Weierstrass semigroups at all the points of a maximal curve with the third largest genus

Peter Beelen, Maria Montanucci and Lara Vicino (lavi@dtu.dk)

Algebra Group, Technical University of Denmark (DTU)

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Department of Applied Mathematics and Computer Science

Outline

- Setting
- A curve with the third largest genus
- Weierstrass semigroups at points in $\mathcal{X}_3(\mathbb{F}_{q^2})$
- Weierstrass semigroups at points in $\mathcal{X}_3 \setminus \mathcal{X}_3(\mathbb{F}_{q^2})$
- Final remarks

Weierstrass semigroups

$\mathcal{X}$ projective, absolutely irreducible, non-singular algebraic curve defined over the finite field $\mathbb{F}_q$

Definition

Let $P \in \mathcal{X}$. An integer $n \geq 0$ is called a pole number of P if there is a function $f \in \mathbb{F}_q(\mathcal{X})$ with $(f)_\infty = nP$. Otherwise n is called a gap number of P.

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The set $H(P)$ of pole numbers of a point P is actually a semigroup, called the Weierstrass semigroup at P.

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By the Weierstrass Gap Theorem, if $g(\mathcal{X}) > 0$, then for each rational $P \in \mathcal{X}$:

- there are exactly $g(\mathcal{X})$ gaps,
- 1 is always a gap,
- the largest gap is $\leq 2g(\mathcal{X}) - 1$.

We denote the set of gaps at P as $G(P)$.

Computing Weierstrass semigroups: strategies

Remark

All $P \in \mathcal{X}$, except a finite number, have the same Weierstrass semigroup. The points with a different semigroup are called Weierstrass points.

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1. Claim: $H := \langle h_1, \ldots, h_k \rangle = H(P)$.

How to prove the claim?

- Show that $H \subseteq H(P)$.
  For all h_i, construct a rational function F_i on $\mathcal{X}$ with
  $$(F_i)_\infty = h_i P.$$
- Show that the genus $g(H) := |\mathbb{N} \setminus H|$ is
  $$g(H) = g(\mathcal{X}).$$

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❷ Claim: $G := \{g_1, \dots, g_j\} = G(P)$,

How to prove the claim?

- Show that $G \subseteq G(P)$.
  For all g_i, construct a regular differential w_i on $\mathcal{X}$ with
  $$v_P(w_i) = g_i - 1.$$
- Show that $|G| = g(\mathcal{X})$.

Maximal curves

Hasse-Weil bound: $|\mathcal{X}(\mathbb{F}_q)| \leq q + 1 + 2g(\mathcal{X})\sqrt{q}$

$\mathcal{X}$ defined over $\mathbb{F}_q$ is $\mathbb{F}_q$-maximal if it attains the Hasse-Weil bound

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Genera of $\mathbb{F}_{q^2}$-maximal curves:

- largest: [Iha82], [RS94]

$$g(\mathcal{X}) = \frac{q(q-1)}{2} \quad \Longleftrightarrow \quad \mathcal{X} \text{ is } \mathbb{F}_{q^2}\text{-isomorphic to the Hermitian curve}$$

- second largest: [FT96], [FGT97], [AT99]

$$g(\mathcal{X}) = \left\lfloor \frac{(q-1)^2}{4} \right\rfloor \Longleftrightarrow \begin{cases} \mathcal{X}\ \mathbb{F}_{q^2}\text{-isom. to } y^q + y = x^{(q+1)/2}, & \text{if } q \text{ odd} \\ \mathcal{X}\ \mathbb{F}_{q^2}\text{-isom. to } y^{q/2} + \ldots + y^2 + y = x^{q+1}, & \text{if } q \text{ even } + \text{ extra condition} \end{cases}$$

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- third largest: [KT02]

$$g(\mathcal{X}) = \left\lfloor \frac{q^2 - q + 4}{6} \right\rfloor \quad \Longleftrightarrow \quad ?$$

The curve $y^{q+1}+x^{2m}+x^m=0$, $q\equiv 2 \pmod 3$

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$$y^{q+1} + x^{2m} + x^m = 0, \quad \text{with genus} \quad g(\mathcal{X}_3) = \frac{q^2 - q + 4}{6}.$$

If $\mathbb{F}_{q^2}(\mathcal{H})$ function field of $\mathcal{H} : u^{q+1} + v^{q+1} + 1 = 0$ and $\mathbb{F}_{q^2}(\mathcal{X}_3)$ function field of $\mathcal{X}_3$, then

