The Cremona group (over the field with two elements)

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and Algebraic geometry Codes
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The plane Cremona group

The plane Cremona group over a field $\mathbf{k}$ is the group

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) = \{f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \text{ birational map defined over } \mathbf{k}\}.$$

That is, it consists of maps $f\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of the form

$$f\colon [x:y:z] \mapsto [f_0(x,y,z) : f_1(x,y,z) : f_2(x,y,z)],$$

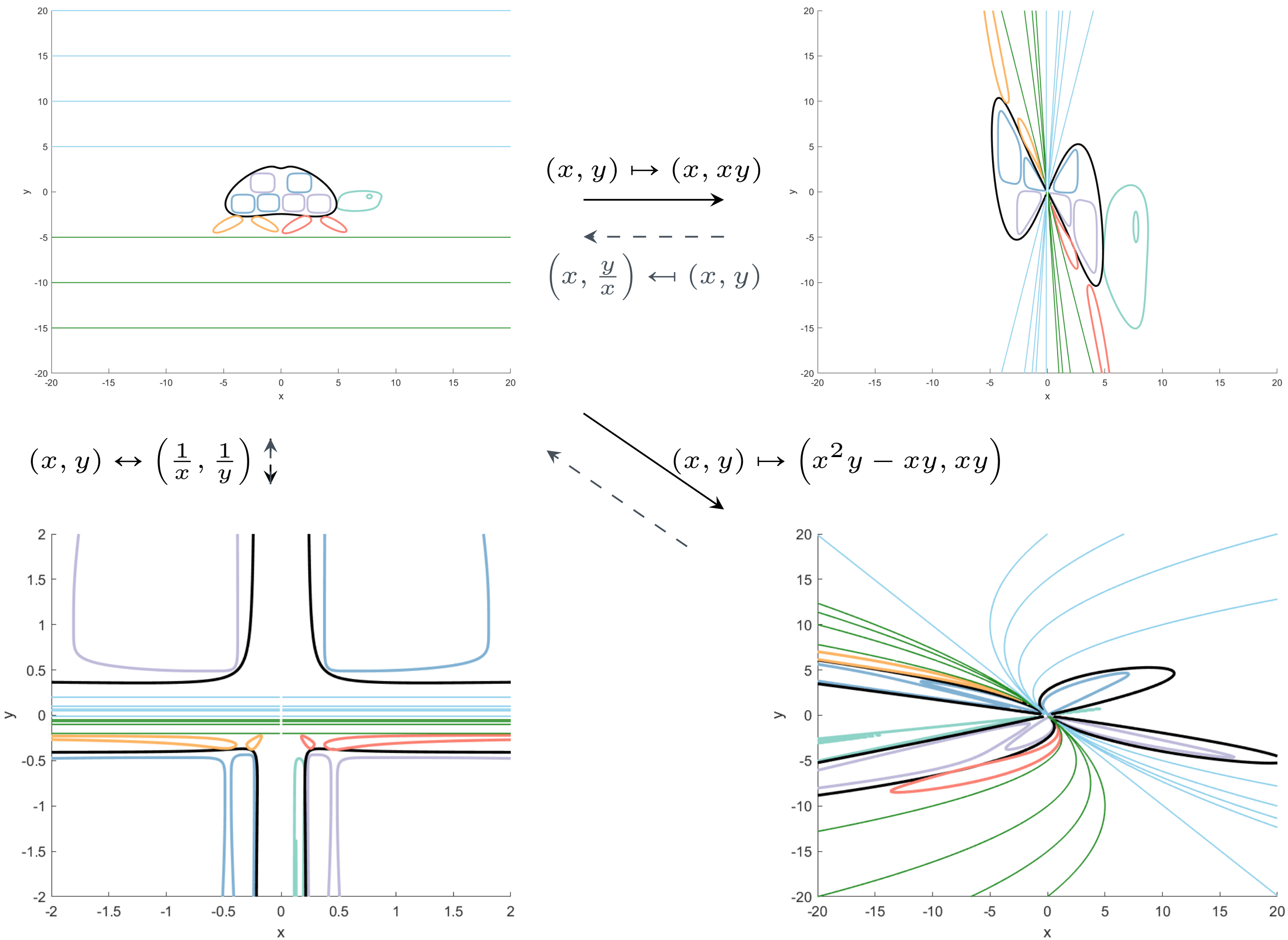
where $f_0, f_1, f_2 \in \mathbf{k}[x,y,z]_d$ are homogeneous polynomials of the same degree d, such that there exists an inverse map of the same shape.

