Algebraic curves with many rational points over finite fields

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DTU Compute

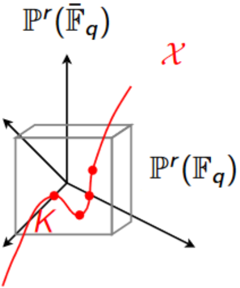
Department of Applied Mathematics and Computer Science

Outline

- Maximal curves over finite fields
  - Notation and terminology
  - Past, present and future research
- Classification and construction
  - The case of $\mathbb{F}_{p^2}$-maximal curves, p prime
  - The second generalized GK curve and its consequences
- Weierstrass semigroups and maximal curves
  - Weierstrass semigroups and the GK curve
  - Decoding AG codes from maximal curves
- Curves with many rational points in coding theory
  - LRC and MRD codes
  - What's next? The project **CREATE**

Notation and terminology

- $\mathcal{X} \subseteq \mathbb{P}^r(\overline{\mathbb{F}}_q)$ projective, geometrically irreducible, non-singular algebraic curve defined over $\mathbb{F}_q$
- g genus of $\mathcal{X}$
  If $r = 2$ then $g = \frac{(d-1)(d-2)}{2}$, where $d = \deg(\mathcal{X})$
- $Aut(\mathcal{X})$ automorphism group of $\mathcal{X}$ over $\overline{\mathbb{F}}_q$ (**Massimo's talk!**)
- $\mathcal{X}(\mathbb{F}_q) = \mathcal{X} \cap \mathbb{P}^r(\mathbb{F}_q)$



Maximal curves

$\mathcal{X}$ defined over $\mathbb{F}_q$

Hasse-Weil bound

$|\mathcal{X}(\mathbb{F}_q)| \leq q + 1 + 2g\sqrt{q}$.

Definition

$\mathcal{X}$ is $\mathbb{F}_q$-maximal if $|\mathcal{X}(\mathbb{F}_q)| = q + 1 + 2g\sqrt{q}$.

Necessary: q square or $g = 0$ (we change notation: $\mathbb{F}_q \to \mathbb{F}_{q^2}$)

Example: $\mathbb{F}_{q^2}$-maximal curve

Hermitian curve

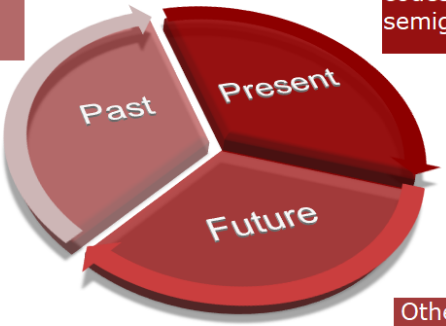
$$\mathcal{H}_q : Y^{q+1} = X^q + X, \quad q = p^h$$

$g = q(q-1)/2, \qquad |\mathcal{H}_q(\mathbb{F}_{q^2})| = q^3 + 1, \qquad Aut(\mathcal{H}_q) \cong PGU(3, q) \to \geq 16g^4$

My research on algebraic curves with many rational points

Classification and construction of maximal curves

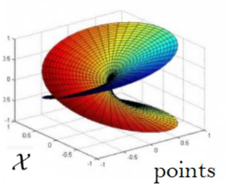
Applications to AG codes: Weierstrass semigroups

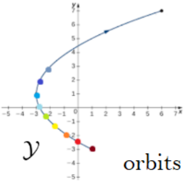


Other applications of curves with many points in coding theory

Coverings and Galois-coverings

- $\mathcal{X} \subseteq \mathbb{P}^r(\mathbb{F}_{q^2})$ and $\mathcal{Y} \subseteq \mathbb{P}^s(\mathbb{F}_{q^2})$
- Non-constant $\phi : \mathcal{X} \to \mathcal{Y} \implies \mathcal{Y}$ is covered by $\mathcal{X}$ (*subcover*)
- $\overline{\mathbb{F}}_{q^2}(\mathcal{X}) : \phi^*(\overline{\mathbb{F}}_{q^2}(\mathcal{Y}))$ is a finite field extension
- $\overline{\mathbb{F}}_{q^2}(\mathcal{X}) : \phi^*(\overline{\mathbb{F}}_{q^2}(\mathcal{Y}))$ Galois $\implies \mathcal{Y}$ is Galois-covered by $\mathcal{X}$ (*Galois-subcover*)





