THE TUTTE POLYNOMIAL OF A q-MATROID

Abstract. We describe a construction of the Tutte polynomial for both matroids and q-matroids based on an appropriate partition of the underlying support lattice into intervals that correspond to prime-free minors, which we call a Tutte partition. We show that such partitions in the matroid case include the class of partitions arising in Crapo's definition of the Tutte polynomial, while not representing a direct q-analogue of such partitions. We propose axioms of q-Tutte-Grothendiek invariance and show that this yields a q-analogue of Tutte-Grothendiek invariance. We establish the connection between the rank polynomial and the Tutte polynomial, showing that one can be obtained from the other by convolution.

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1. Introduction

The Tutte polynomial of a matroid is a fundamental invariant from which several other matroid invariants may be obtained. It is very well-studied and there are numerous papers on the topic; see, for example, [1, 3–5, 8]. In particular, those functions of a matroid M that can be expressed recursively as functions on a pair of disjoint minors of M found by deletion and contraction of M, respectively, may be expressed in terms of the Tutte polynomial of M. The Tutte polynomial of M carries information common to all matroids in the same isomorphism class of M. For example, the number of bases, number of independent sets, and number of spanning sets of a matroid can be found by taking different evaluations of its Tutte polynomial. The characteristic polynomial of M is obtained by an evaluation in one of the variables of its Tutte polynomial, while its rank function is obtained by a linear substitution of variables. The Tutte polynomial is an invariant of matroid duality. In the case of representable matroids, the celebrated MacWilliams duality theorem in coding theory can be retrieved as a special case of a duality result on the Tutte polynomial.

While the origins of the Tutte polynomial lie in graph theory, Crapo defined this invariant for matroids by extending the notions of internal and external activity of the bases of a graph, defined originally by Tutte [10, 18], to the setting of matroids. In [6], Brylawski defined the Tutte-Grothendiek ring in order to study functions invariant under decompositions. Applied to the category of matroids (up to isomorphism) the universal invariant of that ring is the Tutte polynomial as defined by Crapo. In this context, the Tutte-Grothediek invariants satisfy two fundamental recursions with respect to taking matroid minors.

Since the publication of [16], there has been a good deal of interest in studying q-analogues of matroids and polymatroids [7, 9, 11–15, 17]. A q-matroid is defined in the same way as a matroid: it comprises a modular lattice endowed with a rank function satisfying certain rank axioms. The main difference between the two objects is that while a finite matroid consists of a rank function defined on the lattice of subsets of a finite set, the underlying lattice in the case of a q-matroid is the lattice of subspaces of a finite-dimensional vector space. We have chosen to present our results to encompass both the matroid and q-matroid case. In general, M denotes either a matroid or a q-matroid, with the matroid case often emerging as a result of the specialization $q \longrightarrow 1$.

A first attempt to define the Tutte polynomial of a q-matroid has been given in [2]. Building on this preliminary work, we describe a construction of the Tutte polynomial based on an appropriate partition of the underlying support lattice into intervals that correspond to prime-free minors, which we call a Tutte partition. We show that such partitions in the matroid case include the class of partitions arising in [10], while not representing a direct q-analogue of such partitions. Our description yields a Tutte polynomial both for matroids and q-matroids.

We furthermore consider the notion of Tutte-Grothendiek invariance. We define a new set of axioms and call the functions on the equivalence classes of matroids or q-matroids that satisfy this set of axioms q-Tutte-Grothendiek invariants. We show that in the case of a Boolean lattice (the matroid case) these axioms, which are defined in reference to the concept of a matroid being prime-free or not prime-free, are equivalent to the classical axioms of Tutte-Grothendiek invariance. We consider the relation between the Tutte polynomial we define here and the rank polynomial and prove that one can be obtained from the other by convolution. We show that while the Tutte polynomial is a q-Tutte-Grothendiek invariant both in the matroid and q-matroid case, in the case of q-matroid, the rank polynomial is not.

2. Preliminaries

We introduce some notation that will be used throughout this paper.

Notation 1. We let q denote a prime power and we let $\mathbb{F}_q$ denote the finite field of order q. For any positive integer n, we write $[n] := \{1, \ldots, n\}$.

Definition 2. Let n, k be non-negative integers. Let $q \in \mathbb{Z}$.

(1) If $q \neq 1$, then $[n]_q := \frac{q^n - 1}{q-1}$; if $q = 1$, then $[n]_q := n$.

(2) The q-factorial of n is defined by $[n]_q! := \prod_{k=1}^{n} [k]_q$.

