Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$

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A conjecture of Coleman

- A abelian variety over a number field k
  - $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) = \mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ the algebra of $\overline{\mathbb{Q}}$-endomorphisms
- $\mathcal{A}_{g,d} = \{\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \colon A/k$ of dimension g and $[k : \mathbb{Q}] = d\}/\simeq$

Conjecture (Coleman)

The set $\mathcal{A}_{g,d}$ is finite for any $g, d \geq 1$.

- There is even a stronger version for endomorphism rings.
- Very little is known:
  - It is known for $\mathcal{A}_{1,d}$ (elliptic curves)
  - It is known if we restrict to CM abelian varieties: $\mathcal{A}^{\mathrm{CM}}_{g,d}$ is finite (Orr-Skorobogatov 2018)
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  - $I\subseteq M$ a fractional ideal
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- $\{E/\mathbb{C}\colon \mathrm{End}(E)\simeq\mathcal{O}_M\}/\simeq \xleftrightarrow{1:1} \{I\subseteq M \text{ fractional ideals}\}/I\sim\lambda I=\mathrm{Cl}(M)$
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- Heilbronn (1934): $\exists$ finitely many $\mathbb{Q}(\sqrt{-D})$ with $\#\mathrm{Cl}(M) \leq d$
  - $\mathcal{A}_{1,d}$ is finite for all d
- For $d \leq 100$ the set $\mathcal{A}_{1,d}$ is known explicitly (Watkins)
- $\mathcal{A}_{1,1} = \{\mathbb{Q}\} \cup \{\mathbb{Q}(\sqrt{-D})\colon D = 3, 4, 7, 8, 11, 19, 43, 67, 163\}$

The case $\mathcal{A}_{2,1}$ of abelian surfaces over $\mathbb{Q}$

- $\mathcal{A}_{2,1} = \{\mathrm{End}^0_{\overline{\mathbb{Q}}}(A)\colon A/\mathbb{Q},\ \dim(A) = 2\}/\simeq$
- $A/\mathbb{Q}$ an abelian surface
  - geometrically simple if $A_{\overline{\mathbb{Q}}}$ is simple
  - geometrically split if $A_{\overline{\mathbb{Q}}} \sim E_1 \times E_2$
- $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq \begin{cases} \mathbb{Q}, \mathbb{Q}(\sqrt{D}),\ \text{CM field}, B/\mathbb{Q} \text{ division indef. quat. alg.} \\ \mathbb{Q}\times\mathbb{Q}, \mathbb{Q}\times M, M\times M', \mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(M),\quad M = \mathbb{Q}(\sqrt{-D}) \end{cases}$
- The case where A is geometrically simple is open:
  - There are 19 possibilities for the CM field (Murabayashi-Umegaki)
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- $\mathcal{A}^{\mathrm{split}}_{2,1} = \{\mathrm{End}^0_{\overline{\mathbb{Q}}}(A)\colon A/\mathbb{Q},\ \dim(A) = 2,\ A \text{ geom. split}\}$
- $\mathcal{A}^{\mathrm{split}}_{2,1}$ is finite (Shafarevich, 1996)

Our goal

To determine $\mathcal{A}^{\mathrm{split}}_{2,1}$ explicitly

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Our goal

To determine $\mathcal{A}_{2,1}^{\mathrm{split}}$ explicitly

The case $\mathcal{A}_{2,1}^{\text{split}}$

If $A/\mathbb{Q}$ is a geometrically split abelian surface:

1. $A_{\overline{\mathbb{Q}}} \sim E_1 \times E_2$ or $A_{\overline{\mathbb{Q}}} \sim E_1^2$ with E_i non-CM
   - $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq \mathbb{Q} \times \mathbb{Q}$ or $\mathrm{End}^0_{\overline{\mathbb{Q}}}(A) \simeq \mathrm{M}_2(\mathbb{Q})$.
2. $A_{\overline{\mathbb{Q}}} \sim E_1 \times E_2$, and E_1 or E_2 have CM by some M
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   - Fite–Kedlaya–Rotger–Sutherland: $E_i \sim C_i$ for some $C_i/\mathbb{Q}$
   - Thus $\#\mathrm{Cl}(M) = 1 \rightsquigarrow$ finitely many M's
3. $A_{\overline{\mathbb{Q}}} \sim E^2$ and E has CM by some M
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     $\exists\, d \in \mathbb{Z}_{\geq 1}$ s.t. $\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathrm{End}(A_K)$ for some K with $[K:\mathbb{Q}] \leq d$
     $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \simeq \mathbb{Z}^8$ with kernel $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$
     $\mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow \mathrm{GL}_8(\mathbb{Z})$ and so $\mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow \mathrm{GL}_8(\mathbb{Z}/3\mathbb{Z})$
   - E/K and $[K:\mathbb{Q}] \leq d \rightsquigarrow$ finitely many M's since $\mathcal{A}_{1,d}$ is finite.