Two facts:

- $\mathrm{PGL}_3(\mathbf{k}) = \mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^2) \subsetneq \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$,
- $\mathrm{Cr}_2(\mathbf{k}) = \mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2) \simeq \mathrm{Bir}_{\mathbf{k}}(\mathbb{A}^2) \simeq \mathrm{Aut}_{\mathbf{k}}(\mathbf{k}(x,y))$.

Transformations of the plane: Birational maps

Birational map = rational map with an inverse rational map



Generators

Question

What are generators of the plane Cremona group $\text{Bir}_{\mathbf{k}}(\mathbb{P}^2)$?

Theorem (Noether-Castelnuovo ~ 1900)

Let $\mathbf{k}$ be an algebraically closed field. Then,

$$\text{Bir}_{\mathbf{k}}(\mathbb{P}^2) = \langle \text{PGL}_3(\mathbf{k}), \sigma \rangle,$$

where σ is the standard quadratic involution given by

$$[x : y : z] \mapsto [yz : xz : xy] = [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}].$$

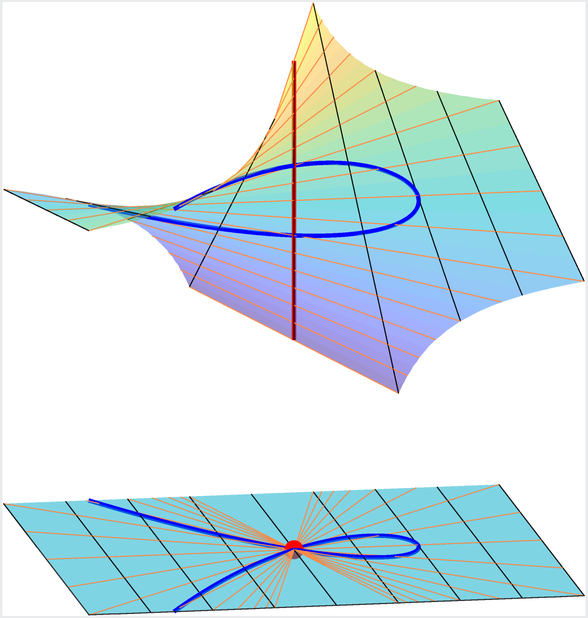
... what if $\mathbf{k}$ is not algebraically closed?

Classical: Factorization into “nice” birational maps

Fact

Let $f\colon X \dashrightarrow Y$ be a birational map between two smooth, projective surfaces. Then, f can be decomposed as

$$
\begin{array}{ccccc}
 & & Z & & \\
 & \swarrow_{\pi_n} & & \searrow^{\tau_m} & \\
 & \vdots & & \vdots & \\
 & \swarrow_{\pi_2} & & \searrow^{\tau_2} & \\
 \swarrow_{\pi_1} & & & & \searrow^{\tau_1} \\
X & & \overset{f}{\dashrightarrow} & & Y,
\end{array}
$$



where all π_i and τ_i are blow-ups of one closed point.

Sarkisov program: Factorization into “nice” birational maps between “nice” surfaces

Theorem (Iskovskikh, Corti, $\sim$ 1995)