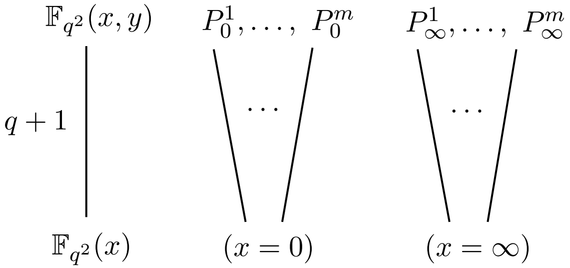
$$\varphi^* : \mathbb{F}_{q^2}(\mathcal{X}_3) \longrightarrow \mathbb{F}_{q^2}(\mathcal{H})$$

defines an unramified Galois extension $\mathbb{F}_{q^2}(u, v)/\mathbb{F}_{q^2}(x, y)$ of degree 3, with

$$x := u^3 \quad \text{and} \quad y := uv.$$

Places of $\mathbb{F}_{q^2}(x,y)$ lying over the zero or the pole of x

Recall: $\mathcal{X}_3$ is the non-singular $\mathbb{F}_{q^2}$-model of $y^{q+1} + x^{2m} + x^m = 0$.



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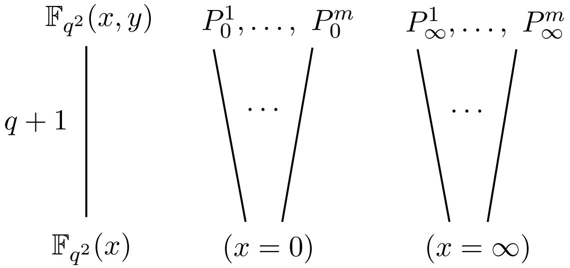

Let $O_0 := \{P_0^1, \dots, P_0^m\}$, $O_\infty := \{P_\infty^1, \dots, P_\infty^m\}$, $O := \bigcup_{a^m+1=0}\{P_{(a,0)}\}$ and

$$\mathcal{O} := O_0 \cup O_\infty \cup O.$$

$|\mathcal{O}| = q+1$ and $\mathcal{O}$ is an orbit of $\mathrm{Aut}(\mathcal{X}_3)$, in its natural action on the points of $\mathcal{X}_3$.

The families of rational functions $\mathcal{P}_i(s)$ and $\mathcal{Q}_j(s)$

We introduce two families of functions that will be crucial for the determination of the Weierstrass semigroups at all the points $P_{(a,b)} \in \mathcal{X}_3 \setminus \mathcal{O}$.

Definition

Let $i, j \in \mathbb{Z}$. Further let $\mathbb{F}$ be a field of characteristic different from three and assume that it contains a primitive cube root of unity, which we will denote by ζ_3. Then we define the following rational functions in $\mathbb{F}(s)$:

$$\mathcal{P}_i(s) := \frac{(s+\zeta_3)^{3i} - (s+\zeta_3^2)^{3i}}{3(\zeta_3 - \zeta_3^2)s(s-1)}$$

and

$$\mathcal{Q}_j(s) := \frac{\left(\frac{1-\zeta_3}{3}\right)(s+\zeta_3)^j + \left(\frac{1-\zeta_3^2}{3}\right)(s+\zeta_3^2)^j}{s-1}.$$

Properties of $\mathcal{P}_i(s)$ and $\mathcal{Q}_j(s)$

Example

Assume $\mathbb{F} = \mathbb{Q}$. Then:

- $\mathcal{P}_0(s) = 0$, $\mathcal{P}_1(s) = 1$, $\mathcal{P}_2(s) = 2s^3 - 3s^2 - 3s + 2$,
- $\mathcal{Q}_2(s) = s + 1$, $\mathcal{Q}_5(s) = s^4 + s^3 - 9s^2 + s + 1$.

For $i, j \geq 0$, $\mathcal{P}_i(s)$ and $Q_{3j+2}(s)$ are polynomials in s.

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Lemma (Beelen - Montanucci - V.)

Let $i, j, \ell \in \mathbb{Z}$. Then

$$\mathcal{P}_i(s)\mathcal{P}_{\ell+j}(s) - \mathcal{P}_j(s)\mathcal{P}_{\ell+i}(s) = \mathcal{P}_{i-j}(s)\mathcal{P}_\ell(s)(s^2 - s + 1)^{3j}$$

and

$$\mathcal{P}_i(s)\mathcal{Q}_{3\ell+3j-1}(s) - \mathcal{P}_j(s)\mathcal{Q}_{3\ell+3i-1}(s) = \mathcal{P}_{i-j}(s)\mathcal{Q}_{3\ell-1}(s)(s^2 - s + 1)^{3j}.$$

A special family of functions on $\mathcal{X}_3$

Let $P_{(a,b)} \in \mathcal{X}_3 \setminus \mathcal{O}$, $\alpha := \frac{a^m}{1+a^m}$ and $D_\infty := \sum_{j=1}^{m} P_\infty^j$.

