# Subcovers of generalized GK curves and their automorphism groups

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COGNAC
algebraic varieties over finite fields and algebraic geometry codes
CIRM

# Outline

1. GK curve and generalizations
2. subcovers of the first generalized GK curve
3. their automorphism groups
4. subcovers of the second generalized GK curve
5. their automorphism groups
6. a characterization of the GK curve

# Algebraic curves over finite fields

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- automorphisms of the curve $\longrightarrow$ automorphisms of the AG codes

# Maximal curves from Galois subcovers

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If the extension $\mathbb{F}_q(\mathcal{X})/\mathbb{F}_q(\mathcal{Y})$ is Galois, with Galois group $G$ :

- $\mathcal{Y}$ is a Galois subcover of $\mathcal{X}$
- $G \equiv$ subgroup of $\mathrm{Aut}_{\mathbb{F}_q}(\mathcal{X}) = \{\mathbb{F}_q\text{-birationalities } \mathcal{X} \to \mathcal{X}\}$

  $\Longrightarrow \mathcal{Y} \equiv$ quotient curve $\mathcal{X}/G = \{\text{orbits of } G \text{ on } \mathcal{X}\}$

# First generalized GK curve

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- $\mathbb{F}_{q^2}$-maximal Hermitian curve $\quad \mathcal{H}_q : \; y^{q+1} = x^q + x$
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- $\mathbb{F}_{q^6}$-maximal GK curve $\quad \mathcal{GK} : \begin{cases} z^{q^2-q+1} = y^{q^2} - y \\ y^{q+1} = x^q + x \end{cases}$
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- $\mathrm{Aut}(\mathcal{GK}) \cong \mathrm{PGU}(3, q) \rtimes C_{q^2-q+1}$
- if $q > 2$, $\mathcal{GK}$ is not a subcover of $\mathcal{H}_{q^3}$ **(first example)**

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$$\mathrm{PGU}(3,q)_{\bar{P}_\infty} = S_{q^3} \rtimes C_{q^2-1} = \{(x,y,z) \mapsto (a^{q+1}x + ab^q y + c,\ ay + b,\ z) \mid a,b,c \in \mathbb{F}_{q^2}, a \neq 0, c^q + c = b^{q+1}\}$$

$$C_m = \{(x,y,z) \mapsto (x,y,\lambda z) \mid \lambda^m = 1\}$$

# Subcovers $\mathcal{Y}_{n,s}$ of the first generalized GK curve

$n \geq 3$ odd, $m = \frac{q^n+1}{q+1}$, $s$ : divisor of $m$

Tafazolian, Teherán-Herrera, Torres (2016):

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- some non-covering results $\quad \mathcal{H}_{q^n} \not\to \mathcal{Y}_{n,s}$

  Technique: find a contradiction to

$$\frac{|\mathcal{H}_{q^n}(\mathbb{F}_{q^{2n}})|}{|\mathcal{Y}_{n,s}(\mathbb{F}_{q^{2n}})|} \leq \deg(\varphi) \leq \frac{2g(\mathcal{H}_{q^n}) - 2}{2g(\mathcal{Y}_{n,s}) - 2}, \qquad \varphi : \mathcal{H}_{q^n} \to \mathcal{Y}_{n,s}$$

# Automorphism group of $\mathcal{Y}_{n,s}$

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- Güneri-Özdemir-Stichtenoth:
  - determine the Weierstrass semigroup $H(P_\infty) = \langle\, q^3,\, qm,\, (q+1)m \,\rangle$
  - show $H(Q) \neq H(P_\infty)$ for all $Q \in \mathcal{GGS}_n(\mathbb{F}_{q^{2n}})$
- Malmskog-Guralnick-Pries:
  - structural results on groups with TI $p$-subgroups
  - use that $m = \frac{q^n+1}{q+1} \;>>\; q$

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**Theorem (Montanucci-Tizziotti-Z.)**

- *If* $3 \nmid n$ *or* $\frac{m}{s} \nmid \frac{q^3+1}{q+1}$

  $\Longrightarrow \mathrm{Aut}(\mathcal{Y}_{n,s}) = S_{q^3} \rtimes C_{(q^2-1)\frac{m}{s}}$ *fixes* $P_\infty$

