The Tutte Polynomial of a q-Matroid

Andrew Fulcher

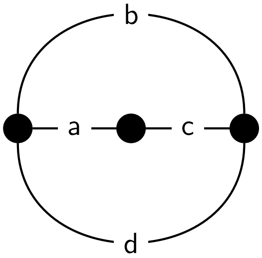
University College Dublin

February 2023

A graphical introduction to matroids

Matroids offer an abstraction of independence in a graph or vector space.

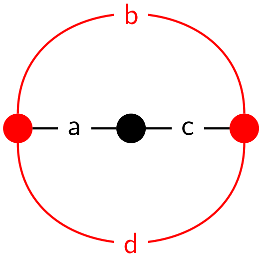
For example, let $S = \{a, b, c, d\}$ be the edge set of a graph:



A graphical introduction to matroids

Matroids offer an abstraction of independence in a graph or vector space.

For example, let $S = \{a, b, c, d\}$ be the edge set of a graph:

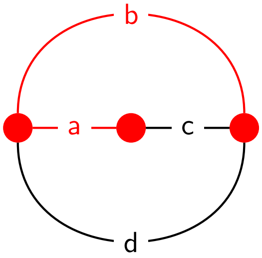


- A spanning tree of $\{b, d\}$ has length one.

A graphical introduction to matroids

Matroids offer an abstraction of independence in a graph or vector space.

For example, let $S = \{a, b, c, d\}$ be the edge set of a graph:



- A spanning tree of $\{a, b\}$ has length two.

Here, the length of the spanning tree of $A \subseteq S$ is called the **rank** of A.

A definition using the rank function

For a set E, let $\mathcal{L}(E)$ be the subset lattice of E.

A definition using the rank function

For a set E, let $\mathcal{L}(E)$ be the subset lattice of E.

Definition

A **matroid** $M = (\mathcal{L}(E), r)$ is a ground set E coupled with a function $r : \mathcal{L}(E) \to \mathbb{Z}$ such that for any $x, y \in \mathcal{L}(E)$:

A definition using the rank function

For a set E, let $\mathcal{L}(E)$ be the subset lattice of E.

Definition

A **matroid** $M = (\mathcal{L}(E), r)$ is a ground set E coupled with a function $r : \mathcal{L}(E) \to \mathbb{Z}$ such that for any $x, y \in \mathcal{L}(E)$:

1. $0 \leqslant r(x) \leqslant h(x)$,

A definition using the rank function

For a set E, let $\mathcal{L}(E)$ be the subset lattice of E.

Definition

A **matroid** $M = (\mathcal{L}(E), r)$ is a ground set E coupled with a function $r : \mathcal{L}(E) \to \mathbb{Z}$ such that for any $x, y \in \mathcal{L}(E)$:

1. $0 \leqslant r(x) \leqslant h(x)$,
2. $r(x) \leqslant r(y)$ if $x \leqslant_{\mathcal{L}} y$,

A definition using the rank function

For a set E, let $\mathcal{L}(E)$ be the subset lattice of E.

Definition

A **matroid** $M = (\mathcal{L}(E), r)$ is a ground set E coupled with a function $r : \mathcal{L}(E) \to \mathbb{Z}$ such that for any $x, y \in \mathcal{L}(E)$:

1. $0 \leqslant r(x) \leqslant h(x)$,
2. $r(x) \leqslant r(y)$ if $x \leqslant_{\mathcal{L}} y$,
3. $r(x \vee y) + r(x \wedge y) \leqslant r(x) + r(y)$.

A definition using the rank function

For a set E, let $\mathcal{L}(E)$ be the subset lattice of E.

Definition

A **matroid** $M = (\mathcal{L}(E), r)$ is a ground set E coupled with a function $r : \mathcal{L}(E) \to \mathbb{Z}$ such that for any $x, y \in \mathcal{L}(E)$:

1. $0 \leqslant r(x) \leqslant h(x)$,
2. $r(x) \leqslant r(y)$ if $x \leqslant_{\mathcal{L}} y$,
3. $r(x \vee y) + r(x \wedge y) \leqslant r(x) + r(y)$.

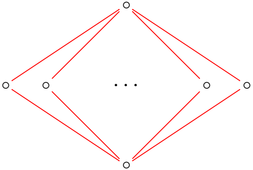
Let $E = \mathbb{F}_q^n$ and $\mathcal{L}(E)$ be its subspace lattice, and we get the definition of a q-**matroid**.

