $A_{\bar{O}} \sim E_{1} \times E_{2}$ and $E_{i}$ can have CM
(3) Here $A_{\overline{\mathbb{Q}}} \sim E^{2}$ with $E$ with CM by $M$ : here is where the work is

## Main Theorem

## Theorem (Fité-G.)

The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:
(1) $\mathbb{Q} \times \mathbb{Q}, \mathrm{M}_{2}(\mathbb{Q})$;
(2) $\mathbb{Q} \times M_{1}, M_{1} \times M_{2}$, with $M_{i}$ quadratic imag. fields of $\# \mathrm{Cl}\left(M_{i}\right)=1$;
(3) $\mathrm{M}_{2}(M)$ with $M$ quadratic imaginary field, $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $M$ distinct from

$$
\mathbb{Q}(\sqrt{-195}), \mathbb{Q}(\sqrt{-312}), \mathbb{Q}(\sqrt{-340}), \mathbb{Q}(\sqrt{-555}), \mathbb{Q}(\sqrt{-715}), \mathbb{Q}(\sqrt{-760})
$$

In particular, the set $\mathcal{A}_{2,1}^{\text {split }}$ has cardinality 92 .
(1) If $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM
(2) If $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ and $E_{i}$ can have CM
(3) Here $A_{\bar{D}} \sim E^{2}$ with $E$ with $C M$ by $M$ : here is where the work is

## Main Theorem

## Theorem (Fité-G.)

The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ is made of:
(1) $\mathbb{Q} \times \mathbb{Q}, \mathrm{M}_{2}(\mathbb{Q})$;
(2) $\mathbb{Q} \times M_{1}, M_{1} \times M_{2}$, with $M_{i}$ quadratic imag. fields of $\# \mathrm{Cl}\left(M_{i}\right)=1$;
(3) $\mathrm{M}_{2}(M)$ with $M$ quadratic imaginary field, $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $M$ distinct from

$$
\mathbb{Q}(\sqrt{-195}), \mathbb{Q}(\sqrt{-312}), \mathbb{Q}(\sqrt{-340}), \mathbb{Q}(\sqrt{-555}), \mathbb{Q}(\sqrt{-715}), \mathbb{Q}(\sqrt{-760})
$$

In particular, the set $\mathcal{A}_{2,1}^{\text {split }}$ has cardinality 92 .
(1) If $A_{\overline{\mathbb{Q}}} \sim E_{1} \times E_{2}$ or $A_{\overline{\mathbb{Q}}} \sim E_{1}^{2}$ with $E_{i}$ non-CM
(2) If $A_{\bar{Q}} \sim E_{1} \times E_{2}$ and $E_{i}$ can have CM
(3) Here $A_{\overline{\mathbb{Q}}} \sim E^{2}$ with $E$ with CM by $M$ : here is where the work is

## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?


Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- K/Q minimal such that $\operatorname{End}\left(A_{\bar{\infty}}\right)=\operatorname{End}\left(A_{K}\right)$
- Main idea:
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\odot}\right)=\operatorname{End}\left(A_{K}\right)$
- Main idea:
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right)$

- Main idea:
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- Main idea:
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- [FKRS12] $\rightsquigarrow \operatorname{Gal}(K / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- Main idea:
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- [FKRS12] $\rightsquigarrow \operatorname{Gal}(K / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- This implies $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- [FKRS12] $\rightsquigarrow \operatorname{Gal}(K / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- This implies $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- Main idea:
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- [FKRS12] $\rightsquigarrow \operatorname{Gal}(K / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- This implies $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- Main idea:
- Show that $\exists N \subseteq K$ with $\operatorname{Gal}(N / M)$ of exponent 2 such that $E$ can be defined over $N$ up to isogeny
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- [FKRS12] $\rightsquigarrow \operatorname{Gal}(K / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- This implies $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- Main idea:
- Show that $\exists N \subseteq K$ with $\operatorname{Gal}(N / M)$ of exponent 2 such that $E$ can be defined over $N$ up to isogeny
- $\operatorname{Gal}(N / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $H_{M} \subseteq N$ so $\operatorname{Gal}\left(H_{M} / M\right)$ as well


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\overline{\mathbb{Q}}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- [FKRS12] $\rightsquigarrow \operatorname{Gal}(K / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- This implies $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- Main idea:
- Show that $\exists N \subseteq K$ with $\operatorname{Gal}(N / M)$ of exponent 2 such that $E$ can be defined over $N$ up to isogeny
- $\operatorname{Gal}(N / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $H_{M} \subseteq N$ so $\operatorname{Gal}\left(H_{M} / M\right)$ as well
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\mathbb{Q}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- [FKRS12] $\rightsquigarrow \operatorname{Gal}(K / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- This implies $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- Main idea:
- Show that $\exists N \subseteq K$ with $\operatorname{Gal}(N / M)$ of exponent 2 such that $E$ can be defined over $N$ up to isogeny
- $\operatorname{Gal}(N / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $H_{M} \subseteq N$ so $\operatorname{Gal}\left(H_{M} / M\right)$ as well
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves
- $\sigma \in \operatorname{Gal}(K / \mathbb{Q}) \rightsquigarrow\left({ }^{\sigma} E\right)^{2} \sim{ }^{\sigma} A_{K}=A_{K} \sim E^{2}$
$\star$ There is an isogeny $\mu_{\sigma}:{ }^{\sigma} E \longrightarrow E$