(Kleiman-Serre, 1987)

If $\mathcal{X}$ is $\mathbb{F}_{q^2}$-maximal and $\mathcal{Y}$ is $\mathbb{F}_{q^2}$-covered by $\mathcal{X}$ then $\mathcal{Y}$ is $\mathbb{F}_{q^2}$-maximal

Conjecture

Every $\mathbb{F}_{q^2}$-maximal curve is (Galois-)covered by the Hermitian curve $\mathcal{H}_q$

Natural Embedding Theorem

(Garcia-Stichtenoth, 2006)

The GS curve $X^9 - X = Y^7$ is $\mathbb{F}_{3^6}$-maximal and not Galois-covered by $\mathcal{H}_{3^3}$.

- Hermitian Variety in $\mathbb{P}^r(\overline{\mathbb{F}}_{q^2})$:
  $\mathcal{H}_{r,q} : X_2^{q+1} + X_3^{q+1} + \ldots + X_r^{q+1} = X_1^q X_0 + X_1 X_0^q$

Natural Embedding Theorem (Korchmáros-Torres, 2001)

Un to isomorphisms, $\mathbb{F}_{q^2}$-maximal curves are

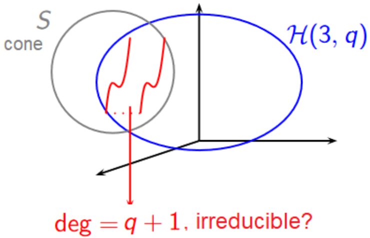
- contained in some $\mathcal{H}_{r,q}$ for some $r \geq 2$
- irreducible of degree $q+1$
- not contained in any hyperplane of $\mathbb{P}^r(\overline{\mathbb{F}}_q)$

Definition

$r \geq 2$ is the (geometrical) Frobenius dimension of $\mathcal{X}$.

- If $r = 2$ then $\mathcal{X}$ is the Hermitian curve (up to isomorphism) $\rightarrow$ $r \geq 3$?

The conjecture is false: GK curve



(Giulietti-Korchmáros, 2009)

Let q be a prime power of a prime p. The GK-curve

$$\mathcal{C} : \begin{cases} Z^{\frac{q^3+1}{q+1}} = Y^{q^2} - Y, \\ X^q + X = Y^{q+1} \rightarrow \text{Hermitian curve!} \end{cases}$$

if $\mathbb{F}_{q^6}$-maximal. If $q > 2$, $\mathcal{C}$ is **not** $\mathbb{F}_{q^6}$-covered by $\mathcal{H}_{q^3}$

Question: Why are both the GK and the GS curve $\mathbb{F}_{q^6}$-maximal?

The case of $\mathbb{F}_{p^2}$-maximal curves

Conjecture, 2000

Every $\mathbb{F}_{p^2}$-maximal curve is a subcover of the Hermitian curve $\mathcal{H}_p$

Note: Known $\mathbb{F}_{p^2}$-maximal curves are Galois-covered by $\mathcal{H}_p$ and have many automorphisms

Theorem (Bartoli-M.-Torres, 2020)

Let $\mathcal{X}$ be an $\mathbb{F}_{p^2}$-maximal curves of genus $g \geq 2$, $p \geq 7$. If $|Aut(\mathcal{X})| > 84(g-1)$ then $\mathcal{X}$ is Galois covered by $\mathcal{H}_p$

- Can Theorem be extended when $|Aut(\mathcal{X})| \leq 84(g-1)$? NO!
- Example: $\mathbb{F}_{71^2}$-maximal Fricke-MacBeath curve

Question (open)

Are there other examples? Is a similar result true for $\mathbb{F}_{p^{2n}}$-maximal curves?