(3) The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{cases} \dfrac{[n]!_q}{[n-k]!_q [k]!_q} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

The q-analogue of a binomial coefficient is called the *q-binomial coefficient* or *Gaussian coefficient*; $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ counts the number of k-dimensional subspaces of the vector space $\mathbb{F}_q^n$.

Notation 3. Throughout, $\mathcal{L}$ will denote a modular lattice ordered under $\leq$ with meet $\wedge$ and join $\vee$. We will write $\mathcal{C}$ to denote the set of covers in $\mathcal{L}$. We write $\mathbf{0} := 0_{\mathcal{L}}$ and $\mathbf{1} := 1_{\mathcal{L}}$. A complement of $v \in \mathcal{L}$ will be denoted by v^c, which may be lattice element $w \in \mathcal{L}$ such that $v \vee w = \mathbf{1}$ and $v \wedge w = \mathbf{0}$. We will let V denote an $\mathbb{F}_q$-vector space of dimension n. We will write $\mathcal{L}(V)$ to denote the lattice of subspaces of V.

There are numerous ways to define a q-matroid, [7, 16]. We will first define the concept of a q-matroid as a $\{0, 1\}$-*weighting* of a lattice. This is essentially the same concept as a *bicolouring*, as described in [2].

Notation 4. For any $x, y \in \mathcal{L}$, we write $x \lessdot y$ if $[x, y]$ is an interval of length one.

Definition 5. A $\{0, 1\}$-weighting of $\mathcal{L}$ is a map $w : \mathcal{C} \longrightarrow \{0, 1\}$. We say that $\mathrm{w}([a, b])$ is the *weight* of the cover $[a, b] \in \mathcal{C}$. For any chain $x_0 \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_k$, we define the weight of the chain to be $\sum_{j=0}^{k-1} \mathrm{w}([x_i, x_{i+1}])$. If $[a, b]$ is a diamond of $\mathcal{L}$, we say that it has has type *full, empty, mixed* or *prime* with respect to w if it satisfies one of the following descriptions:

Full: $\mathrm{w}([u, v]) = 1$ for each $[u, v] \in \mathcal{C}([a, b])$.
Empty: $\mathrm{w}([u, v]) = 0$ for each $[u, v] \in \mathcal{C}([a, b])$.
Mixed: There exists $x \in [a, b], x \neq a, b$ such that $\mathrm{w}([x, b]) = 1$ and $\mathrm{w}([a, x]) = 0$, while $\mathrm{w}([y, b]) = 0$, and $\mathrm{w}([a, y]) = 1$ for every $y \in [a, b], y \neq x$.
Prime: $w([a, x]) = 1$ and $w([x, b]) = 0$ for any $x \in [a, b], x \neq a, b$.

Definition 6. A $\{0, 1\}$-weighting of w of $\mathcal{L}$ is called *matroidal* if each diamond in $\mathcal{L}$ is of type full, empty, mixed, or prime.

Definition 7. Let $r : \mathcal{L} \to \mathbb{Z}_{\geq 0}$ be a function. We define the following *rank axioms*.

(R1) $0 \leq r(x) \leq h(x)$ for all $x \in \mathcal{L}$.

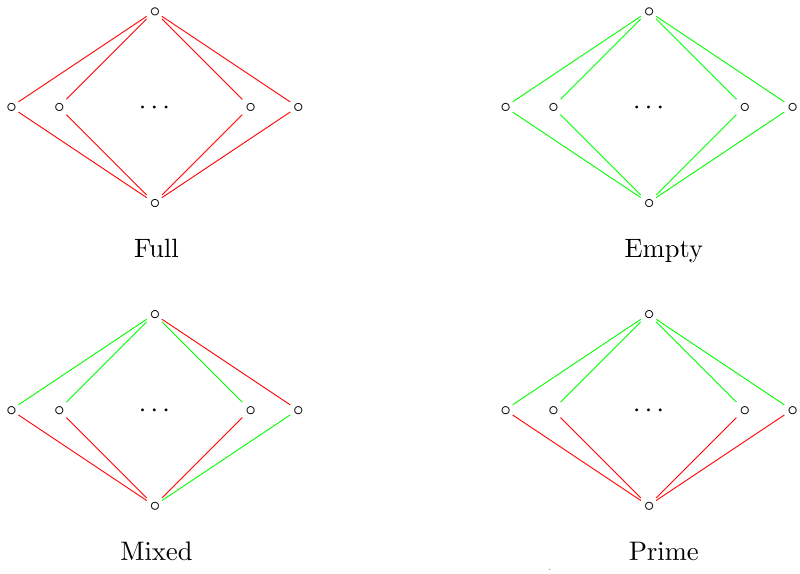

FIGURE 1. Matroidal Diamonds: red covers have weight 1, green covers have weight 0

(R2) For all $x, y \in \mathcal{L}$, $r(x) \leq r(y)$ whenever $x \leq y$.
(R3) For all $x, y \in \mathcal{L}$, $r(x \vee y) + r(x \wedge y) \leq r(x) + r(y)$.