## Squares of CM elliptic curves

## Central question

If $A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$, what are the possible $M$ 's?
Theorem (Fité-G., 2015)
Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, or $\mathrm{C}_{2} \times \mathrm{C}_{2}$

- $K / \mathbb{Q}$ minimal such that $\operatorname{End}\left(A_{\mathbb{Q}}\right)=\operatorname{End}\left(A_{K}\right) \rightsquigarrow E / K$ and $A_{K} \sim E^{2}$
- CM theory $\rightsquigarrow H_{M} \subseteq K$
- [FKRS12] $\rightsquigarrow \operatorname{Gal}(K / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- This implies $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{r}, \mathrm{D}_{r}$ with $r \in\{2,3,4,6\}$
- Main idea:
- Show that $\exists N \subseteq K$ with $\operatorname{Gal}(N / M)$ of exponent 2 such that $E$ can be defined over $N$ up to isogeny
- $\operatorname{Gal}(N / M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}, \mathrm{C}_{2} \times \mathrm{C}_{2}$ and $H_{M} \subseteq N$ so $\operatorname{Gal}\left(H_{M} / M\right)$ as well
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves
- $\sigma \in \operatorname{Gal}(K / \mathbb{Q}) \rightsquigarrow\left({ }^{\sigma} E\right)^{2} \sim{ }^{\sigma} A_{K}=A_{K} \sim E^{2}$
$\star$ There is an isogeny $\mu_{\sigma}:{ }^{\sigma} E \longrightarrow E$
- Cohomology class $c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)$

$$
\star c_{E}(\sigma, \tau)=\mu_{\sigma} \circ{ }^{\sigma} \mu_{\tau} \circ \mu_{\sigma \tau}^{-1}
$$

## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow C_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow C_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $\simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $C_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresnonding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$

Now the question is
of these possible M's, which ones do really occur?

- Give a construction of $A$ 's for some $M$ 's

Weil descent (up to isogeny)
For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow C_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
$\simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $C_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$

Now the question is
of these possible M's, which ones do really occur?

- Give a construction of $A$ 's for some $M$ 's

Weil descent (up to isogeny)
For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow C_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $\simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $c_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$
of these possible M's, which ones do really occur?
- Give a construction of $A$ 's for some $M^{\prime}$ 's


## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow C_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $C_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$
of these possible $M$ 's, which ones do really occur?
- Give a construction of A's for some M's


## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow C_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $\simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $C_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $F_{K} \sim F_{K}^{*}$ for some $E^{*} / N$
of these possible M's, which ones do really occur?
- Give a construction of A's for some M's


## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow C_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- 

$$
\simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)
$$

- $c_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2 - Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$ of these possible M's, which ones do really occur?
- Give a construction of A's for some M's


## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow c_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $\quad \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $c_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$


## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow c_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $\quad \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $c_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$
- Give a construction of $A$ 's for some M's


## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow c_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $\quad \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $c_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$


## Now the question is of these possible M's, which ones do really occur?

## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow c_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $\quad \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $c_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$


## Now the question is

 of these possible M's, which ones do really occur?- Give a construction of A's for some M's


## Weil descent (up to isogeny)

For $L \subseteq K$, there exists $E^{*} / L$ with $E \sim E_{K}^{*} \Longleftrightarrow c_{E \mid \operatorname{Gal}(K / L)}=1$.

- $A_{K} \sim E^{2} \Rightarrow c_{E} \in H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2]$
- Ribet: $P=M^{\times} /\{ \pm 1\}$ and $M^{\times} \simeq\{ \pm 1\} \times P$.
- $H^{2}\left(\operatorname{Gal}(K / M), M^{\times}\right)[2] \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times H^{2}(\operatorname{Gal}(K / M), P)[2]$
- $\quad \simeq H^{2}(\operatorname{Gal}(K / M),\{ \pm 1\}) \times \operatorname{Hom}\left(\operatorname{Gal}(K / M), P / P^{2}\right)$
- $c_{E}$ is trivial when restricted to $\left\langle\sigma^{2}\right\rangle$
- $N \subseteq K$ the corresponding subfield, $\operatorname{Gal}(N / M) \simeq$ of exponent 2
- Weil descent: $E_{K} \sim E_{K}^{*}$ for some $E^{*} / N$


## Now the question is

 of these possible M's, which ones do really occur?- Give a construction of A's for some M's and rule out the other M's


## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\bar{Q}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2}$
- If $\mathrm{Cl}(M)=\mathrm{C}_{2} \times \mathrm{C}_{2}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=4$ and $\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has $\operatorname{dim} 4$


## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\bar{Q}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2} \times \mathrm{C}_{2}$ then $\left[\mathbb{Q}\left(j_{E}\right)\right.$ $\mathbb{Q}]=4$ and $\operatorname{Rese(je)/Q}$ E has dim 4


## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\bar{Q}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2}$



## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2}$
- If $E$ has CM by $\mathcal{O}_{M}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=2$, so we can take $E / \mathbb{Q}(j(E))$



## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2}$
- If $E$ has CM by $\mathcal{O}_{M}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=2$, so we can take $E / \mathbb{Q}(j(E))$
- $A=\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has dimension 2 and $A_{\overline{\mathbb{Q}}} \sim E \times{ }^{\sigma} E$



## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2}$
- If $E$ has CM by $\mathcal{O}_{M}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=2$, so we can take $E / \mathbb{Q}(j(E))$
- $A=\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has dimension 2 and $A_{\overline{\mathbb{Q}}} \sim E \times{ }^{\sigma} E$
- If $E$ has CM, then ${ }^{\sigma} E_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}$ and therefore $A_{\overline{\mathbb{Q}}} \sim E^{2}$


## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2}$
- If $E$ has CM by $\mathcal{O}_{M}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=2$, so we can take $E / \mathbb{Q}(j(E))$
- $A=\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has dimension 2 and $A_{\overline{\mathbb{Q}}} \sim E \times{ }^{\sigma} E$
- If $E$ has CM, then ${ }^{\sigma} E_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}$ and therefore $A_{\overline{\mathbb{Q}}} \sim E^{2}$
- If $\mathrm{Cl}(M)=\mathrm{C}_{2} \times \mathrm{C}_{2}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=4$ and $\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has $\operatorname{dim} 4$
- We will take $E$ to be a Gross $\mathbb{Q}$-curve


## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2}$
- If $E$ has CM by $\mathcal{O}_{M}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=2$, so we can take $E / \mathbb{Q}(j(E))$
- $A=\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has dimension 2 and $A_{\overline{\mathbb{Q}}} \sim E \times{ }^{\sigma} E$
- If $E$ has CM, then ${ }^{\sigma} E_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}$ and therefore $A_{\overline{\mathbb{Q}}} \sim E^{2}$
- If $\mathrm{Cl}(M)=\mathrm{C}_{2} \times \mathrm{C}_{2}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=4$ and $\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has $\operatorname{dim} 4$
- Idea: choose $E$ so that $\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E \sim A^{2}$


## Constructing abelian surfaces: restriction of scalars

## Easy cases

Given $M$ with $\mathrm{Cl}(M) \simeq \mathrm{C}_{1}, \mathrm{C}_{2}$, construct $A / \mathbb{Q}$ abelian surface with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ with CM by $M$.

- If $\mathrm{Cl}(M)=1$
- take $E / \mathbb{Q}$ with CM by $M$ and $A=E \times E$.
- If $\mathrm{Cl}(M)=\mathrm{C}_{2}$
- If $E$ has CM by $\mathcal{O}_{M}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=2$, so we can take $E / \mathbb{Q}(j(E))$
- $A=\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has dimension 2 and $A_{\overline{\mathbb{Q}}} \sim E \times{ }^{\sigma} E$
- If $E$ has CM, then ${ }^{\sigma} E_{\overline{\mathbb{Q}}} \sim E_{\overline{\mathbb{Q}}}$ and therefore $A_{\overline{\mathbb{Q}}} \sim E^{2}$
- If $\mathrm{Cl}(M)=\mathrm{C}_{2} \times \mathrm{C}_{2}$ then $\left[\mathbb{Q}\left(j_{E}\right): \mathbb{Q}\right]=4$ and $\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ has $\operatorname{dim} 4$
- Idea: choose $E$ so that $\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E \sim A^{2}$
- We will take $E$ to be a Gross $\mathbb{Q}$-curve


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
- E/H elliptic curve with CM by M s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / Q)$

Theorem (Shimura-Nakamura)
$\exists$ Gross $\mathbb{Q}$-curve $E / H \Longleftrightarrow \operatorname{Disc}(M) \neq-4 \times($ primes $\equiv 1(\bmod 4))$

- $M=\mathbb{Q}(\sqrt{-D})$ has $\operatorname{Cl}(M) \simeq C_{2} \times C_{2}$ for $D$ in

$$
\begin{aligned}
D^{2,2}= & \{84,120,132,168,195,228,280,312,340,372,408,435, \\
& 483,520,532,555,595,627,708,715,760,795,1012,1435\}
\end{aligned}
$$

- There is Gross $\mathbb{Q}$-curves in all cases except $D=340$
- (Gross) If $\mathcal{E}=\operatorname{Res}_{\mathbb{D}\left(j_{E}\right) / \mathbb{D}} E$ then $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathbb{Q}^{C_{E}}[\operatorname{Gal}(H / M]$
- If End ${ }^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q}) \rightsquigarrow \mathcal{E} \sim A^{2}$ and we're done!
- For $D \neq 340$, Nakamura showed that:
- For each $D$, Gross $\mathbb{Q}$-curves $D$ give rise to 8 cohomology classes
- Gave a method for computing all these cohomology classes $C_{E}$


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
-E/H elliptic curve with CM by $M$ s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / \mathbb{Q})$


## Theorem (Shimura-Nakamura)

$\exists$ Gross $\mathbb{Q}$-curve $E / H \Longleftrightarrow \operatorname{Disc}(M) \neq-4 \times($ primes $\equiv 1(\bmod 4))$

- $M=\mathbb{Q}(\sqrt{-D})$ has $\operatorname{Cl}(M) \simeq C_{2} \times C_{2}$ for $D$ in

$$
\begin{aligned}
D^{2,2}= & \{84,120,132,168,195,228,280,312,340,372,408,435, \\
& 483,520,532,555,595,627,708,715,760,795,1012,1435\}
\end{aligned}
$$

- There is Gross $\mathbb{Q}$-curves in all cases except $D=340$
- (Gross) If $\mathcal{E}=\operatorname{Res}_{\mathbb{D}\left(j_{E}\right) / \mathbb{D}} E$ then $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathbb{Q}^{C_{E}}[\operatorname{Gal}(H / M]$
- If $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q}) \rightsquigarrow \mathcal{E} \sim A^{2}$ and we're done!
- For $D \neq 340$, Nakamura showed that:
- For each $D$, Gross $\mathbb{Q}$-curves $D$ give rise to 8 cohomology classes
- Gave a method for computing all these cohomology classes $C_{E}$