A generalization of the GK-curve

(Garcia-Güneri-Stichtenoth, 2010)

Let q be a prime power, $n \geq 3$ odd. The $\mathbb{F}_{q^{2n}}$-maximal **GGS-curve** is

$$\mathcal{C}_n : \begin{cases} Z^{\frac{q^n+1}{q+1}} = Y^{q^2} - Y, \\ Y^{q+1} = X^q + X. \end{cases}$$

Theorem (Duursma-Mak, 2012)

For $q \geq 3$ and $n \geq 5$ odd, $\mathcal{C}_n$ is not Galois-covered by $\mathcal{H}_{q^n}$

Theorem (Giulietti-M.-Zini, 2016)

For $q = 2$ and $n \geq 5$ odd, $\mathcal{C}_n$ is not Galois-covered by $\mathcal{H}_{2^n}$

- Key steps: If $\mathcal{C}_n \cong \mathcal{H}_{2^n}/G$: $|G| = \frac{2^n+1}{3}$ and G acts semiregularly on $\mathcal{H}_{2^n}$
- (Hartley, 1925): Maximal subgroups of $\mathrm{PGU}(3, 2^n)$ and their action on $\mathcal{H}_{2^n}$

A new family of maximal curves

(Giulietti-Korchmáros, 2009)

$Aut(\mathcal{C})/C_{(q^3+1)/(q+1)} \cong PGU(3,q)$ **entire** $Aut(\mathcal{H}_q)$

(Guralnick-Malmskog-Pries, Güneri-Ozdemir-Stichtenoth, 2012-2013)

$n \geq 5$, $Aut(\mathcal{C}_n)/C_{(q^n+1)/(q+1)} \cong PGU(3,q)_{P_\infty}$ **maximal subgroup** of $Aut(\mathcal{H}_q)$

- (Mitchell 1911, Hartley 1925) Maximal subgroups of $Aut(\mathcal{H}_q)$
- (M.-Zini, 2018) $PGU(3,q)_\ell$ **second largest maximal subgroup** of $Aut(\mathcal{H}_q) \cong PGU(3,q)$

Idea

Find another family $\{\mathcal{X}_n\}_n$ with $\mathcal{X}_3 \cong \mathcal{C}$ such that $Aut(\mathcal{X}_n)/C_{(q^n+1)/(q+1)} \cong PGU(3,q)_\ell$

$$\mathcal{X}_n : \begin{cases} Z^{\frac{q^n+1}{q+1}} = Y\frac{X^{q^2}-X}{X^{q+1}-1}, \\ Y^{q+1} = X^{q+1} - 1. \end{cases}$$

Intuition: $Aut(\mathcal{C}_n)/C_{(q^n+1)/(q+1)} \cong PGU(3,q)_{P_\infty}$

$$\mathcal{C}_n : \begin{cases} Y^{q+1} = X^q + X, \\ Z^{\frac{q^n+1}{q+1}} = Y^{q^2} - Y. \end{cases} \qquad Let\ \alpha \in Aut(\mathcal{H}_q)_{P_\infty}$$

Observation 1

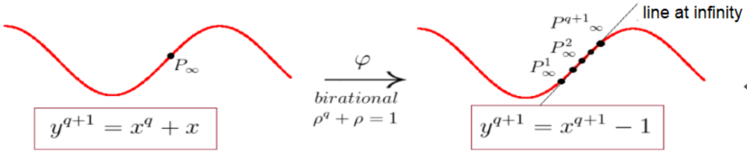
$\alpha(Y^{q^2} - Y) = a(Y^{q^2} - Y)$ for some constant a.