A bicoloured lattice (Bollen, Crapo, Jurrius, 2017)

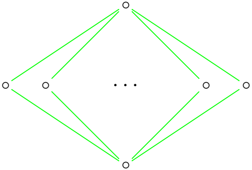
An equivalent definition of a q-matroid $M = (\mathcal{L}, r)$: every interval of length two (diamond) has one of the following forms:

A bicoloured lattice (Bollen, Crapo, Jurrius, 2017)

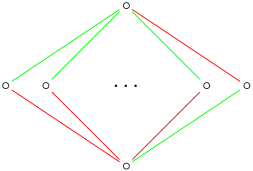
An equivalent definition of a q-matroid $M = (\mathcal{L}, r)$: every interval of length two (diamond) has one of the following forms:



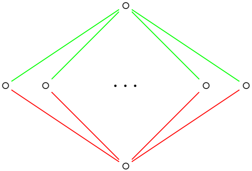
Full



Empty



Mixed



Prime

Minors of a q-matroid

Definition

A **minor** of a matroid $M = (\mathcal{L}, r)$ is a matroid $M' = ([a, b], r_{[a,b]})$ defined on an interval $[a, b] \subseteq \mathcal{L}$ such that $r_{[a,b]}(x) = r(x) - r(a)$ for any $x \in [a, b]$.

Minors of a q-matroid

Definition

A **minor** of a matroid $M = (\mathcal{L}, r)$ is a matroid $M' = ([a, b], r_{[a,b]})$ defined on an interval $[a, b] \subseteq \mathcal{L}$ such that $r_{[a,b]}(x) = r(x) - r(a)$ for any $x \in [a, b]$.

Let $a \in \mathcal{L}$. Three fundamental matroid minors are:

Minors of a q-matroid

Definition

A **minor** of a matroid $M = (\mathcal{L}, r)$ is a matroid $M' = ([a, b], r_{[a,b]})$ defined on an interval $[a, b] \subseteq \mathcal{L}$ such that $r_{[a,b]}(x) = r(x) - r(a)$ for any $x \in [a, b]$.

Let $a \in \mathcal{L}$. Three fundamental matroid minors are:

1. **Contraction:** $M/a = ([a, 1], r_{[a,1]})$.

Minors of a q-matroid

Definition

A **minor** of a matroid $M = (\mathcal{L}, r)$ is a matroid $M' = ([a,b], r_{[a,b]})$ defined on an interval $[a,b] \subseteq \mathcal{L}$ such that $r_{[a,b]}(x) = r(x) - r(a)$ for any $x \in [a,b]$.

Let $a \in \mathcal{L}$. Three fundamental matroid minors are:

1. **Contraction:** $M/a = ([a,1], r_{[a,1]})$.
2. **Restriction:** $M(a) = ([0,a], r_{[0,a]})$.

Minors of a q-matroid

Definition

A **minor** of a matroid $M = (\mathcal{L}, r)$ is a matroid $M' = ([a, b], r_{[a,b]})$ defined on an interval $[a, b] \subseteq \mathcal{L}$ such that $r_{[a,b]}(x) = r(x) - r(a)$ for any $x \in [a, b]$.

Let $a \in \mathcal{L}$. Three fundamental matroid minors are:

1. **Contraction:** $M/a = ([a, 1], r_{[a,1]})$.
2. **Restriction:** $M(a) = ([0, a], r_{[0,a]})$.
3. **Deletion:** $M \backslash a = ([0, a^c], r_{[0,a^c]}) = M(a^c)$.

Minors of a q-matroid

Definition

A **minor** of a matroid $M = (\mathcal{L}, r)$ is a matroid $M' = ([a, b], r_{[a,b]})$ defined on an interval $[a, b] \subseteq \mathcal{L}$ such that $r_{[a,b]}(x) = r(x) - r(a)$ for any $x \in [a, b]$.