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
- $E / H$ elliptic curve with CM by $M$ s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / \mathbb{Q})$

- $M=\mathbb{Q}(\sqrt{-D})$ has $\mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$ for $D$ in
$D^{2,2}=\left\{\begin{array}{lllllll}84 & 120 & 132 & 168 & 195 & 228 & 280 \\ 312 & 340 & 372,408,435\end{array}\right.$ $483,520,532,555,595,627,708,715,760,795,1012,1435\}$
- There is Gross $\mathbb{Q}$-curves in all cases except $D=340$
- (Gross) If $\mathcal{E}=\operatorname{Res}_{\mathbb{O}\left(j_{E}\right) / \mathbb{Q}} E$ then $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathbb{Q}^{C_{E}}[\operatorname{Gal}(H / M]$
- If $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q}) \rightsquigarrow \mathcal{E} \sim A^{2}$ and we're done!
- For $D \neq 340$, Nakamura showed that:
- For each $D$, Gross $\mathbb{Q}$-curves $D$ give rise to 8 cohomology classes
- Gave a method for computing all these cohomology classes $C_{E}$


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
- $E / H$ elliptic curve with CM by $M$ s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / \mathbb{Q})$

Theorem (Shimura-Nakamura)
$\exists$ Gross $\mathbb{Q}$-curve $E / H \Longleftrightarrow \operatorname{Disc}(M) \neq-4 \times($ primes $\equiv 1(\bmod 4))$


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
- $E / H$ elliptic curve with CM by $M$ s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / \mathbb{Q})$

Theorem (Shimura-Nakamura)
$\exists$ Gross $\mathbb{Q}$-curve $E / H \Longleftrightarrow \operatorname{Disc}(M) \neq-4 \times($ primes $\equiv 1(\bmod 4))$

- $M=\mathbb{Q}(\sqrt{-D})$ has $\mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$ for $D$ in

$$
\begin{aligned}
D^{2,2}= & \{84,120,132,168,195,228,280,312,340,372,408,435, \\
& 483,520,532,555,595,627,708,715,760,795,1012,1435\}
\end{aligned}
$$

- There is Gross $\mathbb{Q}$-curves in all cases except $D=340$
- (Gross) If $\mathcal{E}=\operatorname{Res}_{\mathbb{Q}\left(\mathcal{J}_{E}\right) / \mathbb{Q}} E$ then $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathbb{Q}^{C_{E}}[\operatorname{Gal}(H / M]$
- If $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q}) \rightsquigarrow \mathcal{E} \sim A^{2}$ and we're done!
- For $D \neq 340$, Nakamura showed that:
- For each $D$, Gross $\mathbb{Q}$-curves $D$ give rise to 8 cohomology classes
- Gave a method for computing all these cohomology classes $C_{E}$


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
- $E / H$ elliptic curve with CM by $M$ s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / \mathbb{Q})$


## Theorem (Shimura-Nakamura)

$\exists$ Gross $\mathbb{Q}$-curve $E / H \Longleftrightarrow \operatorname{Disc}(M) \neq-4 \times($ primes $\equiv 1(\bmod 4))$

- $M=\mathbb{Q}(\sqrt{-D})$ has $\mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$ for $D$ in

$$
\begin{aligned}
D^{2,2}= & \{84,120,132,168,195,228,280,312,340,372,408,435, \\
& 483,520,532,555,595,627,708,715,760,795,1012,1435\}
\end{aligned}
$$

- There is Gross $\mathbb{Q}$-curves in all cases except $D=340$
- (Gross) If $\left.\mathcal{E}=\operatorname{Res}_{\mathbb{Q}(\mathcal{E})}\right) \mathbb{Q}^{E}$ then $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathbb{Q}^{C_{E}}[\operatorname{Gal}(H / M]$
- If End $(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q}) \rightsquigarrow \mathcal{E} \sim A^{2}$ and we're done!
- For $D \neq 340$, Nakamura showed that:
- For each $D$, Gross $\mathbb{Q}$-curves $D$ give rise to 8 cohomology classes
- Gave a method for computing all these cohomology classes $C_{E}$


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
- $E / H$ elliptic curve with CM by $M$ s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / \mathbb{Q})$


## Theorem (Shimura-Nakamura)

$\exists$ Gross $\mathbb{Q}$-curve $E / H \Longleftrightarrow \operatorname{Disc}(M) \neq-4 \times($ primes $\equiv 1(\bmod 4))$

- $M=\mathbb{Q}(\sqrt{-D})$ has $\mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$ for $D$ in

$$
\begin{aligned}
D^{2,2}= & \{84,120,132,168,195,228,280,312,340,372,408,435, \\
& 483,520,532,555,595,627,708,715,760,795,1012,1435\}
\end{aligned}
$$

- There is Gross $\mathbb{Q}$-curves in all cases except $D=340$
- (Gross) If $\mathcal{E}=\operatorname{Res}_{\mathbb{Q}\left(j_{\mathcal{E}}\right) / \mathbb{Q}} E$ then $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathbb{Q}^{C_{E}}[\operatorname{Gal}(H / M]$


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
- $E / H$ elliptic curve with CM by $M$ s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / \mathbb{Q})$


## Theorem (Shimura-Nakamura)

$\exists$ Gross $\mathbb{Q}$-curve $E / H \Longleftrightarrow \operatorname{Disc}(M) \neq-4 \times($ primes $\equiv 1(\bmod 4))$

- $M=\mathbb{Q}(\sqrt{-D})$ has $\mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$ for $D$ in

$$
\begin{aligned}
D^{2,2}= & \{84,120,132,168,195,228,280,312,340,372,408,435, \\
& 483,520,532,555,595,627,708,715,760,795,1012,1435\}
\end{aligned}
$$

- There is Gross $\mathbb{Q}$-curves in all cases except $D=340$
- (Gross) If $\mathcal{E}=\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ then $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathbb{Q}^{C_{E}}[\operatorname{Gal}(H / M]$
- If $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q}) \rightsquigarrow \mathcal{E} \sim A^{2}$ and we're done!


