An attack on the elliptic curve discrete logarithm problem

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4. This primitive is under constant attack.
5. We present another attack. This attack is not generic and is Las Vegas in nature.

The main theorem

Theorem

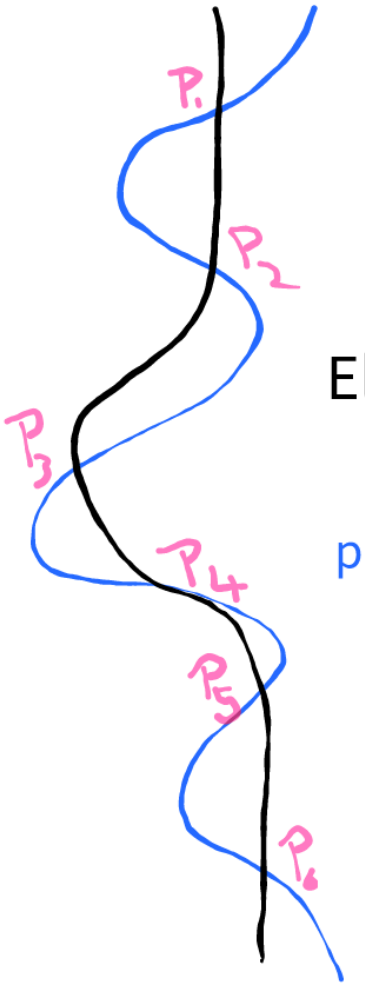
If a plane projective curve $\mathcal{C}$ of degree n' passes through the elliptic curve $\mathcal{E}$ at $3n'$ points $\{P_1, P_2, \ldots, P_{3n'}\}$ then $\sum_{i=1}^{3n'} P_i = \mathcal{O}$.

Proof.

Follows from the Abel-Jacobi's theorem that $P \mapsto [P] - [\mathcal{O}]$ is an isomorphism for an elliptic curve. □

Corollary

The above translates to if $P_i = n_i P$ where $0 < n_i < p$ above then $\sum_{i=1}^{3n'} n_i = 0 \mod p$. Note that $o(P) = p$ is a prime.



Elliptic curve

plane projective curve

1. Let $\mathcal{C} = \sum_{i+j+k=n'} x^i y^j z^k$ be an arbitrary plane projective curve of degree n'.
2. Compute $\mathcal{C}(P)$ as a row in a matrix $\mathcal{M}$. There is a fixed ordering in monomials.

$$\mathcal{M} = \begin{pmatrix} \mathcal{C}(P_1) \\ \mathcal{C}(P_2) \\ \vdots \\ \mathcal{C}(P_{3n'}) \end{pmatrix}$$

left kernel vs. right kernel

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Theorem

The left kernel $\mathcal{K}$ of $\mathcal{M}$ is non-zero if and only if there is a curve $\mathcal{C}$ passing through $3n'$ points of the elliptic curve.

Extra points (the real theorem)

Theorem

Let $k = 3n'$ for some positive integer n'. Let l be another positive integer. Choose positive integers s and t such that $s \neq t$ but $s + t = k + l$. Then construct the matrix $\mathcal{M}$ as described earlier. rows corresponding to $n_i P$ for $i = 1, 2, \ldots s$ and $-n_j Q$ for $j = 1, 2, \ldots, t$.
Let $\mathcal{K}$ be the left-kernel of $\mathcal{M}$. Then the following is true:

The left=kernel $\mathcal{K}$ is of dimension l.

If there is a vector v in $\mathcal{K}$ with l zeros then there is a curve $\mathcal{C}$ passing through $3n'$ points corresponding ot the non-zero points of v.