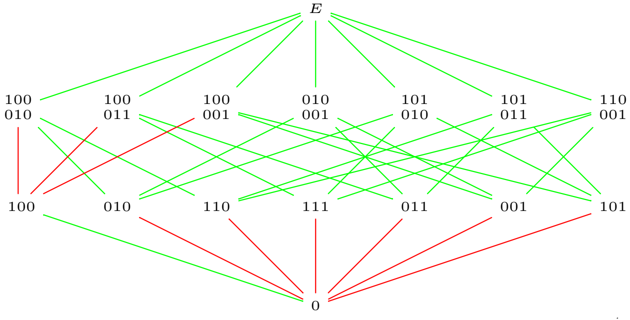
Let $a \in \mathcal{L}$. Three fundamental matroid minors are:

1. **Contraction:** $M/a = ([a, 1], r_{[a,1]})$.
2. **Restriction:** $M(a) = ([0, a], r_{[0,a]})$.
3. **Deletion:** $M \backslash a = ([0, a^c], r_{[0,a^c]}) = M(a^c)$.

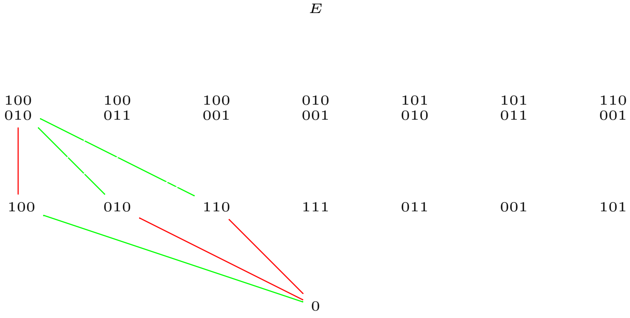
For $a, a^c \in \mathcal{L}$, we have that $a \wedge a^c = 0$ and $a \vee a^c = 1$.

Minors of a q-matroid

Let M be the following q-matroid:

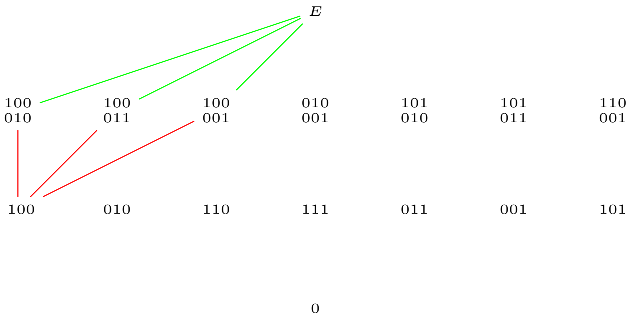


Minors of a q-matroid



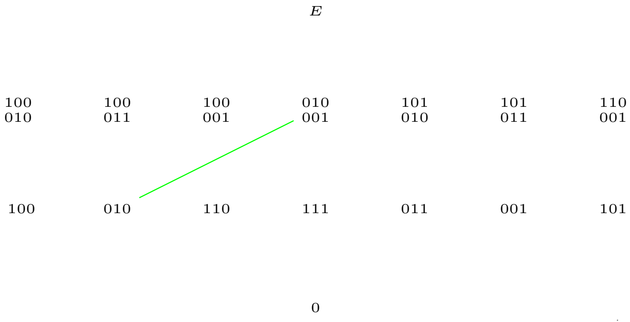
Restriction: $M\left(\begin{smallmatrix}100\\010\end{smallmatrix}\right)$

Minors of a q-matroid



Contraction: $M/100$

Minors of a q-matroid



Restriction and contraction: $M\left(\begin{smallmatrix}010\\001\end{smallmatrix}\right)/010$

Some basic definitions

Let $M = (\mathcal{L}, r)$ be a (q-)matroid.

Some basic definitions

Let $M = (\mathcal{L}, r)$ be a (q-)matroid.

Definition

For $x \in \mathcal{L}$, we define the **nullity** ν of x by $\nu(x) = h(x) - r(x)$.

Some basic definitions

Let $M = (\mathcal{L}, r)$ be a (q-)matroid.

Definition

For $x \in \mathcal{L}$, we define the **nullity** ν of x by $\nu(x) = h(x) - r(x)$.

Definition

For $e \in \mathcal{L}$, if $h(e) = 1$ we call e an **atom**. Furthermore, if $\nu(e) = 1$ we call e a **loop**, and if $r(e^c) = r(M) - 1$ for any $e^c \in \mathcal{L}$ we call e an **isthmus**.

Some basic definitions

Let $M = (\mathcal{L}, r)$ be a (q-)matroid.

Definition

For $x \in \mathcal{L}$, we define the **nullity** ν of x by $\nu(x) = h(x) - r(x)$.