## Case $\mathrm{C}_{2} \times \mathrm{C}_{2}$ : Gross's $\mathbb{Q}$-curves

- $H=H_{M}$ Hilbert class field of $M$.
- A Gross $\mathbb{Q}$-curve is
- $E / H$ elliptic curve with CM by $M$ s.t. ${ }^{\sigma} E \sim E \forall \sigma \in \operatorname{Gal}(H / \mathbb{Q})$


## Theorem (Shimura-Nakamura)

$\exists$ Gross $\mathbb{Q}$-curve $E / H \Longleftrightarrow \operatorname{Disc}(M) \neq-4 \times($ primes $\equiv 1(\bmod 4))$

- $M=\mathbb{Q}(\sqrt{-D})$ has $\mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$ for $D$ in

$$
\begin{aligned}
D^{2,2}= & \{84,120,132,168,195,228,280,312,340,372,408,435, \\
& 483,520,532,555,595,627,708,715,760,795,1012,1435\}
\end{aligned}
$$

- There is Gross $\mathbb{Q}$-curves in all cases except $D=340$
- (Gross) If $\mathcal{E}=\operatorname{Res}_{\mathbb{Q}\left(j_{E}\right) / \mathbb{Q}} E$ then $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathbb{Q}^{C_{E}}[\operatorname{Gal}(H / M]$
- If $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q}) \rightsquigarrow \mathcal{E} \sim A^{2}$ and we're done!
- For $D \neq 340$, Nakamura showed that:
- For each $D$, Gross $\mathbb{Q}$-curves $D$ give rise to 8 cohomology classes
- Gave a method for computing all these cohomology classes $C_{E}$


## Computing the endomorphism algebra of $\mathcal{E}$

- For each $D \neq 340$, we computed $\operatorname{End}^{0}(\mathcal{E})$ for each of the eight representatives of $\mathbb{Q}$-curves with CM by $\mathbb{Q}(\sqrt{-D})$.

$$
\text { For } D \in\{84,120,132,168,228,280,372,408,435,483 \text {, }
$$

## $520,532,595,627,708,795,1012,1435\}$



- Need to show that for the fields $M$ in 2, $A$ does not exist


## Computing the endomorphism algebra of $\mathcal{E}$

- For each $D \neq 340$, we computed $\operatorname{End}^{0}(\mathcal{E})$ for each of the eight representatives of $\mathbb{Q}$-curves with CM by $\mathbb{Q}(\sqrt{-D})$.
(1)

$$
\begin{aligned}
\text { For } D \in & \{84,120,132,168,228,280,372,408,435,483, \\
& 520,532,595,627,708,795,1012,1435\}
\end{aligned}
$$

at least one of the $\mathbb{Q}$-curves has $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q})$.


## Computing the endomorphism algebra of $\mathcal{E}$

- For each $D \neq 340$, we computed $\operatorname{End}^{0}(\mathcal{E})$ for each of the eight representatives of $\mathbb{Q}$-curves with CM by $\mathbb{Q}(\sqrt{-D})$.
(1)

$$
\begin{aligned}
\text { For } D \in & \{84,120,132,168,228,280,372,408,435,483, \\
& 520,532,595,627,708,795,1012,1435\}
\end{aligned}
$$

at least one of the $\mathbb{Q}$-curves has $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) For $D \in\{195,312,555,715,760\}$
all $\mathbb{Q}$-curves have $\operatorname{End}^{0}(\mathcal{E}) \simeq\left\{\begin{array}{l}\text { number field } \\ \text { division quaternion algebra }\end{array}\right.$
$\Rightarrow \mathcal{E}$ is simple over $\mathbb{Q}$ of dimension 4

- Need to show that for the fields $M$ in 2, $A$ does not exist


## Computing the endomorphism algebra of $\mathcal{E}$

- For each $D \neq 340$, we computed $\operatorname{End}^{0}(\mathcal{E})$ for each of the eight representatives of $\mathbb{Q}$-curves with CM by $\mathbb{Q}(\sqrt{-D})$.
(1)

$$
\begin{aligned}
\text { For } D \in & \{84,120,132,168,228,280,372,408,435,483, \\
& 520,532,595,627,708,795,1012,1435\}
\end{aligned}
$$

at least one of the $\mathbb{Q}$-curves has $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) For $D \in\{195,312,555,715,760\}$
all $\mathbb{Q}$-curves have $\operatorname{End}^{0}(\mathcal{E}) \simeq\left\{\begin{array}{l}\text { number field } \\ \text { division quaternion algebra }\end{array}\right.$
$\Rightarrow \mathcal{E}$ is simple over $\mathbb{Q}$ of dimension 4