Main question

What imaginary quadratic fields M do in fact appear?

The case $\mathcal{A}_{2,1}^{\text{split}}$

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Squares of CM elliptic curves

Central question

If $A/\mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^2$ and E has CM by M, what are the possible M's?

Theorem (Fité–G., 2015)

Necessarily $\mathrm{Cl}(M) \simeq \mathrm{C}_1, \mathrm{C}_2$, or $\mathrm{C}_2 \times \mathrm{C}_2$

- $K/\mathbb{Q}$ minimal such that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathrm{End}(A_K) \rightsquigarrow E/K$ and $A_K \sim E^2$
  - CM theory $\rightsquigarrow H_M \subseteq K$
  - [*FKRS*12] $\rightsquigarrow \mathrm{Gal}(K/M) \simeq \mathrm{C}_1, \mathrm{C}_r, \mathrm{D}_r$ with $r \in \{2,3,4,6\}$
  - This implies $\mathrm{Cl}(M) \simeq \mathrm{C}_1, \mathrm{C}_r, \mathrm{D}_r$ with $r \in \{2,3,4,6\}$
- Main idea:
  - Show that $\exists N \subseteq K$ with $\mathrm{Gal}(N/M)$ of exponent 2 such that E can be defined over N up to isogeny
  - $\mathrm{Gal}(N/M) \simeq \mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_2 \times \mathrm{C}_2$ and $H_M \subseteq N$ so $\mathrm{Gal}(H_M/M)$ as well
- Idea of the proof: adapt Ribet's theory of $\mathbb{Q}$-curves
  - $\sigma \in \mathrm{Gal}(K/\mathbb{Q}) \rightsquigarrow ({}^\sigma E)^2 \sim {}^\sigma A_K = A_K \sim E^2$
    - There is an isogeny $\mu_\sigma : {}^\sigma E \longrightarrow E$
  - Cohomology class $c_E \in H^2(\mathrm{Gal}(K/M), M^\times)$
    - $c_E(\sigma, \tau) = \mu_\sigma \circ {}^\sigma\mu_\tau \circ \mu_{\sigma\tau}^{-1}$

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Squares of CM elliptic curves

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of these possible M's, which ones do really occur?

- Give a construction of A's for some M's and rule out the other M's

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Now the question is

of these possible M's, which ones do really occur?

- Give a construction of A's for some M's and rule out the other M's

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Constructing abelian surfaces: restriction of scalars

Easy cases

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- If $\mathrm{Cl}(M) = 1$
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- If $\mathrm{Cl}(M) = \mathrm{C}_2$
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Case $C_2 \times C_2$: Gross's $\mathbb{Q}$-curves

- $H = H_M$ Hilbert class field of M.
- A Gross $\mathbb{Q}$-curve is
  - E/H elliptic curve with CM by M s.t. ${}^\sigma E \sim E\ \forall \sigma \in \mathrm{Gal}(H/\mathbb{Q})$

Theorem (Shimura–Nakamura)

$\exists$ Gross $\mathbb{Q}$-curve $E/H \iff \mathrm{Disc}(M) \neq -4 \times (\text{primes} \equiv 1 \pmod 4)$

- $M = \mathbb{Q}(\sqrt{-D})$ has $\mathrm{Cl}(M) \simeq C_2 \times C_2$ for D in

$$D^{2,2} = \{84, 120, 132, 168, 195, 228, 280, 312, 340, 372, 408, 435, 483, 520, 532, 555, 595, 627, 708, 715, 760, 795, 1012, 1435\}$$