Definition

For $e \in \mathcal{L}$, if $h(e) = 1$ we call e an **atom**. Furthermore, if $\nu(e) = 1$ we call e a **loop**, and if $r(e^c) = r(M) - 1$ for any $e^c \in \mathcal{L}$ we call e an **isthmus**.

Definition

For $x \in \mathcal{L}$, if $h(x) = h(1_{\mathcal{L}}) - 1$ and $r(x) = r(M) - 1$, we call x a **coloop**.

What is the Tutte polynomial?

The Tutte polynomial is a fundamental graph and matroid invariant.

What is the Tutte polynomial?

The Tutte polynomial is a fundamental graph and matroid invariant.

1. It is the most general graph invariant that can be defined by deletion-contraction recurrence.

What is the Tutte polynomial?

The Tutte polynomial is a fundamental graph and matroid invariant.

1. It is the most general graph invariant that can be defined by deletion-contraction recurrence.
2. First defined by Tutte in 1954.

What is the Tutte polynomial?

The Tutte polynomial is a fundamental graph and matroid invariant.

1. It is the most general graph invariant that can be defined by deletion-contraction recurrence.
2. First defined by Tutte in 1954.
3. It was generalised to be defined on matroids by Crapo in a 1969 paper.

What is the Tutte polynomial?

The Tutte polynomial is a fundamental graph and matroid invariant.

1. It is the most general graph invariant that can be defined by deletion-contraction recurrence.
2. First defined by Tutte in 1954.
3. It was generalised to be defined on matroids by Crapo in a 1969 paper.

Similarly, the Tutte polynomial defined on matroids is the most general invariant defined by deletion-contraction recurrence.

A definition of the Tutte polynomial from Brylawski (the one seen in books)

- Let $\mathcal{M}$ denote the set of isomorphism classes of matroids.

A definition of the Tutte polynomial from Brylawski (the one seen in books)

- Let $\mathcal{M}$ denote the set of isomorphism classes of matroids.
- Let M denote an arbitrary matroid in $\mathcal{M}$ and $e \in M$ an arbitrary atom.

A definition of the Tutte polynomial from Brylawski (the one seen in books)

- Let $\mathcal{M}$ denote the set of isomorphism classes of matroids.
- Let M denote an arbitrary matroid in $\mathcal{M}$ and $e \in M$ an arbitrary atom.

The Tutte polynomial of matroids is the unique function $\tau : \mathcal{M} \to \mathbb{Z}[x, y]$ satisfying

A definition of the Tutte polynomial from Brylawski (the one seen in books)

- Let $\mathcal{M}$ denote the set of isomorphism classes of matroids.
- Let M denote an arbitrary matroid in $\mathcal{M}$ and $e \in M$ an arbitrary atom.

The Tutte polynomial of matroids is the unique function $\tau : \mathcal{M} \to \mathbb{Z}[x, y]$ satisfying

1. $\tau(M(e); x, y) = x^{r(e)} y^{\nu(e)}$

A definition of the Tutte polynomial from Brylawski (the one seen in books)

- Let $\mathcal{M}$ denote the set of isomorphism classes of matroids.
- Let M denote an arbitrary matroid in $\mathcal{M}$ and $e \in M$ an arbitrary atom.

The Tutte polynomial of matroids is the unique function $\tau : \mathcal{M} \to \mathbb{Z}[x, y]$ satisfying

1. $\tau(M(e); x, y) = x^{r(e)} y^{\nu(e)}$
2. $\tau(M; x, y) = \tau(M(e); x, y)\tau(M/e; x, y)$ if e is a loop or isthmus

A definition of the Tutte polynomial from Brylawski (the one seen in books)

- Let $\mathcal{M}$ denote the set of isomorphism classes of matroids.
- Let M denote an arbitrary matroid in $\mathcal{M}$ and $e \in M$ an arbitrary atom.

The Tutte polynomial of matroids is the unique function $\tau : \mathcal{M} \rightarrow \mathbb{Z}[x, y]$ satisfying

1. $\tau(M(e); x, y) = x^{r(e)} y^{\nu(e)}$
2. $\tau(M; x, y) = \tau(M(e); x, y)\tau(M/e; x, y)$ if e is a loop or isthmus
3. $\tau(M; x, y) = \tau(M/e; x, y) + \tau(M \backslash e; x, y)$ otherwise

A q-matroid partition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid and $e \in M$ be an atom.