Observation 2

Any such α can be lifted in $Aut(\mathcal{C}_n)$ just by defining $\alpha(Z) = \lambda Z$, $\lambda^{(q^n+1)/(q+1)} = a$

$$\alpha\left(Z^{\frac{q^n+1}{q+1}}\right) = \lambda^{\frac{q^n+1}{q+1}} Z^{\frac{q^n+1}{q+1}} = a(Y^{q^2} - Y) = \alpha(Y^{q^2} - Y)$$

The construction of $\{\mathcal{X}_n\}_n$, n odd



$$\begin{cases} \varphi(x) = \frac{x+1}{(1-\rho)x-\rho}, \\ \varphi(y) = \frac{y}{(1-\rho)x-\rho}. \end{cases}$$

- $\tilde{\varphi}(X,Y,Z) = (\varphi(X), \varphi(Y), -Z/(1-\rho)X-\rho)$ defines a birational map

$$\mathcal{C} : \begin{cases} Y^{q+1} = X^q + X, \\ Z^{\frac{q^3+1}{q+1}} = Y^{q^2} - Y. \end{cases} \quad \mapsto \tilde{\varphi}(\mathcal{C}) := \mathcal{X}_3 : \begin{cases} Y^{q+1} = X^{q+1} - 1, \\ Z^{\frac{q^3+1}{q+1}} = Y \frac{X^{q^2} - X}{X^{q+1} - 1}. \end{cases}$$

- Generalization (as for the GGS): $\mathcal{X}_n : \begin{cases} Y^{q+1} = X^{q+1} - 1, \\ Z^{\frac{q^n+1}{q+1}} = Y \frac{X^{q^2} - X}{X^{q+1} - 1}. \end{cases}$

The construction of $\{\mathcal{X}_n\}_n$, n odd

Theorem (Beelen-M., Journal of the London Math. Soc., 2018)

1. $\mathcal{X}_3$ is isomorphic to the GK curve $\mathcal{C}$,
2. $\mathcal{X}_n$ is $\mathbb{F}_{q^{2n}}$-maximal for every q and $n \geq 3$ odd,
3. For every $n \geq 5$ and $q \geq 3$ $\mathcal{X}_n$ is not Galois-covered by $\mathcal{H}_{q^n}$,
4. For every $n \geq 5$, $Aut(\mathcal{X}_n)/C_{(q^n+1)/(q+1)} \cong PGU(3,q)_\ell$, ℓ line at infinity,
5. $g(\mathcal{X}_n) = g(\mathcal{C}_n)$ but $\mathcal{X}_n$ and $\mathcal{C}_n$ are isomorphic if and only if $n = 3$

- (Beelen-M., 2020) New other maximal curves as Galois-subcovers of $\mathcal{X}_n$
- (M.- Pallozzi Lavorante, 2020) Weierstrass semigroups and codes

Questions

- Can we use the Natural Embedding Theorem for $r = 4$?
- What about other properties? (New PhD: Jonathan Tilling Niemann)

From the generalized GK curves

(Tafazolian-Teherán-Herrera-Torres, 2016)

Galois-subcovers of the GGS curve, $m = \frac{q^n+1}{q+1}$, s divisor of m, $q = p^a$, $\bar{q} = p^b$ with $b \mid a$, $c^{q-1} = -1$

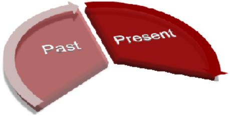
$$\mathcal{Y}_{n,s} : \begin{cases} Z^{m/s} = Y^{q^2} - Y \\ Y^{q+1} = X^q + X \end{cases} \qquad \mathcal{X}_{a,b,n,s} : \begin{cases} Z^{m/s} = Y^{q^2} - Y \\ cY^{q+1} = X + X^{\bar{q}} + \cdots + X^{q/\bar{q}} \end{cases}$$

