Additive polycyclic codes over $\mathbb{F}_4$ II: The general case

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- Introduction to linear and additive polycyclic codes

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- An additive code of length n over a finite field $\mathbb{F}_q$ is a subgroup of $\mathbb{F}_q^n$.
- Additive codes over the finite field $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ where $\alpha^2 + \alpha + 1 = 0$ were introduced in [4] because of their role in constructing quantum codes from classical codes.

Introduction to linear and additive polycyclic codes

- A linear code of length n over a finite field $\mathbb{F}_q$ is a subspace of $\mathbb{F}_q^n$.
- An additive code of length n over a finite field $\mathbb{F}_q$ is a subgroup of $\mathbb{F}_q^n$.
- Additive codes over the finite field $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ where $\alpha^2 + \alpha + 1 = 0$ were introduced in [4] because of their role in constructing quantum codes from classical codes.
- Define the cyclic shift operator $\sigma : \mathbb{F}_4^n \to \mathbb{F}_4^n$ by $\sigma((c_0, c_1, \ldots, c_{n-1})) = (c_{n-1}, c_0, c_1, \ldots, c_{n-2})$. Cyclic additive codes over $\mathbb{F}_4$ are additive codes over $\mathbb{F}_4$ such that if $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, then $\sigma(c) \in C$.

- Let C be a linear code over a finite field $\mathbb{F}_q$. Linear right and left polycyclic codes over finite fields were introduced in [3].
- These codes are a generalization of cyclic codes over finite fields. Their properties and structures are studied in details in [3],[2],[1].
- In [2], we have studied these codes in the special case when $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_2^n$, i.e., when the vector a only has binary entries.
- In this extended abstract, we are interested in studying the structure and the properties of additive right and left polycyclic codes induced by a vector $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_4^n$.

Definition

Let $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_4^n$ and define the mapping $T : \mathbb{F}_4^n \longrightarrow \mathbb{F}_4^n$ by

$$T(c_0, c_1, \ldots, c_{n-1}) = (0, c_0, c_1, \ldots, c_{n-2}) + c_{n-1}(a_0, a_1, \ldots, a_{n-1}).$$

Definition

Let C be an additive code over $\mathbb{F}_4$. C is called an additive right polycyclic code induced by $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_4^n$, if for any $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, we have $T(c) \in C$.

Definition

Let $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_4^n$ and define the mapping $L : \mathbb{F}_4^n \longrightarrow \mathbb{F}_4^n$ by

$$L(c_0, c_1, \ldots, c_{n-1}) = (c_1, c_2, \ldots, c_{n-1}, 0) + c_0 (a_0, a_1, \ldots, a_{n-1}).$$

Definition

Let C be an additive code over $\mathbb{F}_4$. C is called an additive left polycyclic code induced by $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_4^n$, if for any $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, we have $L(c) \in C$.

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- If $a = (\alpha, 0, 0, \ldots, 0)$ or $a = (\alpha + 1, 0, 0, \ldots, 0)$, then right polycyclic codes induced by a are just the usual constacyclic codes.
  Proof: If C is an additive right polycyclic code induced by $a = (\alpha, 0, 0, \ldots, 0)$ and $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, then $T(c) = (0, c_0, c_1, \ldots, c_{n-2}) + c_{n-1}(\alpha, 0, 0, \ldots, 0) = (\alpha c_{n-1}, c_0, \ldots, c_{n-2})$ and hence the code is a linear polycyclic code. The same is true if $a = (\alpha + 1, 0, 0, \ldots, 0)$

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  2. $C = \langle \alpha g_1 + g_2 \rangle$ where the polynomial $\alpha g_1 + g_2$ is a nonbinary polynomial of degree less than n in C in which g_1 and g_2 are binary polynomials and g_1 is of minimal degree, $g_1 | (x^n - a)$, and $(x^n - a) \mid \left(\frac{x^n - a}{g_1} g_2 \right)$, or

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  3. $C = \langle \alpha g_1 + g_2, b \rangle$, where g_1, g_2 and b are polynomials that satisfy the conditions in 1 and 2, $\deg g_2 < \deg b$ and $b | \left(\frac{x^n - a}{g_1} g_2 \right)$.

The structure of additive polycyclic codes induced by a nonbinary polynomial

- Let $a = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{F}_4^n$. As usual, the vector a can be represented as $a(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in \mathbb{F}_4[x]$.

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- The ring $R_n = \mathbb{F}_4[x] / \langle x^n - a(x) \rangle$ is an $\mathbb{F}_2[x]$-module.
- Let $c(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \in \mathbb{F}_4[x] / \langle x^n - a(x) \rangle$. Then,

$$
\begin{aligned}
xc(x) &= c_0x + c_1x^2 + \ldots + c_{n-2}x^{n-1} + c_{n-1}x^n \\
&= c_0x + c_1x^2 + \ldots + c_{n-2}x^{n-1} + \\
&\quad c_{n-1}\left(a_0 + a_1x + \ldots + a_{n-1}x^{n-1}\right) \\
&= c_{n-1}a_0 + x\left(c_0 + c_{n-1}a_1\right) + \ldots + x^{n-1}\left(c_{n-2} + c_{n-1}a_{n-1}\right).
\end{aligned}
$$

- Hence, the vector representation of $xc(x)$ is $\left(c_{n-1}a_0, c_0 + c_{n-1}a_1, \ldots, c_{n-2} + c_{n-1}a_{n-1}\right) = (0, c_0, \ldots, c_{n-2}) + c_{n-1}(a_0, \ldots, a_{n-1})$.

Lemma

C is an additive right polycyclic code induced by a if and only if C is an $\mathbb{F}_2[x]$-submodule of R_n.