A q-matroid partition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid and $e \in M$ be an atom.

Definition (Byrne, F., 2022)

We denote by $\mathcal{S}_M(e)$ a partition of M into minors such that:

A q-matroid partition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid and $e \in M$ be an atom.

Definition (Byrne, F., 2022)

We denote by $\mathcal{S}_M(e)$ a partition of M into minors such that:

1. $|\mathcal{S}_M(e)|$ is minimal.

A q-matroid partition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid and $e \in M$ be an atom.

Definition (Byrne, F., 2022)

We denote by $\mathcal{S}_M(e)$ a partition of M into minors such that:

1. $|\mathcal{S}_M(e)|$ is minimal.
2. $M/e, M(e^c) \in \mathcal{S}_M(e)$.

A q-matroid partition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid and $e \in M$ be an atom.

Definition (Byrne, F., 2022)

We denote by $\mathcal{S}_M(e)$ a partition of M into minors such that:

1. $|\mathcal{S}_M(e)|$ is minimal.
2. $M/e, M(e^c) \in \mathcal{S}_M(e)$.
3. All loops of M are contained in $M(e^c)$.

A q-matroid partition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid and $e \in M$ be an atom.

Definition (Byrne, F., 2022)

We denote by $\mathcal{S}_M(e)$ a partition of M into minors such that:

1. $|\mathcal{S}_M(e)|$ is minimal.
2. $M/e, M(e^c) \in \mathcal{S}_M(e)$.
3. All loops of M are contained in $M(e^c)$.
4. All coloops of M are contained in M/e.

A q-matroid partition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid and $e \in M$ be an atom.

Definition (Byrne, F., 2022)

We denote by $\mathcal{S}_M(e)$ a partition of M into minors such that:

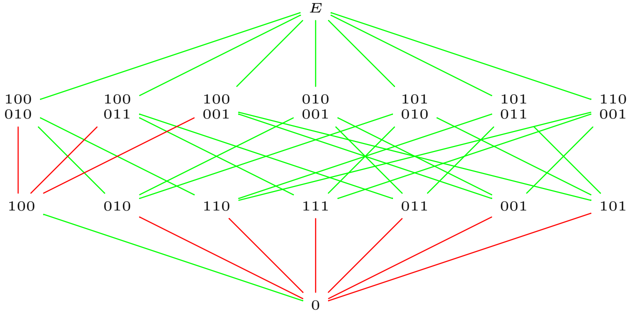
1. $|\mathcal{S}_M(e)|$ is minimal.
2. $M/e, M(e^c) \in \mathcal{S}_M(e)$.
3. All loops of M are contained in $M(e^c)$.
4. All coloops of M are contained in M/e.

Theorem (Byrne, F., 2022)

$\mathcal{S}_M(e)$ exists if and only if M contains a prime diamond.

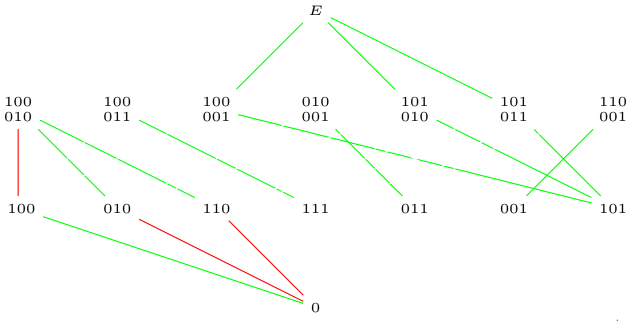
Example of an $\mathcal{S}_M(e)$

Let $\mathbb{F}_2^3$ be the ground space of the q-matroid M and let the following lattice represent the rank structure of M.



Example of an $\mathcal{S}_M(e)$

Let $\mathbb{F}_2^3$ be the ground space of the q-matroid M and let the following lattice represent the rank structure of M.



We have that M can be decomposed into $\mathcal{S}_M(101)$.

No further decomposition into $\mathcal{S}_M(e)$ is possible.

A q-analogue of Brylawski's (book) definition

Definition (Byrne, F., 2022)

A q-analogue of Brylawski's (book) definition

Definition (Byrne, F., 2022)

The **Tutte polynomial of a (q-)matroid** is the unique function $\tau : \mathcal{M} \to \mathbb{Z}[x, y]$ satisfying

A q-analogue of Brylawski's (book) definition

Definition (Byrne, F., 2022)

The **Tutte polynomial of a (q-)matroid** is the unique function $\tau : \mathcal{M} \to \mathbb{Z}[x, y]$ satisfying

- $\tau(M(e)) = x^{r(e)} y^{\nu(e)}$ for any $e \in M$.