- 1 gives an $A / \mathbb{Q}$ for all $M$ with $\mathrm{Cl}(M)=\mathrm{C}_{2} \times \mathrm{C}_{2}$ except for:
- $\operatorname{Disc}(M)=-195,-312,-555,-715,-760\left(\right.$ never get $\left.\mathrm{M}_{2}(\mathbb{Q})\right)$
- $\operatorname{Disc}(M)=-340$ (there is no Gross $\mathbb{Q}$-curve)
- Need to show that for the fields $M$ in 2, $A$ does not exist


## Computing the endomorphism algebra of $\mathcal{E}$

- For each $D \neq 340$, we computed $\operatorname{End}^{0}(\mathcal{E})$ for each of the eight representatives of $\mathbb{Q}$-curves with CM by $\mathbb{Q}(\sqrt{-D})$.
(1)

$$
\begin{aligned}
\text { For } D \in & \{84,120,132,168,228,280,372,408,435,483, \\
& 520,532,595,627,708,795,1012,1435\}
\end{aligned}
$$

at least one of the $\mathbb{Q}$-curves has $\operatorname{End}^{0}(\mathcal{E}) \simeq \mathrm{M}_{2}(\mathbb{Q})$.
(2) For $D \in\{195,312,555,715,760\}$
all $\mathbb{Q}$-curves have $\operatorname{End}^{0}(\mathcal{E}) \simeq\left\{\begin{array}{l}\text { number field } \\ \text { division quaternion algebra }\end{array}\right.$
$\Rightarrow \mathcal{E}$ is simple over $\mathbb{Q}$ of dimension 4

- 1 gives an $A / \mathbb{Q}$ for all $M$ with $\mathrm{Cl}(M)=\mathrm{C}_{2} \times \mathrm{C}_{2}$ except for:
- $\operatorname{Disc}(M)=-195,-312,-555,-715,-760\left(\right.$ never get $\left.\mathrm{M}_{2}(\mathbb{Q})\right)$
- $\operatorname{Disc}(M)=-340$ (there is no Gross $\mathbb{Q}$-curve)
- Need to show that for the fields $M$ in 2, $A$ does not exist


## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\mathbb{Q}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place.
- $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}, \mathrm{D}_{r}, r=3,4,6$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$
- $H \subseteq K$ and $\operatorname{Gal}(H / M) \simeq \mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2} \Rightarrow K=H$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{D}_{r}$ with $r \in\{3,4,6\}$
- A does not exist either, but the argument gets more technical
- Can assume $\operatorname{Gal}(K / M)$ has an element $\beta$ of order $r=4$ or $r=6$.


## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place. - $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}, \mathrm{D}_{r}, r=3,4,6$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$

- If $\operatorname{Gal}(K / M) \simeq \mathrm{D}_{r}$ with $r \in\{3,4,6\}$
- A does not exist either, but the argument gets more technical - Can assume $\operatorname{Gal}(K / M)$ has an element $\beta$ of order $r=4$ or $r=6$.


## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\mathbb{Q}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place.
- If $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$

- If $\operatorname{Gal}(K / M) \simeq \mathrm{D}_{r}$ with $r \in\{3,4,6\}$
- A does not exist either, but the argument gets more technical - Can assume $\operatorname{Gal}(K / M)$ has an element $\beta$ of order $r=4$ or $r=6$.


## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\mathbb{Q}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place.
- $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}, \mathrm{D}_{r}, r=3,4,6$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{D}_{r}$ with $r \in\{3,4,6\}$
- A does not exist either, but the argument gets more technical - Can assume $\operatorname{Gal}(K / M)$ has an element $\beta$ of order $r=4$ or $r=6$.


## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\mathbb{Q}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place.
- $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}, \mathrm{D}_{r}, r=3,4,6$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$
- $H \subseteq K$ and $\operatorname{Gal}(H / M) \simeq \mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2} \Rightarrow K=H$



## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\mathbb{Q}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place.
- $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}, \mathrm{D}_{r}, r=3,4,6$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$
- $H \subseteq K$ and $\operatorname{Gal}(H / M) \simeq \mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2} \Rightarrow K=H$
- Then $E$ is a Gross $\mathbb{Q}$-curve, but this is a contradiction:
- But the simple factors of Res H/O $E$ are of dimension 4.
- A does not exist either, but the argument gets more technical
- Can assume $\operatorname{Gal}(K / M)$ has an element $\beta$ of order $r=4$ or $r=6$.


## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\mathbb{Q}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place.
- $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}, \mathrm{D}_{r}, r=3,4,6$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$
- $H \subseteq K$ and $\operatorname{Gal}(H / M) \simeq \mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2} \Rightarrow K=H$
- Then $E$ is a Gross $\mathbb{Q}$-curve, but this is a contradiction:
- $\operatorname{Hom}\left(A_{H}, E\right) \neq 0 \Rightarrow \operatorname{Hom}\left(A, \operatorname{Res}_{H / \mathbb{Q}} E\right) \neq 0$.
- A does not exist either, but the argument gets more technical


## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\mathbb{Q}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place.
- $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}, \mathrm{D}_{r}, r=3,4,6$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$
- $H \subseteq K$ and $\operatorname{Gal}(H / M) \simeq \mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2} \Rightarrow K=H$
- Then $E$ is a Gross $\mathbb{Q}$-curve, but this is a contradiction:
- $\operatorname{Hom}\left(A_{H}, E\right) \neq 0 \Rightarrow \operatorname{Hom}\left(A, \operatorname{Res}_{H / \mathbb{Q}} E\right) \neq 0$.
- But the simple factors of $\operatorname{Res}_{H / \mathbb{Q}} E$ are of dimension 4.


## Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve $E / H$ we know that $\operatorname{Res}_{H / \mathbb{Q}} E$ does not have any factor of dimension 2.
- Suppose that $\exists A / \mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^{2}$ and $E$ has $C M$ by $M$.
- $K=$ minimal field where this decomposition takes place.
- $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}, \mathrm{D}_{r}, r=3,4,6$
- If $\operatorname{Gal}(K / M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2}$
- $H \subseteq K$ and $\operatorname{Gal}(H / M) \simeq \mathrm{Cl}(M) \simeq \mathrm{C}_{2} \times \mathrm{C}_{2} \Rightarrow K=H$
- Then $E$ is a Gross $\mathbb{Q}$-curve, but this is a contradiction:
- $\operatorname{Hom}\left(A_{H}, E\right) \neq 0 \Rightarrow \operatorname{Hom}\left(A, \operatorname{Res}_{H / \mathbb{Q}} E\right) \neq 0$.
- But the simple factors of $\operatorname{Res}_{H / \mathbb{Q}} E$ are of dimension 4.
- If $\operatorname{Gal}(K / M) \simeq \mathrm{D}_{r}$ with $r \in\{3,4,6\}$
- A does not exist either, but the argument gets more technical
- Can assume $\operatorname{Gal}(K / M)$ has an element $\beta$ of order $r=4$ or $r=6$.


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$

- Key: $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right) \otimes \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)^{*} \simeq \operatorname{End}^{0}\left(A_{K}\right)$ as $\operatorname{Gal}(K / M)$-rep's
- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $C_{E *}(\bar{\beta}, \bar{\beta})= \pm d$ with d a proper divisor of DiscM. Contradiction!
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{ }-340)$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve - Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
- Key: $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right) \otimes \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)^{*} \simeq \operatorname{End}^{0}\left(A_{K}\right)$ as $\operatorname{Gal}(K / M)$-rep's
- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $C_{E *}(\bar{\beta}, \bar{\beta})= \pm d$ with d a proper divisor of DiscM. Contradiction!
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Key: $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right) \otimes \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)^{*} \simeq \operatorname{End}^{0}\left(A_{K}\right)$ as $\operatorname{Gal}(K / M)$-rep's
- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $C_{E *}(\bar{\beta}, \bar{\beta})= \pm d$ with $d$ a proper divisor of DiscM. Contradiction!
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{ }-340)$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
$\rightarrow \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)$ is not a $\operatorname{Gal}(L / M)$ representation:

- Key: $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right) \otimes \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)^{*} \simeq \operatorname{End}^{0}\left(A_{K}\right)$ as $\operatorname{Gal}(K / M)$-rep's
- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $C_{E *}(\bar{\beta}, \bar{\beta})= \pm d$ with
$d$ a proper divisor of DiscM. Contradiction!
- Extra arqument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
- $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)$ is not a $\operatorname{Gal}(L / M)$ representation:

$$
\phi: E_{L}^{*} \rightarrow A_{L} \rightsquigarrow{ }^{\sigma} \phi:{ }^{\sigma} E_{L}^{*} \rightarrow A_{L}
$$


so we can define

- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $C_{E_{*}}(\bar{\beta}, \bar{\beta})= \pm d$ with
d a proper divisor of DiscM. Contradiction!
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
- $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)$ is not a $\operatorname{Gal}(L / M)$ representation:

$$
\phi: E_{L}^{*} \rightarrow A_{L} \rightsquigarrow{ }^{\sigma} \phi:{ }^{\sigma} E_{L}^{*} \rightarrow A_{L}
$$

- But we have $\mu_{\sigma}:{ }^{\sigma} E_{L}^{*} \rightarrow E_{L}^{*}$ so we can define

$$
\rho_{\sigma}(\phi)={ }^{\sigma} \phi \circ \mu_{\sigma}^{-1}: E_{L}^{*} \rightarrow A_{L}
$$

- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $C_{E *}(\bar{\beta}, \bar{\beta})= \pm d$ with d a proper divisor of DiscM. Contradiction!
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
- $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)$ is not a $\operatorname{Gal}(L / M)$ representation:

$$
\phi: E_{L}^{*} \rightarrow A_{L} \rightsquigarrow{ }^{\sigma} \phi:{ }^{\sigma} E_{L}^{*} \rightarrow A_{L}
$$

- But we have $\mu_{\sigma}:{ }^{\sigma} E_{L}^{*} \rightarrow E_{L}^{*}$ so we can define $\rho_{\sigma}(\phi)={ }^{\sigma} \phi \circ \mu_{\sigma}^{-1}: E_{L}^{*} \rightarrow A_{L}$
- $\rho_{\sigma} \rho_{\tau}=C_{E^{*}}(\sigma, \tau) \rho_{\sigma \tau}$ projective representation ( $C_{E^{*}}$-representation)
- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $C_{E *}(\bar{\beta}, \bar{\beta})= \pm d$ with
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
- $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)$ is not a $\operatorname{Gal}(L / M)$ representation:

$$
\phi: E_{L}^{*} \rightarrow A_{L} \rightsquigarrow{ }^{\sigma} \phi:{ }^{\sigma} E_{L}^{*} \rightarrow A_{L}
$$