We say that a rank function is *matroidal* if it satisfies the rank axioms (R1)-(R3).

Definition 8. Let $M = (\mathcal{L}, r)$ be a matroid or a q-matroid. Let $a, b \in \mathcal{L}$ such that $a \leq b$. The function $r_{[a,b]} : [a, b] \to \mathbb{Z}$ is defined by $r_{[a,b]}(x) = r(x) - r(a)$ for all $x \in [a, b]$. The function $r_{[a,b]}$ is a matroidal rank function on $[a, b] \subseteq \mathcal{L}$ and $([a, b], r_{[a,b]})$ is called a *minor* of M.

Notation 9. Let $M = (\mathcal{L}, r)$ be a matroid or q-matroid and let $a, b \in \mathcal{L}$ with $a \leq b$. For $[a, b] \subseteq \mathcal{L}$ we write $M([a, b])$ to denote the minor $([a, b], r_{[a,b]})$. We will also define $M(a) := M([\mathbf{0}, a])$, and define $M/a := M([a, \mathbf{1}])$. If M is a matroid, we use $M - a$ to denote the restriction of M to $[\mathbf{0}, a^c] \subseteq \mathcal{L}$.

Definition 10. Let r be a matroidal rank function on a support lattice $\mathcal{L}$. For any $x \in \mathcal{L}$, the *nullity* of x is defined to be $\nu(x) = h(x) - r(x)$. For an interval $[a, b] \subseteq \mathcal{L}$ the function $\nu_{[a,b]} : [a, b] \to \mathbb{Z}$ is defined by $\nu_{[a,b]}(x) = \nu(x) - \nu(a)$ for all $x \in [a, b]$.

Definition 11. Let $M = (\mathcal{L}, r)$ be a matroid or a q-matroid and let $x \in \mathcal{L}$. If $h(x) = 1$ and $r(x) = 0$, then x is called a *loop* of M. If x is a coatom of $\mathcal{L}$ such that $r(x) = r(M) - 1$, then x is called a *coloop* of M.

A cryptomorphism between q-matroids defined via a rank function or via a $\{0, 1\}$-weighting on the subspace lattice was shown in [2, Theorem 2]. It's easy to see that the same result holds for matroids. We hence have the following statement.

Theorem 12 ([2]).

(1) Let $r : \mathcal{L} \to \mathbb{Z}_{\geq 0}$ be a matroidal rank function. Define a map w_r on the set of covers $\mathcal{C}$ of $\mathcal{L}$ by $\mathrm{w}_r([a, b]) := r(b) - r(a)$ for all $[a, b] \in \mathcal{C}$. Then w_r is a matroidal $\{0, 1\}$-weighting of $\mathcal{L}$.
(2) Let $\mathrm{w} : \mathcal{C} \longrightarrow \{0, 1\}$ be a matroidal $\{0, 1\}$-weighting. Define a map $r_w : \mathcal{L} \to \mathbb{Z}_{\geq 0}$ by $r_{\mathrm{w}}(x) := \sum_{i=1}^{t} \mathrm{w}([x_{i-1}, x_i])$, where $\mathbf{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_t = x$ is a maximal chain in $[\mathbf{0}, x]$. Then r_{w} is a matroidal rank function.
(3) Furthermore, $\mathrm{w}_{r_{\mathrm{w}}} = \mathrm{w}$ and $r_{\mathrm{w}_r} = r$.

Definition 13. Let r_1 and r_2 be a pair of matroidal rank functions with support lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. We say that r_1 and r_2 are *lattice-equivalent* if there exists a lattice isomorphism

$\varphi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ such that $r_1(x) = r_2(\varphi(x))$ for all $x \in \mathcal{L}_1$. In this case we write $M_1 \cong M_2$, where $M_1 = (\mathcal{L}_1, r_1)$ and $M_2 = (\mathcal{L}_2, r_2)$.