A q-analogue of Brylawski's (book) definition

Definition (Byrne, F., 2022)

The **Tutte polynomial of a (q-)matroid** is the unique function $\tau : \mathcal{M} \to \mathbb{Z}[x, y]$ satisfying

- $\tau(M(e)) = x^{r(e)} y^{\nu(e)}$ for any $e \in M$.
- $\tau(M) = \tau(M(e))\tau(M/e)$ if $\mathcal{S}_M(e)$ doesn't exist.

A q-analogue of Brylawski's (book) definition

Definition (Byrne, F., 2022)

The **Tutte polynomial of a (q-)matroid** is the unique function $\tau : \mathcal{M} \to \mathbb{Z}[x, y]$ satisfying

- $\tau(M(e)) = x^{r(e)} y^{\nu(e)}$ for any $e \in M$.
- $\tau(M) = \tau(M(e))\tau(M/e)$ if $\mathcal{S}_M(e)$ doesn't exist.
- $\tau(M) = \sum_{N \in \mathcal{S}_M(e)} \tau(N)$ otherwise.

A q-analogue of Brylawski's (book) definition

Definition (Byrne, F., 2022)

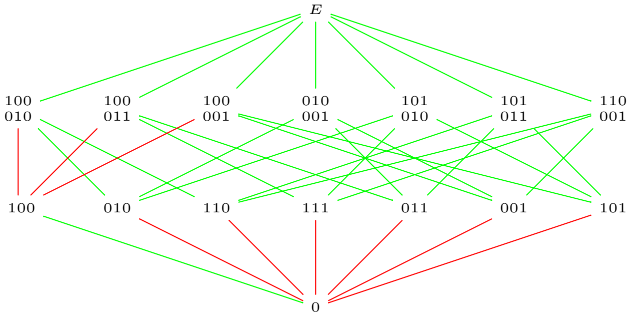
The **Tutte polynomial of a (q-)matroid** is the unique function $\tau : \mathcal{M} \to \mathbb{Z}[x, y]$ satisfying

- $\tau(M(e)) = x^{r(e)} y^{\nu(e)}$ for any $e \in M$.
- $\tau(M) = \tau(M(e))\tau(M/e)$ if $\mathcal{S}_M(e)$ doesn't exist.
- $\tau(M) = \sum_{N \in \mathcal{S}_M(e)} \tau(N)$ otherwise.

We call a function $f : \mathcal{M} \to \mathbb{Z}[x, y]$ a q-**TG invariant** if it satisfies the latter two properties of the above definition.

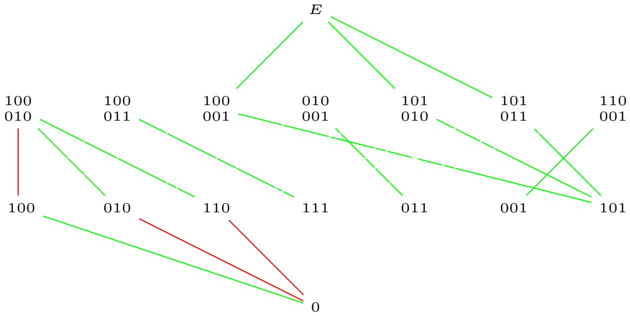
An example of a Tutte polynomial

Let $\mathbb{F}_2^3$ be the ground space of the q-matroid M and let the following lattice represent the rank structure of M.



An example of a Tutte polynomial

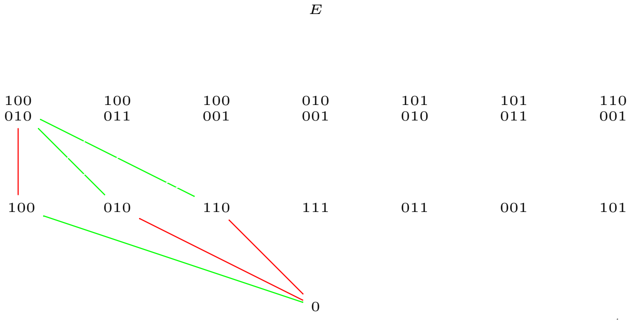
Let $\mathbb{F}_2^3$ be the ground space of the q-matroid M and let the following lattice represent the rank structure of M.