- *If* $3 \mid n$ *and* $\frac{m}{s} \mid \frac{q^3+1}{q+1}$

  $\Longrightarrow \mathcal{Y}_{n,s} \cong \mathcal{GK}/C_{\frac{q^2-q+1}{m/s}}$ , $\mathrm{Aut}(\mathcal{Y}_{n,s}) = \mathrm{PGU}(3,q) \rtimes C_{m/s}$

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Notice: if $3 \mid n \Longrightarrow \mathbb{F}_{q^{2n}} = \mathbb{F}_{q^{6d}}$ with $d$ odd

$\Longrightarrow$ the $\mathbb{F}_{q^6}$-maximal curve $\mathcal{GK}$ is also $\mathbb{F}_{q^{2n}}$-maximal

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$$\Longrightarrow \mathrm{Aut}(\mathcal{Y}_{n,s})/C_{m/s} \leq \mathrm{Aut}(\mathcal{Y}_{n,s}/C_{m/s}) = \mathrm{Aut}(\mathcal{H}_q) = \mathrm{PGU(3,q)}$$

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$$\Longrightarrow \text{either } \frac{\mathrm{Aut}(\mathcal{Y}_{n,s})}{C_{m/s}} \cong \mathrm{Aut}(\mathcal{Y}_{n,s})_{P_\infty} \text{ or } \frac{\mathrm{Aut}(\mathcal{Y})_{n,s}}{C_{m/s}} \cong \mathrm{PGU}(3,q)$$

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$$\Longrightarrow \text{either } \frac{\mathrm{Aut}(\mathcal{Y}_{n,s})}{C_{m/s}} \cong \mathrm{Aut}(\mathcal{Y}_{n,s})_{P_\infty} \text{ or } \frac{\mathrm{Aut}(\mathcal{Y})_{n,s}}{C_{m/s}} \cong \mathrm{PGU}(3,q)$$

- Prove: $\frac{\mathrm{Aut}(\mathcal{Y})_{n,s}}{C_{m/s}} \cong \mathrm{PGU}(3,q) \iff 3 \mid n$ and $\frac{m}{s} \mid \frac{q^3+1}{q+1}$

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- lift of $\tau$ + fundamental equation $\rightarrow$ construct elements of $H(P_\infty)$
- $H(P_\infty)$ is known (Tafazolian, Teherán-Herrera, Torres)

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$$\mathcal{X}_{\bar{q},n,s} : \begin{cases} z^{m/s} = y^{q^2} - y \\ cy^{q+1} = x + x^{\bar{q}} + \cdots + x^{q/\bar{q}} \end{cases}$$

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- $\mathrm{Aut}(\mathcal{X}_{\bar{q},n,s})$ ?

# Automorphism group of $\mathcal{X}_{\bar{q},n,s}$

$$\mathcal{Y}_{n,s} : \begin{cases} w^{m/s} = v^{q^2} - v \\ v^{q+1} = u^q + u \end{cases} \qquad \mathcal{X}_{\bar{q},n,s} : \begin{cases} z^{m/s} = y^{q^2} - y \\ cy^{q+1} = x + x^{\bar{q}} + \cdots + x^{q/\bar{q}} \end{cases}$$

- $\mathcal{X}_{\bar{q},n,s} = \mathcal{Y}_{n,s}/E_{\bar{q}}$ , $\quad E_{\bar{q}} = \left\{ (u,v,w) \mapsto (u + \frac{\alpha}{c}, v, w) \mid \alpha \in \mathbb{F}_{\bar{q}} \right\}$

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**Theorem (Montanucci-Tizziotti-Z.)**

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$$\bar{q} = q, \quad q^2 \mid \left(\frac{m}{s} - 1\right) \quad \Longrightarrow \mathcal{X}_{q,n,s} : \quad z^{m/s} = y^{q^2} - y$$