It is straightforward to check that if w is a matroidal $\{0,1\}$-weighting of $\mathcal{L}$ then its dual satisfies $\mathrm{w}^*([\varphi(y), \varphi(x)]) = 1 - \mathrm{w}([x, y])$ for every cover $[x, y] \in \mathcal{C}$. Observe also that $r^*(\varphi(x)) = \nu(\mathbf{1}) - \nu(x)$ for all $x \in \mathcal{L}$.

3. Tutte Partitions

In this section we introduce a polynomial invariant of M, which we refer to as its q-Tutte polynomial. Recall that M may be a matroid or a q-matroid. Our definition of a q-Tutte polynomial will arise from a family of partitions of the support lattice $\mathcal{L}$ of M. Following [2], we will consider partitions of M into *prime-free* minors.

Definition 14. We say that M is *prime-free* if M has no prime diamonds in $\mathcal{L}$.

Definition 15. Let $\mathcal{P}$ be a partition of the elements of $\mathcal{L}$ such that every element of $\mathcal{P}$ is an interval. We say that $\mathcal{P}$ is a *Tutte partition* of $\mathcal{L}$ if the following properties hold for every $[a, b] \in \mathcal{P}$:

1. $r(a) = h(a)$,
2. $r(b) = r(\mathbf{1})$,
3. $[a, b]$ is prime-free.

Central to our strategy to define the q-Tutte polynomial is the notion of a totally clopen element. Such an element only exists if M is prime-free, in which case it determines the rank of every element of M.

Definition 16. An element $z \in \mathcal{L}$ is called a *clopen* element of M if for every $a \in \mathcal{L}$ such that z covers a we have $\mathrm{w}([a, z]) = 0$ and for every cover b of z we have $\mathrm{w}([z, b]) = 1$. We say that z is a *totally clopen* element of M if any cover $[x, y] \in \mathcal{C}$ satisfies $\mathrm{w}([x, y]) = 1$ if $x \geq z$, and $\mathrm{w}([x, y]) = 0$ if $y \leq z$.

Lemma 17. *Suppose that $z \in \mathcal{L}$ is a totally clopen element of M. Then for any cover $[x, y]$ in $\mathcal{L}$ we have:*

$$\mathrm{w}([x, y]) = \begin{cases} 0 & \text{if } x \wedge z \lessdot y \wedge z, \\ 1 & \text{if } x \vee z \lessdot y \vee z. \end{cases}$$

Furthermore, z is the unique totally clopen element of M and the complements of z are the bases of M.

We give a precise characterization of M for which a totally clopen element exists.

Theorem 18. *There exists a totally clopen element $z \in \mathcal{L}$ in M if and only if M is prime-free.*

Next we provide a partition of $\mathcal{L}$ that will be useful in proving the main result of this section, which uses induction.

Lemma 19. *Let $e \in \mathcal{L}$ be an atom and let $e^c \in \mathcal{L}$ be a complement of e. There exist $q^{n-1} - 1$ intervals $[e_k, \bar{e}_k]$ of $\mathcal{L}$ such that for each k, e_k is an atom, $\bar{e}_k$ is a coatom, and*

$$[e, \mathbf{1}] \,\dot\cup\, [\mathbf{0}, e^c] \,\dot\cup\, [e_1, \bar{e}_1] \,\dot\cup\, \cdots \,\dot\cup\, [e_{q^{n-1}-1}, \bar{e}_{q^{n-1}-1}]$$

forms a partition of the elements of $\mathcal{L}$.

Definition 20. Let $e \in \mathcal{L}$ be an atom and let $\mathcal{P}$ be a partition of $\mathcal{L}$ of the form

$$[e, \mathbf{1}] \,\dot\cup\, [\mathbf{0}, e^c] \,\dot\cup\, [e_1, \bar{e}_1] \,\dot\cup\, \cdots \,\dot\cup\, [e_{q^{n-1}-1}, \bar{e}_{q^{n-1}-1}],$$

where for each k, $e_k \leq \bar{e}_k$ for an atom e_k and a coatom $\bar{e}_k$. We say that $\mathcal{P}$ is a *minimal q-partition*, and denote such a partition by $\mathcal{S}(e)$.

Note that $\mathcal{S}(e)$ is not necessarily unique for each atom e, but it is sufficient for our purposes to choose one such partition for a given e. We call it *minimal* because it is a non-trivial partition into intervals with a minimal number of elements.