- for some values of the parameters $\mathcal{X}_{a,b,n,s}$ and $\mathcal{Y}_{n,s}$ are not covered by $\mathcal{H}_{q^n}$
- **Reason**: values of $g(\mathcal{Y}_{n,s})$ and $g(\tilde{\mathcal{X}}_{a,b,n,s})$
- **Observation:** the groups inducing $\mathcal{X}_{a,b,n,s}$ and $\mathcal{Y}_{n,s}$ exist also in $Aut(\mathcal{X}_n)$ and $g(\mathcal{X}_n) = g(\mathcal{C}_n)$

Idea

Create the anologue curves $\tilde{\mathcal{X}}_{a,b,n,s}$ and $\tilde{\mathcal{Y}}_{n,s}$ as subcovers of $\mathcal{X}_n$

The curves $\tilde{\mathcal{Y}}_{n,s}$ and $\tilde{\mathcal{X}}_{a,b,n,s}$



$n \geq 3$ odd, $m = \frac{q^n+1}{q+1}$, $s \mid m$, $\quad q = p^a$, $\bar{q} = p^b$, $b \mid a$

$$\tilde{\mathcal{Y}}_{n,s} : \begin{cases} Z^{m/s} = Y \frac{X^{q^2} - X}{X^{q+1} - 1} \\ Y^{q+1} = X^{q+1} - 1 \end{cases} \qquad \tilde{\mathcal{X}}_{a,b,n,s} := \tilde{\mathcal{Y}}_{n,s}/E_{\bar{q}}$$

where $E_{\bar{q}} \leq \mathrm{Aut}(\tilde{\mathcal{Y}}_{n,s})$ is the lift to $\tilde{\mathcal{Y}}_{n,s}$ of the stabilizer of $P \in \tilde{\mathcal{H}}_q(\mathbb{F}_{q^2})$,
$\tilde{\mathcal{H}}_q : \; y^{q+1} = x^{q+1} - 1$

(M.-Tizziotti-Zini, 2023)

$g(\tilde{\mathcal{Y}}_{n,s}) = g(\mathcal{Y}_{n,s})$ but $\tilde{\mathcal{Y}}_{n,s} \not\cong \mathcal{Y}_{n,s}$

$g(\tilde{\mathcal{X}}_{a,b,n,s}) = g(\mathcal{X}_{a,b,n,s})$, $\tilde{\mathcal{X}}_{a,b,n,s} \not\cong \mathcal{X}_{a,b,n,s}$

$\Longrightarrow$ new $\mathbb{F}_{q^{2n}}$-maximal curves not covered by $\mathcal{H}_{q^n}$

More info: Giovanni's Talk!

Weierstrass semigroups and maximal curves

Let $\mathcal{X}$ be a curve and $P \in \mathcal{X}$

Definition: Weierstrass semigroup at P

$H(P) = \{\rho \in \mathbb{Z}_{\geq 0} \mid$ there exists a rat. func. f with $(f)_\infty = \rho P\}$

Weierstrass gap Theorem

$G(P) = \mathbb{N}_0 \setminus H(P)$ contains exactly $g(\mathcal{X})$ elements called gaps

Theorem

If $\mathcal{X}$ is $\mathbb{F}_{q^2}$-maximal and $P \in \mathcal{X}(\mathbb{F}_{q^2})$ then $r \leq$ number of elements in $H(P)$ less than $q+1$

- The structure of $H(P)$ is almost always the same: Weierstrass points
- Main ingredient to construct AG codes!
- Hermitian curve: $r = 2 \rightarrow$ (Garcia-Viana, 1986)
- GK curve: $r = 3 \rightarrow$?