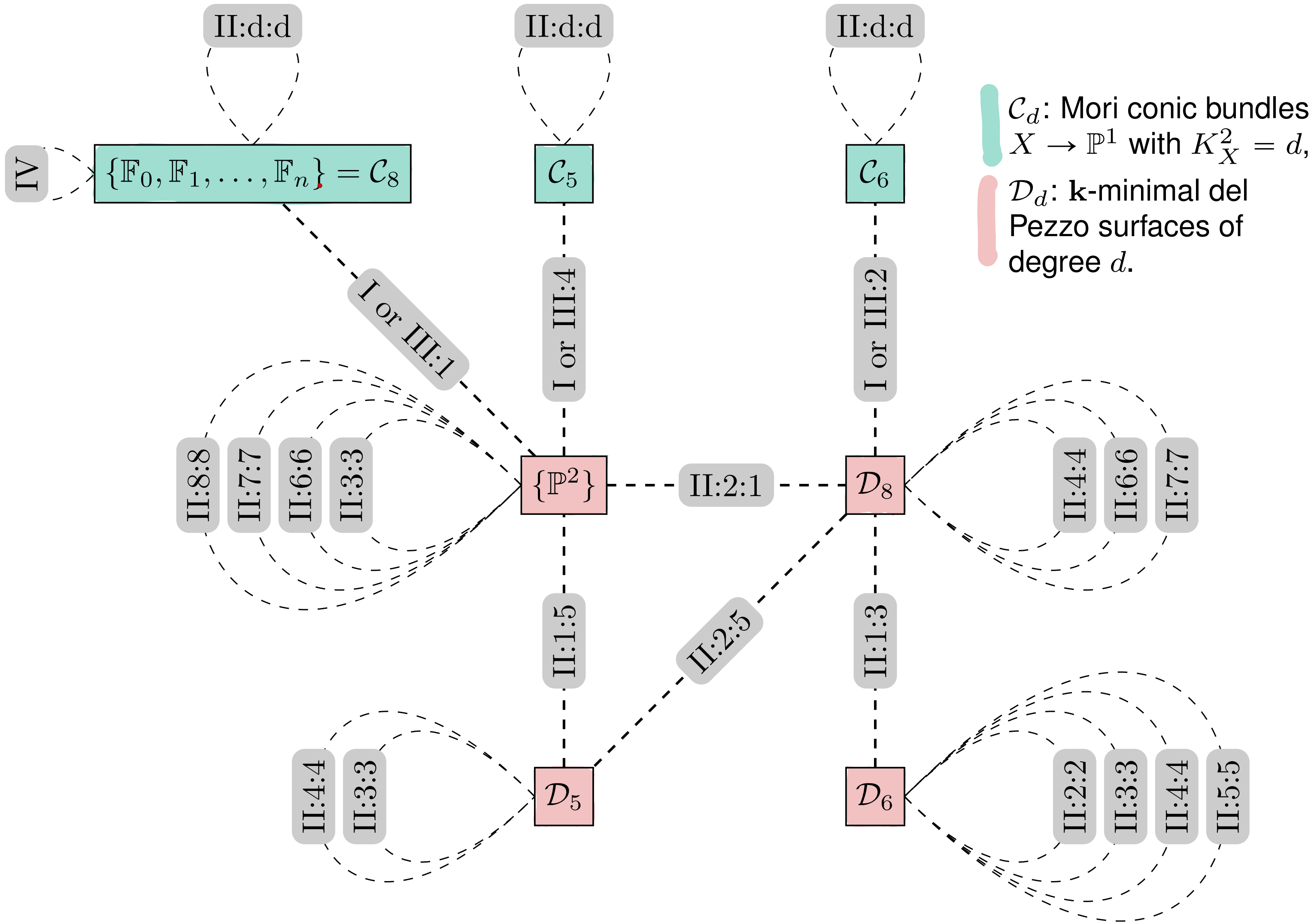
Let $\mathbf{k}$ be a perfect field. Then any birational map between two Mori fibre spaces can be decomposed into a sequence of Sarkisov links (and isomorphisms of Mori fibre spaces).