Remark 21. In the case that $\mathcal{L}$ is a Boolean lattice, the minimal partition $\mathcal{S}(e)$ yields a partition of M into two minors: one of which contains e and one that does not contain e.

Theorem 22. *M has a Tutte partition.*

Notation 23. For an interval $A = [a, b] \subseteq \mathcal{L}$, we will let $\mathbf{1}_A := b$ and $\mathbf{0}_A := a$.

We are now ready to give a definition of a Tutte polynomial.

Definition 24. Let $\mathcal{P}$ be a Tutte partition of M. We define the *Tutte polynomial* of M with respect to the partition $\mathcal{P}$ to be

$$\tau_{\mathcal{P}}(M; x, y) = \sum_{A \in \mathcal{P}} x^{r_A(\mathbf{1}_A)} y^{\nu_A(\mathbf{0}_A)}.$$

While the definition we give here is given with respect to a partition $\mathcal{P}$, we will see in Section 5 that the Tutte polynomial is independent of the choice of Tutte partition.

Example 25. Let $M = (\mathcal{L}(\mathbb{F}_2^3), r)$ be the q-matroid with support lattice shown on the left of Figure 2. The weight-0 edges of M have coloured green and the weight-1 edges have been coloured red. On the right of Figure 2, we see a Tutte partition $\mathcal{P}$ of M into 5 prime-free minors. The diamond $[e, \mathbf{1}] = [\langle 101 \rangle, \mathbf{1}]$ is empty and has rank 0 and nullity 2, which contributes the term y^2 to $\tau_{\mathcal{P}}(M, x, y)$. The minor with support lattice $[\mathbf{0}, e^c] = [\mathbf{0}, \langle 100, 010 \rangle]$ has rank 1 and nullity 1, which corresponds to the term xy. The remaining 3 intervals support minors of length 1, rank 0, and nullity 1.

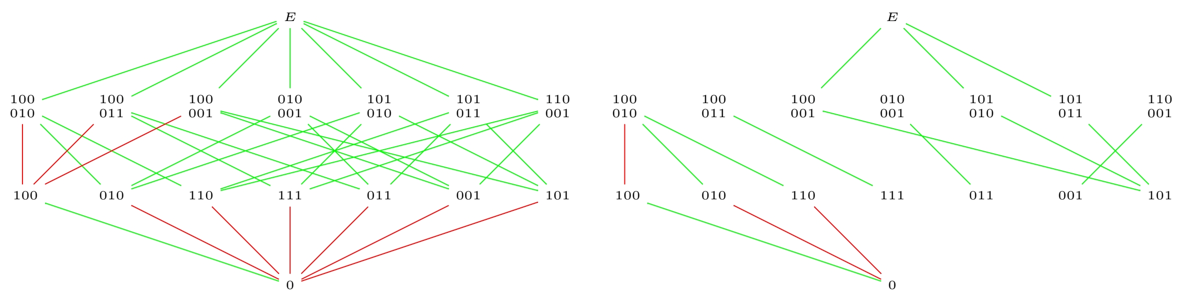

FIGURE 2. The Tutte Polynomial $\tau_{\mathcal{P}}(M; x, y) = xy + y^2 + 3y$.

4. An Inversion Formula

We establish an inversion formula, which we will use in Section **??** to show that the rank polynomial of M can be derived from a q-Tutte polynomial of M and vice-versa.

Definition 26. Let n, m, i, j be non-negative integers. We define:

$$\alpha_q(n, m; i, j) := \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ j \end{bmatrix}_q q^{(n-i)(m-j)}.$$

Definition 27. Let a, b, c, d be non-negative integers. Define:

$$\beta_q(a, b; c, d) := (-1)^{(a-c)+(b-d)} \begin{bmatrix} a \\ c \end{bmatrix}_q \begin{bmatrix} b \\ d \end{bmatrix}_q q^{\binom{|(a-c)-(b-d)|}{2}} (1 + q^{|(a-c)-(b-d)|} - q^{\max(a-c, b-d)}).$$

Definition 28. Let $\gamma(i, j; a, b)$ be non-negative integers for all non-negative integers i, j, a, b. We define $\gamma : \mathbb{Z}[x, y] \longrightarrow \mathbb{Z}[x, y]$ by:

$$\gamma * f := \sum_{i=0}^{u} \sum_{j=0}^{v} f_{i,j} \sum_{a=0}^{i} \sum_{b=0}^{j} \gamma(i, j; a, b) x^a y^b,$$

for all $f(x,y) = \sum_{i=0}^{u}\sum_{j=0}^{v} f_{i,j}x^i y^j \in \mathbb{Z}[x,y]$.