Weierstrass semigroups on the GK curve $\mathcal{C}$

Present

- (Giulietti-Korchmáros, 2009) $H(P) = \langle q^3 - q^2 + q, q^3, q^3 + 1 \rangle$, $P \in \mathcal{C}(\mathbb{F}_{q^2})$
- (Fanali-Giulietti, 2010) $H(P)$, $P \in \mathcal{C}(\mathbb{F}_{q^6}) \setminus \mathcal{C}(\mathbb{F}_{q^2})$ and $q \leq 3$
- (Duursma, 2011) $H(P)$, $P \in \mathcal{C}(\mathbb{F}_{q^6}) \setminus \mathcal{C}(\mathbb{F}_{q^2})$ and $q \leq 9$

Conjecture

Let $P \in \mathcal{C}(\mathbb{F}_{q^6}) \setminus \mathcal{C}(\mathbb{F}_{q^2})$. Then

$$H(P) = \langle q^3 - q + 1, q^3 + 1, q^3 + i(q^4 - q^3 - q^2 + q - 1) \mid i = 0, \ldots, q - 1 \rangle$$

- Nothing known for $P \notin \mathcal{C}(\mathbb{F}_{q^6})$

(Beelen-M., 2018)

- $H(P)$ for all $P \in \mathcal{C}$ $\rightarrow$ conjecture proven!
- The set of Weierstrass points of the GK curve is $\mathcal{C}(\mathbb{F}_{q^6})$

Why was it a conjecture?

Let $P = P_{(a,b,c)} \in \mathcal{C}(\mathbb{F}_{q^6}) \setminus \mathcal{C}(\mathbb{F}_{q^2})$ and

$$T := \langle q^3 - q + 1, q^3 + 1, q^3 + i(q^4 - q^3 - q^2 + q - 1) \mid i = 0, \dots, q-1 \rangle$$

- We need functions $f_\rho \in \mathbb{F}_{q^6}(\mathcal{C})$ with $(f_\rho)_\infty = \rho P$, ρ generator of T
- An explicit description of $f_\rho(x, y, z)$ can be really complicated
- Example $q = 3$: $f_{q^4 - q^2 + q - 1} = f_{74}$ is

$$(2x^3b^{72} + 2x^3b^{64} + 2x^3b^{56} + 2x^3b^{48} + 2x^3b^{40} + 2x^3b^{32} + 2x^3b^{24} + 2x^3b^{16} + 2x^3b^{8} + 2x^3 + 2x^2yb^{27} + x^2yb^3 + x^2b^{36}$$

$$+2x^2b^{12} + xy^2b^{54} + 2xy^2b^6 + xya^3b^{27} + 2xya^3b^3 + xyb^{63} + 2xyb^{39} + 2xa^3b^{36} + xa^3b^{12} + xb^{72} + xb^{48} + xb^{24} + y^3b^{73} + y^3b^{65}$$

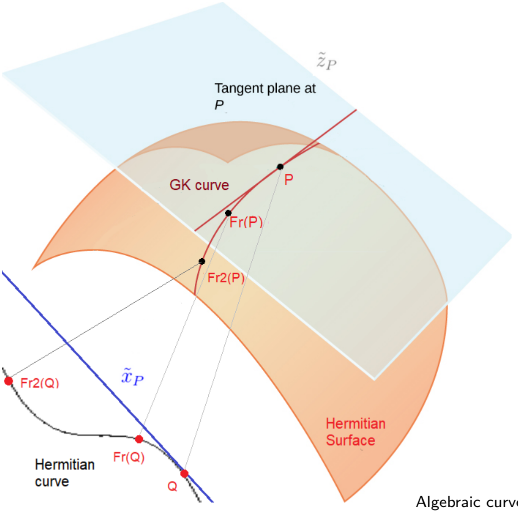
$$+y^3b^{49} + y^3b^{41} + 2y^3b^{33} + y^3b^{25} + y^3b^{17} + 2y^3b^9 + y^2a^3b^{54} + 2y^2a^3b^6 + 2y^2b^{66} + y^2b^{18} + 2ya^6b^{27} + ya^6b^3 + ya^3b^{63}$$

$$+2a^9b^{72} + 2a^9b^{64} + 2a^9b^{56} + 2a^9b^{48} + 2a^9b^{40} + 2a^9b^{32} + 2a^9b^{24} + 2a^9b^{16} + 2a^9b^8 + 2a^9 + a^6b^{36} + 2a^6b^{12} + a^3b^{72} + a^3b^{48}$$

$$+2ya^3b^{39} + 2yb^{75} + 2yb^{51} + 2yb^{27} + a^3b^{24} + b^{84} + 2b^{12})/(-a^{27} - x + b^{27}y + c^{27}z)^3$$