We have that M can be decomposed into $\mathcal{S}_M(101)$

An example of a Tutte polynomial

Let $\mathbb{F}_2^3$ be the ground space of the q-matroid M and let the following lattice represent the rank structure of M.

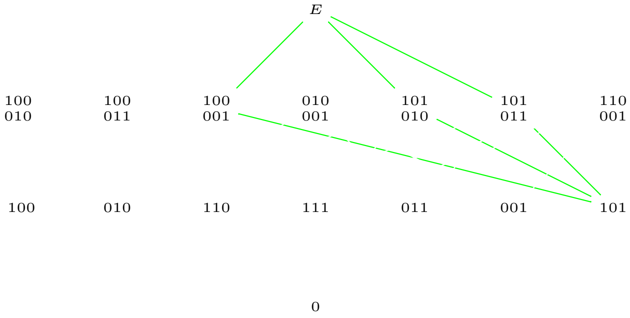


We have that M can be decomposed into $\mathcal{S}_M(101)$, which gives us

$$\tau(M; x, y) = xy + y^2 + 3y.$$

An example of a Tutte polynomial

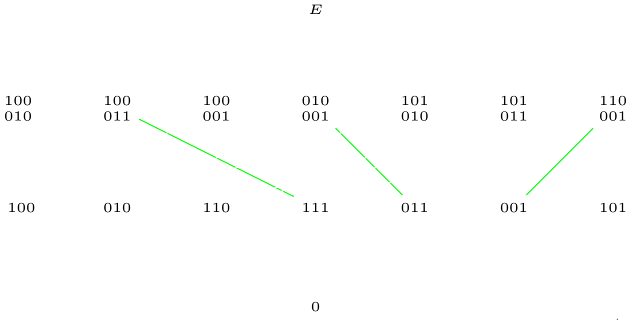
Let $\mathbb{F}_2^3$ be the ground space of the q-matroid M and let the following lattice represent the rank structure of M.



We have that M can be decomposed into $\mathcal{S}_M(101)$, which gives us $\tau(M; x, y) = xy + y^2 + 3y$.

An example of a Tutte polynomial

Let $\mathbb{F}_2^3$ be the ground space of the q-matroid M and let the following lattice represent the rank structure of M.



We have that M can be decomposed into $\mathcal{S}_M(101)$, which gives us
$$\tau(M; x, y) = xy + y^2 + 3y.$$

Connections with the rank polynomial

Connections with the rank polynomial

Definition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid. The **rank polynomial** ρ of M is defined by

$$\rho(M; x, y) = \sum_{z \in \mathcal{L}} x^{r(M) - r(z)} y^{\nu(z)}.$$

Connections with the rank polynomial

Definition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid. The **rank polynomial** ρ of M is defined by

$$\rho(M; x, y) = \sum_{z \in \mathcal{L}} x^{r(M) - r(z)} y^{\nu(z)}.$$

If M is a matroid, then $\rho(M; x, y) = \tau(M; x + 1, y + 1)$.

Connections with the rank polynomial

Definition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid. The **rank polynomial** ρ of M is defined by

$$\rho(M; x, y) = \sum_{z \in \mathcal{L}} x^{r(M)-r(z)} y^{\nu(z)}.$$

If M is a matroid, then $\rho(M; x, y) = \tau(M; x+1, y+1)$.

One way to see this is by considering that ρ is a q-TG invariant when defined on matroids. When defined on q-matroids, ρ is not a q-TG invariant!

Connections with the rank polynomial

Definition

Let $M = (\mathcal{L}, r)$ be a (q-)matroid. The **rank polynomial** ρ of M is defined by

$$\rho(M; x, y) = \sum_{z \in \mathcal{L}} x^{r(M)-r(z)} y^{\nu(z)}.$$

If M is a matroid, then $\rho(M; x, y) = \tau(M; x+1, y+1)$.

One way to see this is by considering that ρ is a q-TG invariant when defined on matroids. When defined on q-matroids, ρ is not a q-TG invariant!

If M is a q-matroid, then $\rho(M; x, y) \neq \tau(M; x+1, y+1)$ in general.

Connections with the rank polynomial (ctd.)