Theorem (Beelen - Montanucci - V.)

Let $P_{(a,b)}$ be a point of $\mathcal{X}_3 \setminus \mathcal{O}$ such that $\alpha^2 - \alpha + 1 \neq 0$.
Let i be the smallest positive integer such that $\mathcal{P}_{i+1}(\alpha) = 0$ and $\mathcal{P}_j(\alpha) \neq 0$ for all $j \in \mathbb{Z}$ with $1 \leq j \leq i$.
Then:

- $\exists\ f_i \in L((3i+3)D_\infty)$ *such that* $v_{P_{(a,b)}}(f_i) = 3i+3$,
- $\forall\ j \in \mathbb{Z}$ *with* $0 \leq j \leq i-1$, $\exists\ f_j \in L((3j+3)D_\infty)$ *with* $v_{P_{(a,b)}}(f_j) = 3j+2$.

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Idea:

- Construct by hand f_0, f_1 and f_2.
- Construct inductively

$$f_j := -\frac{\mathcal{P}_j f_{j-2} f_1 - \mathcal{P}_2 \mathcal{P}_{j-1} f_{j-1} f_0}{(\alpha^2 - \alpha + 1)^2 \mathcal{P}_{j-2}}.$$

The Fundamental Equation

As $\mathcal{X}_3$ is an $\mathbb{F}_{q^2}$-maximal curve, the Fundamental Equation ([HKT08][1]) implies that, for all $i = 1, \dots, m$ and $P \in \mathcal{X}_3$, there exists a function $f_{P,i} \in \mathbb{F}_{q^2}(x, y)$ such that

$$(f_{P,i}) = qP + \Phi(P) - (q + 1)P_\infty^i,$$

with Φ the $\mathbb{F}_{q^2}$-Frobenius map.

[1] J.W.P. Hirschfeld – G. Korchmáros – F. Torres. *Algebraic Curves over a Finite Field*. Princeton Series in Applied Mathematics, Princeton, 2008.

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Consequence: for all $P \in \mathcal{X}_3(\mathbb{F}_{q^2})$, both $q+1$ and q are contained in $H(P)$.

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Weierstrass semigroup at $P \in \mathcal{O}$

Theorem (Beelen - Montanucci - V.)

Let $P \in \mathcal{X}_3(\mathbb{F}_{q^2})$ be a point such that $P \in \mathcal{O}$. Then

$$H(P) = \langle q-2, q, q+1 \rangle.$$

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Idea: we prove $H(P_{(a,0)}) = \langle q-2, q, q+1 \rangle$ for $P_{(a,0)} \in \mathcal{O}$, with $a^m + 1 = 0$.

- note that

$$\left(\frac{1}{x-a}\right) = 3\sum_{j=1}^{m} P_\infty^j - (q+1)P_{(a,0)}, \qquad \left(\frac{y^3}{x(x-a)}\right) = 3\sum_{\tilde{a}^m+1=0,\ \tilde{a}\neq a} P_{(\tilde{a},0)} - (q-2)P_{(a,0)},$$

$$\left(\frac{y}{x-a}\right) = \sum_{j=1}^{m} P_0^j + \sum_{j=1}^{m} P_\infty^j + \sum_{\tilde{a}^m+1=0,\ \tilde{a}\neq a} P_{(\tilde{a},0)} - qP_{(a,0)}$$

- $\langle q-2, q, q+1 \rangle$ is telescopic $\rightsquigarrow$ explicit formula for the genus

Weierstrass semigroups at the points $P_{(a,b)} \in \mathcal{X}_3(\mathbb{F}_{q^2}) \setminus \mathcal{O}$

Theorem (Beelen - Montanucci - V.)