Proving the conjecture: $T \subseteq H(P)$

- $\tilde{z}_P$ exists with $(\tilde{z}_P) = (q^3+1)P - (q^3+1)P_\infty$ (Fundamental Equality)
- $\tilde{x}_P = -a^q - x + b^q y$ (tangent line at $P|Q$ on the Hermitian curve)
- $P \notin \mathcal{C}(\mathbb{F}_{q^2})$: $k = 1, 2$, k-Frobenius twist of $\tilde{x}_P$: $\tilde{x}_P^{(k)} = -a^{q^{2k+1}} - x + b^{q^{2k+1}} y$



Proving the conjecture: $T \subseteq H(P)$

Present

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(Lemma, Beelen-M., 2018)

Let

$$f_i = \frac{\tilde{x}_P^{qi} \cdot \tilde{x}_P^{(2)}}{(\tilde{x}_P^{(1)})^i \cdot \tilde{z}_P^{q-i+1}}, \qquad i = 1, \dots, q-1.$$

Then $1/\tilde{z}_P$, $(y-b)/\tilde{z}_P$, $\tilde{x}_P/\tilde{z}_P$ and f_i give $T \subseteq H(P)$

- (M.- Pallozzi Lavorante, 2020) $H(P)$ where $P \in \mathcal{X}_n(\mathbb{F}_{q^2})$
- (Bartoli-M.-Zini, 2020) $H(P)$ at every P: Suzuki curve
- (Beelen-Landi-M., 2021) $H(P)$ at every P: Skabelund curve
- (Beelen-M.-Vicino, 2023) $H(P)$ at every P: third largest genus (**Lara's talk!**)

Question

Can this method also work for the Ree curve?

The Fundamental Equality: list decoding

- List decoding: the decoder gives a short list of messages that might have been encoded
- Guruswami-Sudan list decoding: $c = (f(P_1), \ldots, f(P_n))$ find a polynomial Q such that $Q(f) = 0$
- **Main steps**: Interpolation step+root finding (optimize?)
- **Theoretically**: find a set of conditions that ensure Q exists and has an nice form

(Beelen-M., 202?)

Efficient Guruswami-Sudan list decoding algorithm for AG codes from maximal curves using the fundamental equation

$$Q(z) = Q_{q+1} z_{q+1} + Q_q z^q + Q_1 z + Q_0$$

- We are still working on the algorithmic complexity/optimization!

Algebraic curves in Coding Theory

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(Bartoli-M.-Quoos, 2021)

LRC codes from automorphisms of curves of genus $g \geq 1$ (maximal/p-rank zero curves have special automorphism groups **Massimo's talk**)

(Bartoli-M.-Zini, 2022)

Quantum codes from "Swiss curves"

(Bartoli-Csajbók-M., Marino-M.-Zullo, Bartoli-M., Zanella-M., 2021/2022)

Algebraic curves methods to construct/classify asymptotically **MRD codes** (maximum/exceptional scattered polynomials)

- **Dream:** find a big picture for all these ad-hoc constructions
- With some extra help: develop the right algebraic curves theory/construct the right curves

Algebraic curves in Information Theory: a treasure yet to discover 2023/2028

CREATE
ALGEBRAIC CURVES IN INFORMATION THEORY

2023: Postdoc (MRD codes)

2024: PhD (Algebraic curves)

2026: Postdoc (LRC codes)

THE VELUX FOUNDATIONS

VILLUM FONDEN VELUX FONDEN

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