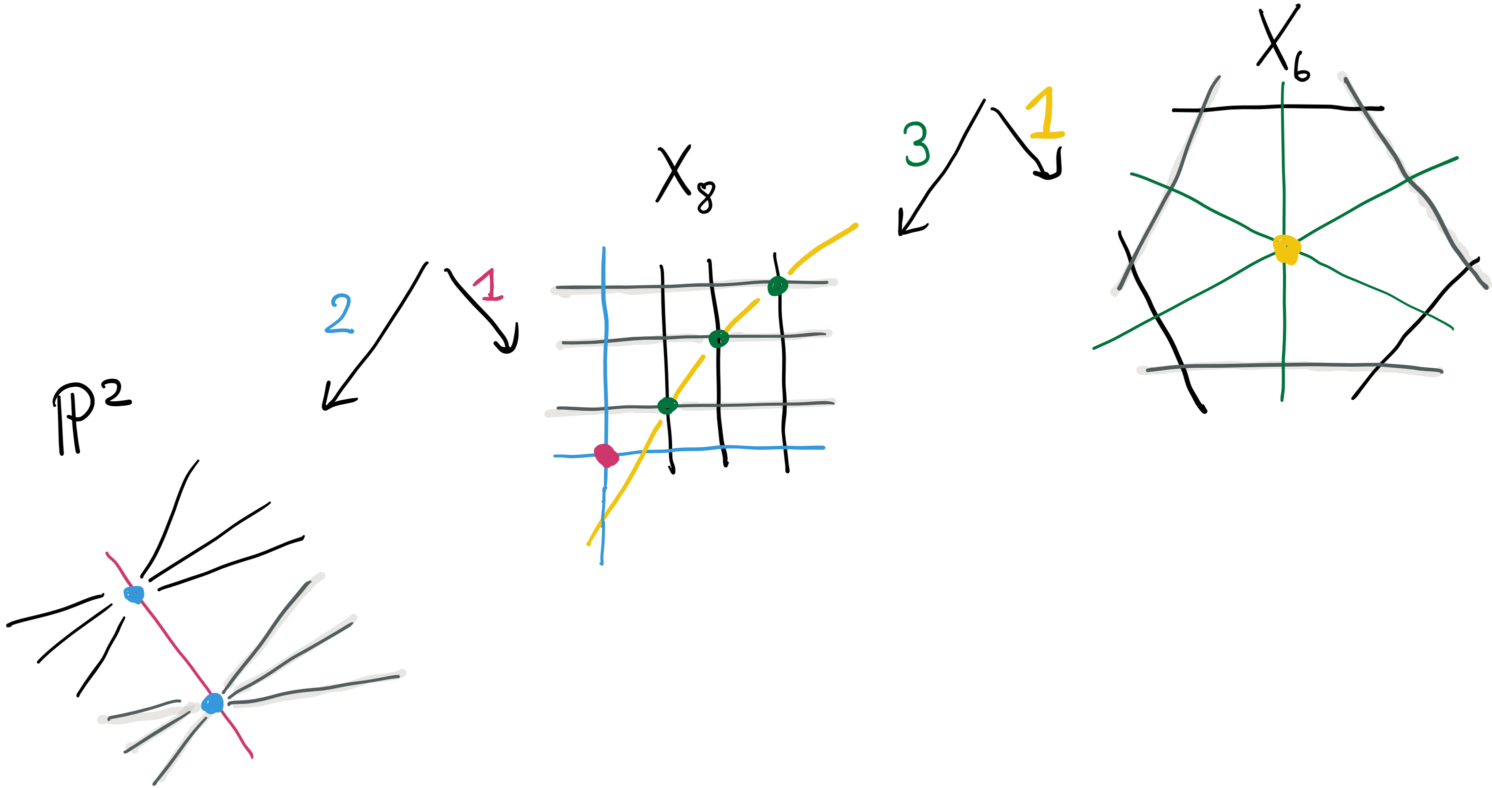
- Mori fibre space = rank 1 fibration;
- Sarkisov link = rank 2 fibration.