Let $P_{(a,b)} \in \mathcal{X}_3(\mathbb{F}_{q^2}) \setminus \mathcal{O}$. Then

1. *if, for all j with $0 \leq j \leq m-2$, $\mathcal{P}_{j+1}(\alpha) \neq 0$, then*

$$H(P_{(a,b)}) = \langle q, q+1, (q-1)+j(q-2) \mid j = 0, \dots, m-2 \rangle;$$

2. *if $\exists\, i$, $1 \leq i \leq m-2$, such that $\mathcal{P}_j(\alpha) \neq 0$ for all j with $1 \leq j \leq i$ and $\mathcal{P}_{i+1}(\alpha) = 0$, then*

$$H(P_{(a,b)}) = \langle q, q+1, (q-1)+j(q-2), (q-1)+i(q-2)-1 \mid j = 0, \dots, i-1 \rangle.$$

Idea: Let $P_{(\bar{a},0)}$ point with $\bar{a}^m + 1 = 0$ and define $F_j := \frac{f_j \cdot f^{j+1}_{P_{(\bar{a},0)},1}}{f^{j+1}_{P_{(a,b)},1} \cdot (x-\bar{a})^{j+1}}$

$$\Longrightarrow \quad (F_j) = E_j - ((q-1)+j(q-2))P_{(a,b)}$$

Regular differentials and gaps

Link ([Sal06][2]): $\mathcal{X}$ curve of genus g defined over a field $\mathbb{K}$, $P \in \mathcal{X}$ and w regular differential on $\mathcal{X}$

$$\Longrightarrow v_P(w) + 1 \text{ is a gap at } P.$$

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Our strategy: for $P \in \mathcal{X}_3 \setminus \mathcal{X}_3(\mathbb{F}_{q^2})$, claim that $G(P)$ is $G := \{g_1, \dots, g_j\}$ and show

$$G \subseteq G(P) \quad \wedge \quad |G| = g(\mathcal{X}_3).$$

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$$G \subseteq G(P) \quad \wedge \quad |G| = g(\mathcal{X}_3).$$

- Show $\left(\frac{y}{x(x^m+1)} \cdot dx\right) = (q-2)D_\infty$;
- then, if $h \in L((q-2)D_\infty) \quad \implies \quad v_P(h) + 1 \in G(P)$, as

$$w := h \cdot \frac{y}{x(x^m+1)} \cdot dx \quad \text{is regular and} \quad v_P(w) = v_P(h).$$

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Theorem (Beelen - Montanucci - V.)

Let $P_{(a,b)} \in \mathcal{X}_3 \setminus \mathcal{X}_3(\mathbb{F}_{q^2})$.

1. *If $\mathcal{P}_j(\alpha) \neq 0$ for all $j = 2, \dots, m-1$, then $H(P_{(a,b)})$ is*

$$H_{gen} = \{0, (j+1)(q-3)+2+k, (m-1)q+2, \dots \mid j = 0, \dots, m-2,\ k = 0, \dots, 3j+1\}.$$

2. *If $\exists\ j$ with $1 \leq j \leq m-2$ and $\mathcal{P}_{j+1}(\alpha) = 0$, let i be the smallest such integer. Then*

$$H(P_{(a,b)}) = \left(H_{gen} \setminus \left\{(m-2-i-\ell(i+1))q + (\ell+1)(3i+3)+1 \mid \ell = 0, \dots, \left\lfloor \frac{m-2-i}{i+1} \right\rfloor \right\}\right) \cup \left\{(m-2-i-\ell(i+1))q + (\ell+1)(3i+3) \mid \ell = 0, \dots, \left\lfloor \frac{m-2-i}{i+1} \right\rfloor \right\}.$$

Remarks and future work

Some remarks:

- For $q = 2, 5, 8$, there are no non-$\mathbb{F}_{q^2}$-rational Weierstrass points (note: for $q = 8$, $\mathcal{X}_3 \cong GK$).
- For $q > 8$, there are non-$\mathbb{F}_{q^2}$-rational Weierstrass points:

$$m \geq 4 \quad \Longrightarrow \quad \exists \quad j \quad \text{such that} \quad 1 \leq j \leq m-2 \quad \wedge \quad p \nmid j+1 \quad \wedge \quad 3(j+1) \nmid q+1$$

$$\Longrightarrow \text{roots of } \mathcal{P}_{j+1}(s) \text{ lie in } \mathbb{F}_{q^{2e}} \setminus \mathbb{F}_{q^2},\ e \geq 2.$$

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- Counting the number of roots of the $\mathcal{P}_i(s)$, we can determine the number of different semigroups that occur for $\mathbb{F}_{q^2}$-rational points, namely:

$$\#\ \text{semigroups} \ = \#\ \text{divisors of } \frac{q+1}{3}.$$

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Future work:

- One-point AG codes on $\mathcal{X}_3$?
- Weierstrass semigroups on other known curves with the third genus?

Thank you for your attention!

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