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- Example: Let $n = 3$, $a = (\alpha, 1, \alpha) = \alpha + x + \alpha x^2$ and $x^3 - a = x^3 - \alpha - x - \alpha x^2$
- Consider the code $C = \langle b \rangle = \langle x + 1 \rangle$ .Notice that the following codewords are in the code C
- $x + 1 = (1, 1, 0)$
- $(0, 1, 1) + 0\,(\alpha, 1, \alpha) = (0, 1, 1)$
- $(0, 0, 1) + (\alpha, 1, \alpha) = (\alpha, 1, \alpha + 1)$
- This implies that the code C has 8 codewords with generating set $\{(1, 1, 0), (0, 1, 1), (\alpha, 1, \alpha + 1)\}$.
- If $C = \langle x + 1 \rangle$ is induced by the binary polynomial $a = x$ and $x^3 - a = x^3 - x$, then C will have 4 codewords.

- Let C be an additive right polycyclic code induced by a. Then C is invariant under right multiplication by the square matrix

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}.$$

The relationship between linear and Additive polycyclic codes

- Recall that If $a = (\alpha, 0, 0, \ldots, 0)$ or $a = (\alpha + 1, 0, 0, \ldots, 0)$, then right polycyclic codes induced by a are just the usual constacyclic codes.

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Lemma

Let C be an additive right polycyclic code of length n induced by $a = \alpha x^i + \alpha$, then C is a right polycyclic linear code.

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Lemma

Let C be an additive right polycyclic code of length n induced by $a = \alpha x^i + \alpha$, then C is a right polycyclic linear code.

- Proof: Suppose that $a = \alpha x^i + \alpha$ and let $c \in C$. Let $n = ki + l$ such that k and l are natural numbers and $l < i$. We have $x^n c = \left(\alpha x^i + \alpha\right) c = c\alpha x^i + c\alpha \in C$. Hence, $x^{n+1}c = c\alpha x^{i+1} + c\alpha x \in C$ and $x^{n+2}c = c\alpha x^{i+2} + c\alpha x^2 \in C$. By continuing the same process we have $x^{2n-2i}c = c\alpha x^{n-i} + c\alpha x^{n-2i} \in C$ and $x^{2n-i}c = c\alpha x^n + c\alpha x^{n-i} = c\alpha x^i + cx^i + c\alpha + c + c\alpha x^{n-i} \in C$. Since $c\alpha x^i + c\alpha + cx^i + c \in C$, $c\alpha x^{n-i} \in C$. Hence $c\alpha x^{n-2i} \in C$.

- If $l = 0$, then by continuing the same process we have $c\alpha x^i \in C$. Since $c\alpha x^i + c\alpha \in C$, $c\alpha \in C$. If $l \neq 0$, then by continuing the same process we will have $c\alpha x^l \in C$ such that $l < i$. Hence $c\alpha x^i \in C$. Since $c\alpha x^i + c\alpha \in C$, $c\alpha \in C$. This implies that the code C is a linear code.

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Lemma

Let C be an additive right polycyclic code of length n induced by $a = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + \alpha$, where $a_i \in \mathbb{F}_2$ for all $i = 1, 2, \ldots, n-1$. Then C is a linear polycyclic code.

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- **Proof:** Notice that $(x^n - u)\, c = \alpha vc \in C$ for any $c \in C$.
- Since $d = \gcd(x^n - u, v) \bmod 2$, then $x^n - u = k_1 d$ and $v = k_2 d$ for some binary polynomials k_1 and k_2. Moreover, there are two polynomials z_1 and z_2 such that $d = z_1 (x^n - u) + z_2 v$. Hence,

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- $\alpha dc = \alpha (z_1 (x^n - u) + z_2 v)\, c =$
  $\alpha (z_1 (x^n - u + \alpha v) + z_1 \alpha v + z_2 v)\, c = z_1 vc + z_1 \alpha vc + \alpha z_2 vc \in C.$

Lemma

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- **Proof:** If v is a unit $\bmod (x^n - u)$, then there is a binary polynomial p such that $pv = 1 \bmod (x^n - u)$.
- Or $pv = 1 + j(x^n - u) \Rightarrow 1 = pv + j(x^n - u)$. Hence
- $\alpha c = \alpha c(pv + j(x^n - u)) = p\alpha vc + \alpha cj(x^n - u + \alpha v) + \alpha cj(\alpha v) = p\alpha vc + jvc + j\alpha vc \in C$. Therefore, C is a linear polycyclic code.

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- Let $a = \alpha x + \alpha^2$ and $x^2 - a = (x+1)\left(x + \alpha^2\right)$. Then the code $C = \langle x + \alpha^2 \rangle = \left\{(0,0), (\alpha^2, 1)\right\}$ is an additive right polycyclic code which is not a linear code over $\mathbb{F}_4$.

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- Let $a = \alpha x^2 + x + \alpha$ and $x^3 - a = (x + 1)^2 (x + \alpha)$. Then the code $C = \langle x + \alpha \rangle = \left\{(0,0,0), (\alpha, 1, 0), (0, \alpha, 1), (\alpha, \alpha^2, 1)\right\}$ is an additive right polycyclic code which is not a linear code over $\mathbb{F}_4$.

Conclusion and ongoing research

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- We have shown that additive right polycyclic code induced by certain nonbinary elements a in $\mathbb{F}_4$ are linear polycyclic codes induced by a.

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- We have provided examples of additive right polycyclic code induced by certain nonbinary a that are not linear polycyclic codes.
- We are working on the generator polynomials of these codes.
- We are also working on the applications of these codes to construct optimal codes.

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