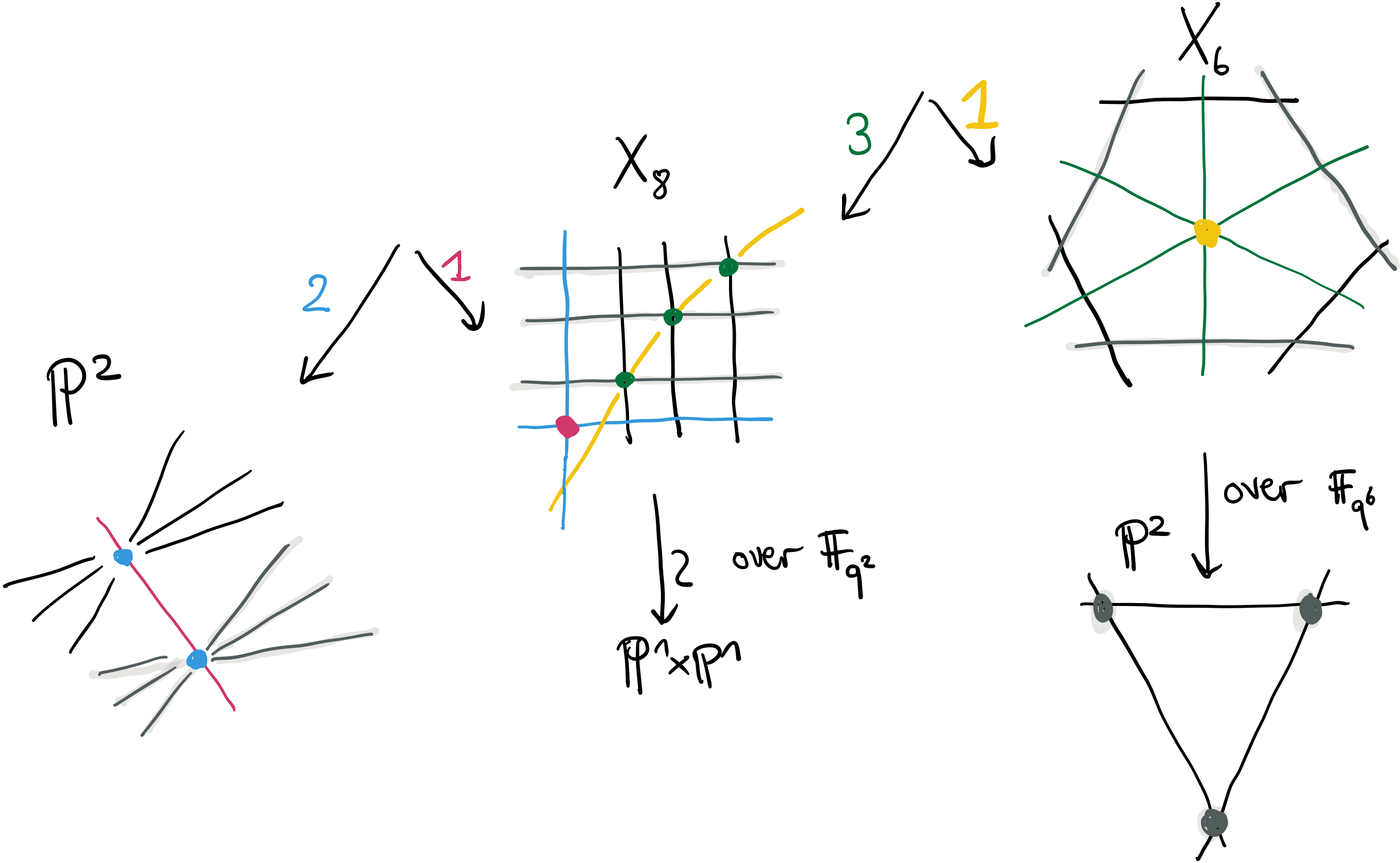
Definition

Let X be a smooth projective surface. A surjective morphism $\pi \colon X \to B$ with connected fibres is a rank r fibration, if B smooth with $\dim B \leqslant 1$ and

- $\rho(X/B) = r$, and
- $-K_X$ is π-ample.