Theorem 29. *Let a,b,c,d,e,f be non-negative integers. Then*

$$\sum_{c=0}^{a}\sum_{d=0}^{b} \alpha_q(a,b;c,d)\beta_q(c,d;e,f) = \delta_a^e \delta_b^f.$$

In particular, for any $f \in \mathbb{Z}[x,y]$, we have

$$\alpha_q * (\beta_q * f) = f = \beta_q * (\alpha_q * f).$$

5. The Rank Polynomial and the Tutte Polynomial

Definition 30. The *rank polynomial* of M is defined to be

$$\rho(M;x,y) := \sum_{z\in\mathcal{L}} x^{r(\mathbf{1})-r(z)} y^{\nu(z)} \in \mathbb{Z}[x,y].$$

Proposition 31 ([2])**.** *Let M be prime-free. Let $h(\mathbf{1}) = n$, $r(\mathbf{1}) = \rho$, and $\nu(\mathbf{1}) = \nu$. The number of elements in $\mathcal{L}$ with rank-lack i and nullity j is $\alpha_q(\rho,\nu;i,j)$.*

Corollary 32. *Let M be a prime-free and have rank ρ and nullity ν. Then the rank polynomial of M is:*

$$\rho(M;x,y) = \sum_{i=0}^{\rho}\sum_{j=0}^{\nu} \alpha_q(\rho,\nu;i,j)x^i y^j.$$

Theorem 33. *Let $\mathcal{P}$ be a Tutte partition of M and let $\tau_{\mathcal{P}}(M;x,y) = \sum_{i=0}^{\rho}\sum_{j=0}^{\nu} \tau_{i,j}x^i y^j$. Then the rank polynomial of M is:*

$$\rho(M;x,y) = \alpha_q * \tau_{\mathcal{P}}(M;x,y).$$

From Theorem 29, we obtain that for any Tutte partition $\mathcal{P}$, $\tau_{\mathcal{P}}(M;x,y)$ is determined by the rank polynomial of M.

Corollary 34. *Let $\mathcal{P}$ be a Tutte partition of M. Then*

$$\beta_q * \rho(M;x,y) = \tau_{\mathcal{P}}(M;x,y).$$

In particular, the Tutte polynomial $\tau_{\mathcal{P}}(M;x,y)$ of M is independent of the choice of Tutte partition $\mathcal{P}$.

Therefore, we denote the q-Tutte polynomial of M by $\tau(M;x,y)$.

6. q-Tutte-Grothendieck Invariance

In the classical theory of matroids, Tutte-Grothendiek invariance is fundamental to the study of the Tutte polynomial [4–6].

We define a notion of such invariance in the more general setting of this paper and show that we recover the original T-G invariance for matroids by letting $q \to 1$.

Notation 35. We let $\mathcal{M}$ denote either the class of all q-matroids or the class of all matroids.

Definition 36. Let f be a function defined on $\mathcal{M}$. We say that f is a *q-Tutte-Grothendieck invariant* (q-T-G) if it satisfies the following three properties:

(q-P1): For all $M_1, M_2 \in \mathcal{M}$, if $M_1 \cong M_2$ then $f(M_1) = f(M_2)$,
(q-P2): If M is prime-free, then $f(M) = f(M(e))f(M/e)$ for any atom $e \in \mathcal{L}$,

(q-P3): If M is not prime-free, and e is an atom of M that is independent and contained in every coloop of M, then there exists a minimal q-partition $\mathcal{S}(e)$ such that

$$f(M) = f(M(e^c)) + f(M/e) + \sum_{k=1}^{q^{n-1}-1} f(M([e_k, \bar{e}_k])).$$

Proposition 37. *The Tutte polynomial $\tau(M; x, y)$ is a q-T-G invariant.*

We now arrive at a definition of the q-Tutte polynomial as a q-T-G invariant.

Theorem 38. *The q-Tutte polynomial is the unique function τ from $\mathcal{M}$ to the ring $\mathbb{Z}[x, y]$ satisfying the following properties:*

(1) $\tau(M(e); x, y) = x$ if $e \in \mathcal{L}$ is an isthmus of M, and $\tau(M(e); x, y) = y$ if $e \in \mathcal{L}$ is a loop of M,
(2) $\tau(M; x, y)$ is a q-T-G invariant.

In fact the q-T-G invariance axioms of Definition 36 are equivalent to the classical T-G axioms when we restrict to the class of matroids.

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