- Hermitian curve, $\mathrm{Aut}(\mathcal{X}_{q,n,s}) = \mathrm{PGU}(3, q^2)$
- Quotient of the Hermitian curve, $\mathrm{Aut}(\mathcal{X}_{q,n,s})/C_{m/s} = \mathrm{PGL}(2, q^2)$
- $\mathrm{Aut}(\mathcal{X}_{q,n,s})$ fixes $P_\infty$, larger Sylow $p$-subgroup

# Second generalized GK curve

Beelen-Montanucci 2018: $n$ odd, $\mathbb{F}_{q^{2n}}$-**maximal curve**

$$\mathcal{BM}_n : \begin{cases} z^m = y\frac{x^{q^2}-x}{x^{q+1}-1} \\ y^{q+1} = x^{q+1} - 1 \end{cases} \qquad m = \frac{q^n+1}{q+1}$$

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$$\mathrm{Aut}(\mathcal{BM}_n) = \langle \mathrm{PGU}(3,q)_\ell \,,\, C_{m/s} \rangle \cong \mathrm{SL}(2,q) \rtimes C_{q^n+1}$$

$\mathrm{Aut}(\mathcal{BM}_n)$ is the lift of the stabilizer $\mathrm{PGU}(3,q)_\ell$

of a $(q+1)$-secant $\ell$ to $\qquad y^{q+1} = x^{q+1} - 1$

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**Theorem (Montanucci-Tizziotti-Z.)**

- *If* $3 \nmid n$ *or* $\frac{m}{s} \nmid \frac{q^3+1}{q+1} \implies \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s}) \cong \mathrm{SL}(2,q) \rtimes C_{(q^n+1)/s}$
- *If* $3 \mid n$ *and* $\frac{m}{s} \mid \frac{q^3+1}{q+1}$

$$\implies \quad \tilde{\mathcal{Y}}_{n,s} \cong \mathcal{GK}/C_{\frac{q^2-q+1}{m/s}}, \quad \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s}) = \mathrm{PGU}(3,q) \rtimes C_{m/s}$$

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$$\implies \quad \tilde{\mathcal{Y}}_{n,s} \cong \mathcal{GK}/C_{\frac{q^2-q+1}{m/s}}, \quad \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s}) = \mathrm{PGU}(3,q) \times C_{m/s}$$

In the first case: $\quad g(\tilde{\mathcal{Y}}_{n,s}) = g(\mathcal{Y}_{n,s}), \quad \tilde{\mathcal{Y}}_{n,s} \not\cong \mathcal{Y}_{n,s}$

$\implies$ new $\mathbb{F}_{q^{2n}}$-maximal curves not covered by $\mathcal{H}_{q^n}$

# Subcovers $\tilde{\mathcal{X}}_{\bar{q},n,s}$ of the second generalized GK curve

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where $E_{\bar{q}} \leq \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s})$ is the lift to $\tilde{\mathcal{Y}}_{n,s}$ with $E_{\bar{q}}(z) = z$
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$$\mathrm{Aut}(\tilde{\mathcal{X}}_{\bar{q},n,s}) \;=\; \mathrm{N}_{\mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s})}(E_{\bar{q}})/E_{\bar{q}} \;\cong\; (E_q/E_{\bar{q}}) \rtimes C_{(q+1)(\bar{q}-1)\frac{m}{s}}$$

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$\Longrightarrow$ new $\mathbb{F}_{q^{2n}}$-maximal curves not covered by $\mathcal{H}_{q^n}$

# Conclusion

- Subcovers $\mathcal{Y}_{n,s}$ , $\mathcal{X}_{a,b,n,s}$ , $\tilde{\mathcal{Y}}_{n,s}$ , $\tilde{\mathcal{X}}_{a,b,n,s}$
  of the first (GGS) and second (BM) generalized GK curve
- their **automorphism groups**
- new maximal curves not covered by the Hermitian curve
- a characterization of the curves

$$\mathcal{GK}/C_s \in \{ \mathcal{Y}_{n,s} , \mathcal{X}_{a,b,n,s} , \tilde{\mathcal{Y}}_{n,s} , \tilde{\mathcal{X}}_{a,b,n,s} \}$$

by

$$\mathrm{PGU}(3,q) \leq \mathrm{Aut}(\mathcal{GK}/C_s)$$

Merci pour votre attention!