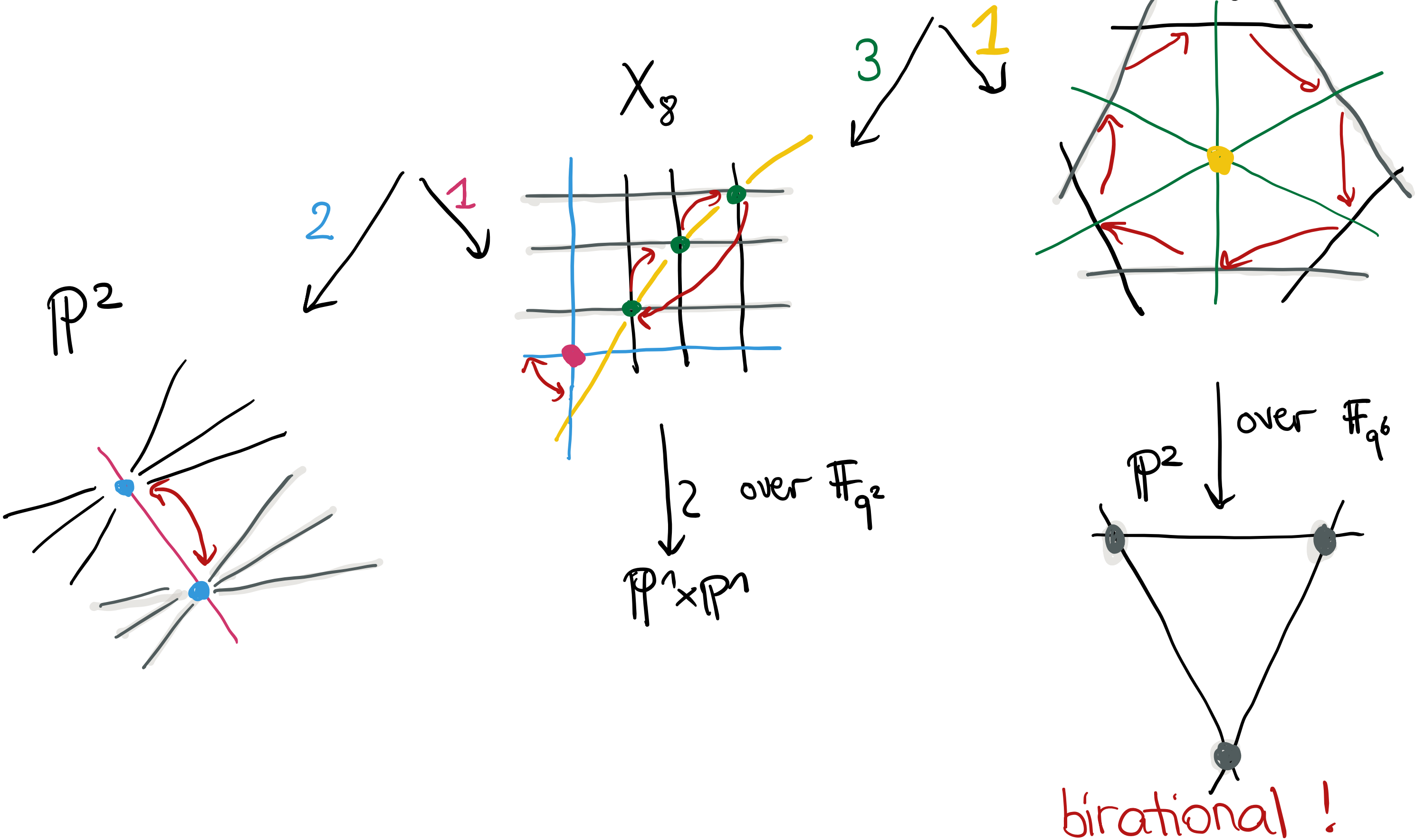

II:d:d
II:d:d
II:d:d
IV
$\{\mathbb{F}_0, \mathbb{F}_1, \ldots, \mathbb{F}_n\} = \mathcal{C}_8$
$\mathcal{C}_5$
$\mathcal{C}_6$
$\mathcal{C}_d$: Mori conic bundles $X \to \mathbb{P}^1$ with $K_X^2 = d$,
$\mathcal{D}_d$: $\mathbf{k}$-minimal del Pezzo surfaces of degree d.
I or III:1
I or III:4
I or III:2
II:8:8
II:7:7
II:6:6
II:3:3
$\{\mathbb{P}^2\}$
II:2:1
$\mathcal{D}_8$
II:4:4
II:6:6
II:7:7
II:1:5
II:2:5
II:1:3
II:4:4
II:3:3
$\mathcal{D}_5$
$\mathcal{D}_6$
II:2:2
II:3:3
II:4:4
II:5:5