- But we have $\mu_{\sigma}:{ }^{\sigma} E_{L}^{*} \rightarrow E_{L}^{*}$ so we can define

$$
\rho_{\sigma}(\phi)={ }^{\sigma} \phi \circ \mu_{\sigma}^{-1}: E_{L}^{*} \rightarrow A_{L}
$$

- $\rho_{\sigma} \rho_{\tau}=C_{E^{*}}(\sigma, \tau) \rho_{\sigma \tau}$ projective representation ( $C_{E^{*}}$-representation)
- Key: $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right) \otimes \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)^{*} \simeq \operatorname{End}^{0}\left(A_{K}\right)$ as $\operatorname{Gal}(K / M)$-rep's
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has CM by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
- $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)$ is not a $\operatorname{Gal}(L / M)$ representation:

$$
\phi: E_{L}^{*} \rightarrow A_{L} \rightsquigarrow{ }^{\sigma} \phi:{ }^{\sigma} E_{L}^{*} \rightarrow A_{L}
$$

- But we have $\mu_{\sigma}:{ }^{\sigma} E_{L}^{*} \rightarrow E_{L}^{*}$ so we can define

$$
\rho_{\sigma}(\phi)={ }^{\sigma} \phi \circ \mu_{\sigma}^{-1}: E_{L}^{*} \rightarrow A_{L}
$$

- $\rho_{\sigma} \rho_{\tau}=C_{E^{*}}(\sigma, \tau) \rho_{\sigma \tau}$ projective representation ( $C_{E^{*}}$-representation)
- Key: $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right) \otimes \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)^{*} \simeq \operatorname{End}^{0}\left(A_{K}\right)$ as $\operatorname{Gal}(K / M)$-rep's
- Using this we show that $c_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has $C M$ by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
- $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)$ is not a $\operatorname{Gal}(L / M)$ representation:

$$
\phi: E_{L}^{*} \rightarrow A_{L} \rightsquigarrow{ }^{\sigma} \phi:{ }^{\sigma} E_{L}^{*} \rightarrow A_{L}
$$

- But we have $\mu_{\sigma}:{ }^{\sigma} E_{L}^{*} \rightarrow E_{L}^{*}$ so we can define

$$
\rho_{\sigma}(\phi)={ }^{\sigma} \phi \circ \mu_{\sigma}^{-1}: E_{L}^{*} \rightarrow A_{L}^{L}
$$

- $\rho_{\sigma} \rho_{\tau}=C_{E^{*}}(\sigma, \tau) \rho_{\sigma \tau}$ projective representation ( $C_{E^{*}}$-representation)
- Key: $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right) \otimes \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)^{*} \simeq \operatorname{End}^{0}\left(A_{K}\right)$ as $\operatorname{Gal}(K / M)$-rep's
- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $c_{E *}(\bar{\beta}, \bar{\beta})= \pm d$ with d a proper divisor of DiscM. Contradiction!


## Ruling out abelian surfaces: projective representations

- $A / \mathbb{Q}$ with $A_{K} \sim E^{2}$ and $E$ has $C M$ by $M$
- ${ }^{\sigma} E \sim E$ for all $\sigma \in \operatorname{Gal}(K / \mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
- Need to relate $A$ to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let $E^{*} / H$ be a Gross $\mathbb{Q}$-curve with $E_{L}^{*} \sim E_{L}$
- $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)$ is not a $\operatorname{Gal}(L / M)$ representation:

$$
\phi: E_{L}^{*} \rightarrow A_{L} \rightsquigarrow{ }^{\sigma} \phi:{ }^{\sigma} E_{L}^{*} \rightarrow A_{L}
$$

- But we have $\mu_{\sigma}:{ }^{\sigma} E_{L}^{*} \rightarrow E_{L}^{*}$ so we can define

$$
\rho_{\sigma}(\phi)={ }^{\sigma} \phi \circ \mu_{\sigma}^{-1}: E_{L}^{*} \rightarrow A_{L}
$$

- $\rho_{\sigma} \rho_{\tau}=C_{E^{*}}(\sigma, \tau) \rho_{\sigma \tau}$ projective representation ( $C_{E^{*}}$-representation)
- Key: $\operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right) \otimes \operatorname{Hom}^{0}\left(E_{L}^{*}, A_{L}\right)^{*} \simeq \operatorname{End}^{0}\left(A_{K}\right)$ as $\operatorname{Gal}(K / M)$-rep's
- Using this we show that $C_{E^{*}}(\bar{\beta}, \bar{\beta}) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $c_{E *}(\bar{\beta}, \bar{\beta})= \pm d$ with $d$ a proper divisor of DiscM. Contradiction!
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too


## A natural question

## Theorem (Fité-G.)

The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ consists of 92 algebras.

Question
Which of the 92 endomorphism algebras arise from Jacobians of genus 2 curves defined over $\mathbb{Q}$ ?

## A natural question

## Theorem (Fité-G.)

The set $\mathcal{A}_{2,1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ consists of 92 algebras.

## Question

Which of the 92 endomorphism algebras arise from Jacobians of genus 2 curves defined over $\mathbb{Q}$ ?

## A natural question

## Theorem (Fité-G.)

The set $\mathcal{A}_{2.1}^{\text {split }}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ consists of 92 algebras.

## Question

Which of the 92 endomorphism algebras arise from Jacobians of genus 2 curves defined over $\mathbb{Q}$ ?

## Spoiler

Not all of them.

# Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ 

Francesc Fité (UB) Xevi Guitart (UB)

## COUNT 2023