- There is Gross $\mathbb{Q}$-curves in all cases except $D = 340$
- (Gross) If $\mathcal{E} = \mathrm{Res}_{\mathbb{Q}(j_E)/\mathbb{Q}} E$ then $\mathrm{End}^0(\mathcal{E}) \simeq \mathbb{Q}^{c_E}[\mathrm{Gal}(H/M)]$
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- For $D \neq 340$, Nakamura showed that:
  - For each D, Gross $\mathbb{Q}$-curves D give rise to 8 cohomology classes
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Case $\mathrm{C}_2 \times \mathrm{C}_2$: Gross's $\mathbb{Q}$-curves

- $H = H_M$ Hilbert class field of M.
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$\exists$ Gross $\mathbb{Q}$-curve $E/H \iff \mathrm{Disc}(M) \neq -4 \times (\text{primes} \equiv 1 \pmod 4)$

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$$D^{2,2} = \{84, 120, 132, 168, 195, 228, 280, 312, 340, 372, 408, 435,\\ 483, 520, 532, 555, 595, 627, 708, 715, 760, 795, 1012, 1435\}$$

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Computing the endomorphism algebra of $\mathcal{E}$

- For each $D \neq 340$, we computed $\mathrm{End}^0(\mathcal{E})$ for each of the eight representatives of $\mathbb{Q}$-curves with CM by $\mathbb{Q}(\sqrt{-D})$.
  1. $$\text{For } D \in \{84, 120, 132, 168, 228, 280, 372, 408, 435, 483, 520, 532, 595, 627, 708, 795, 1012, 1435\}$$

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Ruling out abelian surfaces

- $M \in D^{2,2}$ s.t. for any Gross $\mathbb{Q}$-curve E/H we know that $\mathrm{Res}_{H/\mathbb{Q}}E$ does not have any factor of dimension 2.
- Suppose that $\exists\ A/\mathbb{Q}$ with $A_{\overline{\mathbb{Q}}} \sim E^2$ and E has CM by M.
- $K =$ minimal field where this decomposition takes place.
  - $\mathrm{Gal}(K/M) \simeq \mathrm{C}_2 \times \mathrm{C}_2, \mathrm{D}_r$, $r = 3, 4, 6$
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Ruling out abelian surfaces: projective representations

- $A/\mathbb{Q}$ with $A_K \sim E^2$ and E has CM by M
  - ${}^\sigma E \sim E$ for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, but $H \subsetneq K \rightsquigarrow$ not a Gross $\mathbb{Q}$-curve
  - Need to relate A to a Gross $\mathbb{Q}$-curve (let us suppose there is one)
- Let E^*/H be a Gross $\mathbb{Q}$-curve with $E^*_L \sim E_L$
  - $\mathrm{Hom}^0(E^*_L, A_L)$ is not a $\mathrm{Gal}(L/M)$ representation:

$$\phi\colon E^*_L \to A_L \rightsquigarrow {}^\sigma\phi\colon {}^\sigma E^*_L \to A_L$$

  - But we have $\mu_\sigma : {}^\sigma E^*_L \to E^*_L$ so we can define $\rho_\sigma(\phi) = {}^\sigma\phi \circ \mu_\sigma^{-1} : E^*_L \to A_L$
  - $\rho_\sigma\rho_\tau = c_{E^*}(\sigma,\tau)\rho_{\sigma\tau}$ projective representation (c_{E^*}-representation)
- Key: $\mathrm{Hom}^0(E^*_L, A_L) \otimes \mathrm{Hom}^0(E^*_L, A_L)^* \simeq \mathrm{End}^0(A_K)$ as $\mathrm{Gal}(K/M)$-rep's
- Using this we show that $c_{E^*}(\bar\beta, \bar\beta) \in \pm 1$
- The cocycles of Gross's $\mathbb{Q}$-curves satisfy that $c_{E*}(\bar\beta, \bar\beta) = \pm d$ with d a proper divisor of $\mathrm{Disc} M$. Contradiction!
- Extra argument using c-representations rules out $\mathbb{Q}(\sqrt{-340})$ too

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Theorem (Fité–G.)

The set $\mathcal{A}_{2,1}^{\text{split}}$ of $\overline{\mathbb{Q}}$-endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$ consists of 92 algebras.

Question

Which of the 92 endomorphism algebras arise from Jacobians of genus 2 curves defined over $\mathbb{Q}$?

Spoiler

Not all of them.

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Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$

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