$\mathbb{P}^2$
2
1
X_8
3
1
X_6

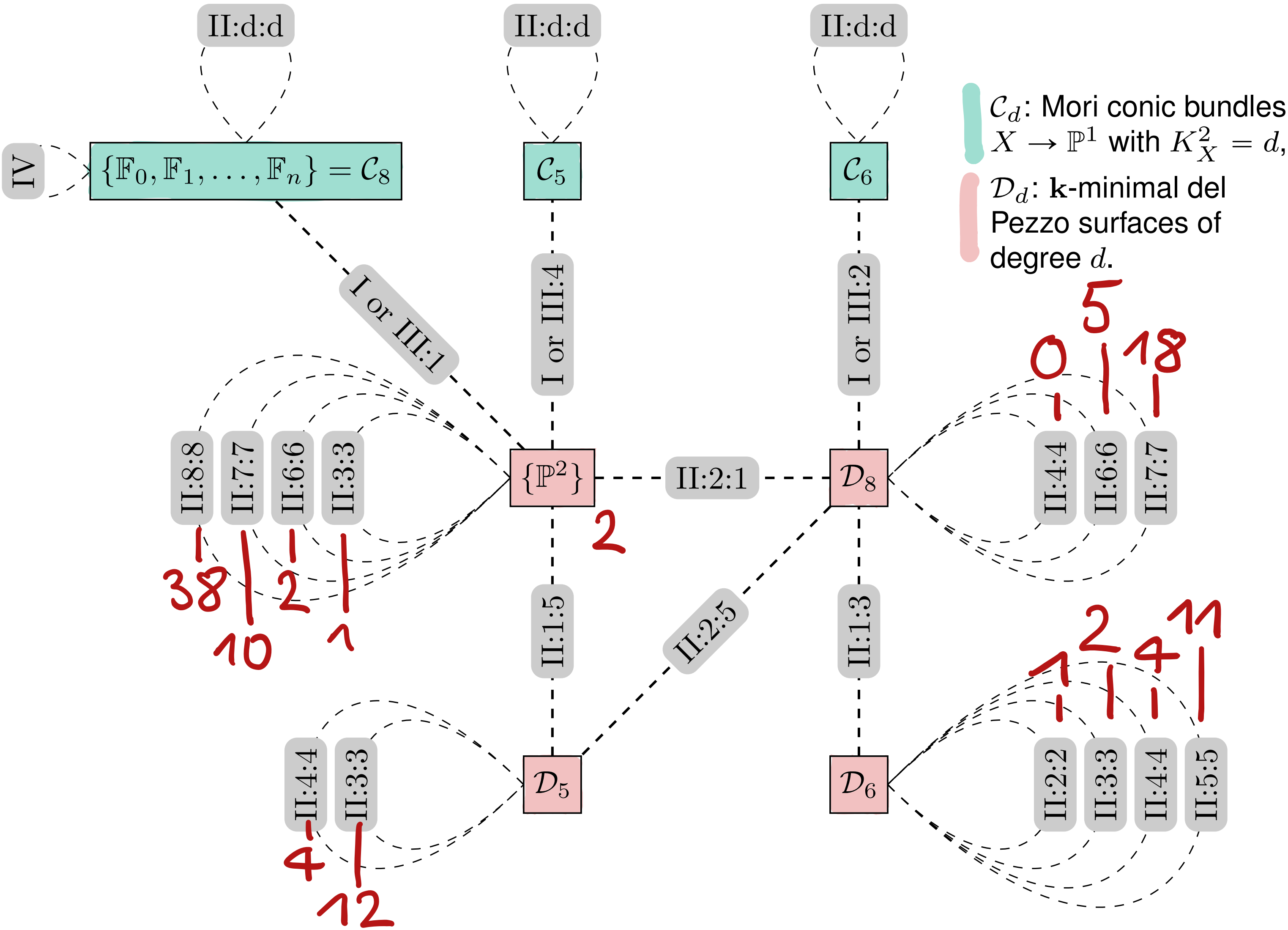


X_8
X_6
2
1
3
1
$\mathbb{P}^2$
2 over $\mathbb{F}_{q^2}$
$\mathbb{P}^1 \times \mathbb{P}^1$
over $\mathbb{F}_{q^6}$
$\mathbb{P}^2$