The following was a conjecture by Bollen, Crapo, and Jurrius in 2017.

Connections with the rank polynomial (ctd.)

The following was a conjecture by Bollen, Crapo, and Jurrius in 2017.

Theorem (Byrne, F., 2022)

Let M be a (q-)matroid with $\rho(M;x,y) = \sum_{a,b} \rho_{a,b} x^a y^b$.

Connections with the rank polynomial (ctd.)

The following was a conjecture by Bollen, Crapo, and Jurrius in 2017.

Theorem (Byrne, F., 2022)

Let M be a (q-)matroid with $\rho(M; x, y) = \sum_{a,b} \rho_{a,b} x^a y^b$. The Tutte polynomial of M is $\tau(M; x, y) = \sum_{a,b} \rho_{a,b} \sum_{c,d} \beta_q(a, b; c, d) x^c y^d$

Connections with the rank polynomial (ctd.)

The following was a conjecture by Bollen, Crapo, and Jurrius in 2017.

Theorem (Byrne, F., 2022)

Let M be a (q-)matroid with $\rho(M;x,y)=\sum_{a,b}\rho_{a,b}x^a y^b$.The Tutte polynomial of M is $\tau(M;x,y)=\sum_{a,b}\rho_{a,b}\sum_{c,d}\beta_q(a,b;c,d)x^c y^d$, where

$$\beta_q(a,b;c,d)=(-1)^{(a-c)+(b-d)}\begin{bmatrix}a\\c\end{bmatrix}_q\begin{bmatrix}b\\d\end{bmatrix}_q q^{\binom{|(a-c)-(b-d)|}{2}}(1+ q^{|(a-c)-(b-d)|}-q^{\max(a-c,b-d)}).$$

Connections with the rank polynomial (ctd.)

The following was a conjecture by Bollen, Crapo, and Jurrius in 2017.

Theorem (Byrne, F., 2022)

Let M be a (q-)matroid with $\rho(M;x,y) = \sum_{a,b} \rho_{a,b} x^a y^b$. The Tutte polynomial of M is $\tau(M;x,y) = \sum_{a,b} \rho_{a,b} \sum_{c,d} \beta_q(a,b;c,d) x^c y^d$, where

$$\beta_q(a,b;c,d) = (-1)^{(a-c)+(b-d)} \begin{bmatrix} a \\ c \end{bmatrix}_q \begin{bmatrix} b \\ d \end{bmatrix}_q q^{\binom{|(a-c)-(b-d)|}{2}} (1 + q^{|(a-c)-(b-d)|} - q^{\max(a-c,b-d)}).$$

Set $q = 1$ to get the matroid case.

In summary...

In summary...

- $\beta_q : \mathbb{Z}[x, y] \to \mathbb{Z}[x, y]$ is an invertible map, and defines a ring homomorphism iff $q = 1$.

In summary...

- $\beta_q : \mathbb{Z}[x, y] \to \mathbb{Z}[x, y]$ is an invertible map, and defines a ring homomorphism iff $q = 1$.
- We have that ρ is a q-TG invariant (application of Brylawski, 1972) iff $q = 1$.

In summary...

- $\beta_q : \mathbb{Z}[x, y] \rightarrow \mathbb{Z}[x, y]$ is an invertible map, and defines a ring homomorphism iff $q = 1$.
- We have that ρ is a q-TG invariant (application of Brylawski, 1972) iff $q = 1$.
- There are numerous equivalent definitions of the Tutte polynomial for matroids.

In summary...

- $\beta_q : \mathbb{Z}[x, y] \to \mathbb{Z}[x, y]$ is an invertible map, and defines a ring homomorphism iff $q = 1$.
- We have that ρ is a q-TG invariant (application of Brylawski, 1972) iff $q = 1$.
- There are numerous equivalent definitions of the Tutte polynomial for matroids.
- By using a lattice-theoretic argument we can naturally generalise the Tutte polynomial.

In summary...

- $\beta_q : \mathbb{Z}[x, y] \to \mathbb{Z}[x, y]$ is an invertible map, and defines a ring homomorphism iff $q = 1$.
- We have that ρ is a q-TG invariant (application of Brylawski, 1972) iff $q = 1$.
- There are numerous equivalent definitions of the Tutte polynomial for matroids.
- By using a lattice-theoretic argument we can naturally generalise the Tutte polynomial.

Thank you.