Frobenius



X_6
3
1
X_8
2
1
$\mathbb{P}^2$
2 over $\mathbb{F}_{q^2}$
$\mathbb{P}^1 \times \mathbb{P}^1$
over $\mathbb{F}_{q^6}$
$\mathbb{P}^2$
birational !



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$\mathcal{D}_5$
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II:3:3
II:4:4
II:5:5

Generators of the plane Cremona group over $\mathbb{F}_2$

Theorem (S. 2020)

The plane Cremona group over the field with two elements, $\mathbb{F}_2$, is generated by three infinite families of birational maps

- Λ_1, *which can be parametrized by* $\mathbb{F}_2(t)$,
- Λ_2 *and* Λ_4, *which can both be parametrized by a conic in* $\mathbb{A}^2(\mathbb{F}_2(t))$,

and a set of 111 *additional birational maps.*

We give an explicit description of the three infinite families and of the 111 birational maps.

Generators of the three infinite families

- Λ_1 is the set that consists of all birational maps in $\mathrm{Bir}_{\mathbb{F}_2}(\mathbb{P}^2)$ that are of the shape

$$[x:y:z] \dashrightarrow [xz^d : yp(y,z) : zp(y,z)]$$

  where $p \in \mathbb{F}_2[y,z]$ is a homogeneous polynomial of degree d.
- Λ_2, respectively Λ_4, is the set of involutions of the form

$$[x:y:z] \dashrightarrow [x : \lambda x + y : \mu x + z],$$

  where $(\lambda, \mu) \in (\mathbb{F}_2(t))^2$ satisfy
  - $(\lambda^2+\lambda)t + \mu^2 + \mu = 0$ and $t = \frac{x^2+xz+z^2}{y^2+xy}$, respectively
  - $(\lambda^2+\mu)t + \mu^2 + \lambda = 0$ and $t = \frac{x^2+xy+z^2}{y^2+xz}$.

Example: Explicit Sarkisov link over $\mathbb{F}_2$

Sarkisov links $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with base point of degree 6 are given by $[x:y:z] \mapsto [f_0 : f_1 : f_2]$, where

$$
\begin{aligned}
f_0 =& x^5 + x^4y + xy^4 + x^2y^2z + x^3z^2 + x^2yz^2 + xy^2z^2 \\
&+ x^2z^3 + xyz^3 + yz^4, \\
f_1 =& x^4y + x^3y^2 + xy^4 + y^5 + xy^3z + x^2yz^2 + xy^2z^2 \\
&+ y^3z^2 + xyz^3 + y^2z^3 + xz^4, \\
f_2 =& x^3y^2 + x^4z + x^3yz + x^2y^2z + y^4z + x^3z^2 + x^2yz^2 \\
&+ xy^2z^2 + xz^4 + yz^4 + z^5,
\end{aligned}
$$

respectively

$$
\begin{aligned}
f_0 =& x^4y + x^3y^2 + xy^4 + x^4z + x^3yz + x^2y^2z + xy^2z^2 + xyz^3 + xz^4 + z^5, \\
f_1 =& x^5 + x^4y + x^2y^3 + y^5 + xy^3z + x^2yz^2 + y^3z^2 + xz^4 + yz^4 + z^5, \\
f_2 =& x^5 + x^3yz + x^2y^2z + y^4z + x^2yz^2 + xy^2z^2 + y^2z^3 + yz^4 + z^5.
\end{aligned}
$$

Generation by involution

Corollary

$\mathrm{Bir}_{\mathbb{F}_2}(\mathbb{P}^2)$ *is generated by involutions.*

Theorem (Lamy-S., 2021)

For every perfect field $\mathbf{k}$, $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^2)$ *is generated by involutions.*

Thank you!