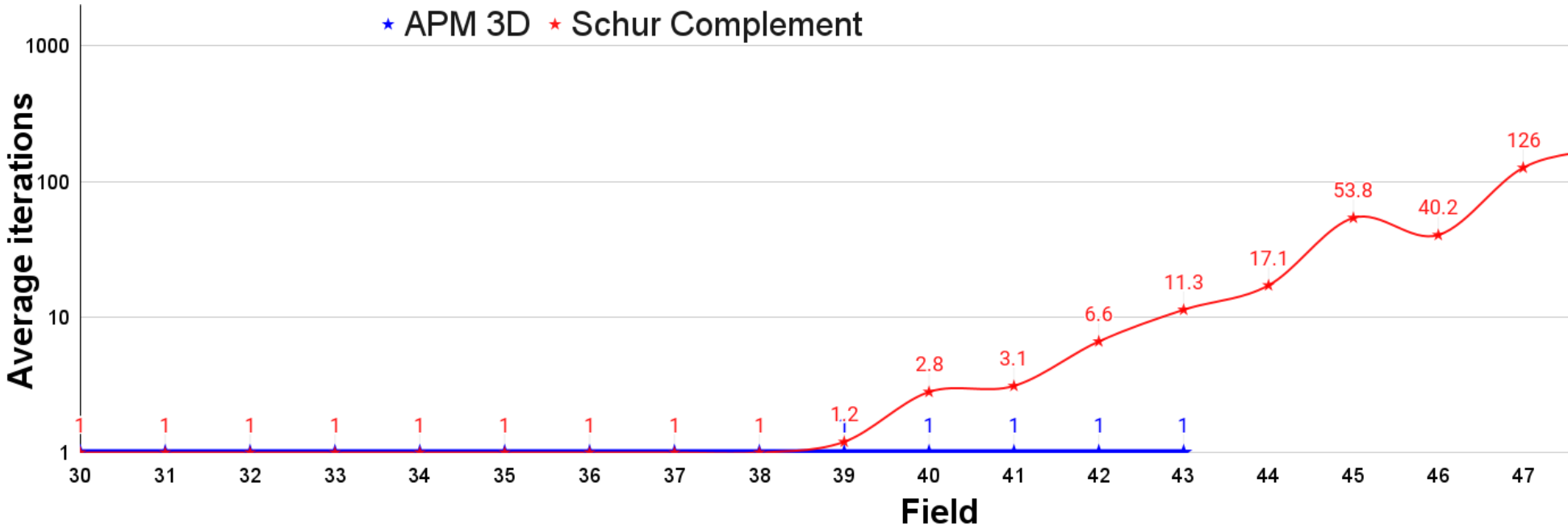
zero minors to solve ECDLP

$$\mathcal{K} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & 0 & 0 & \dots & 0 & 1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

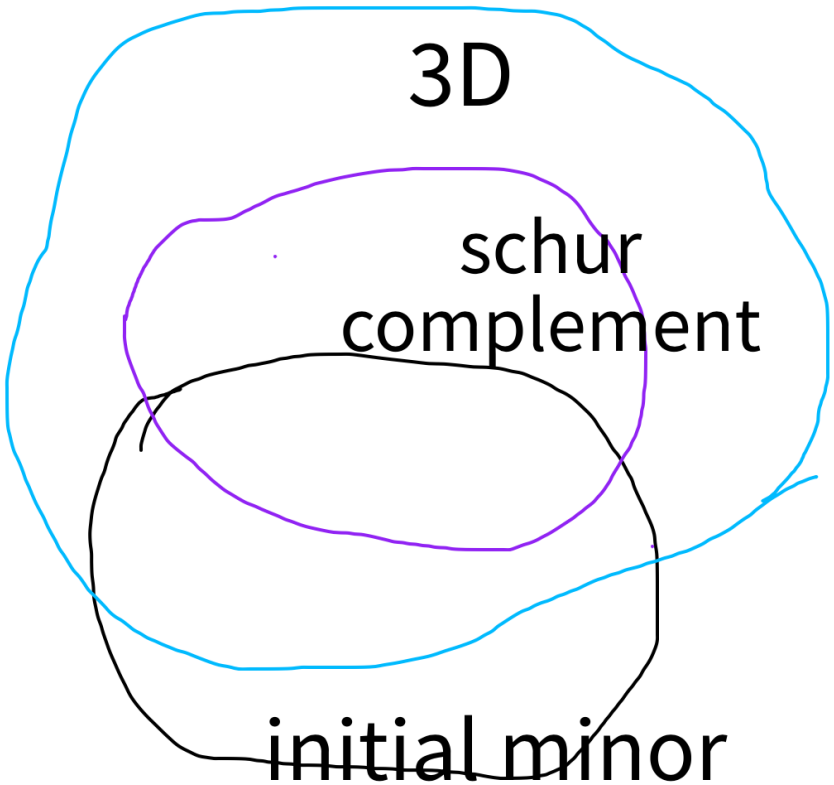
$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

Here $\mathcal{K}$ is a basis of the left kernel and the rows are the basis vectors of that subspace. Find a zero minor in $\mathcal{A}$.

A graph – are we on the right track?



Last slide – what is going on?



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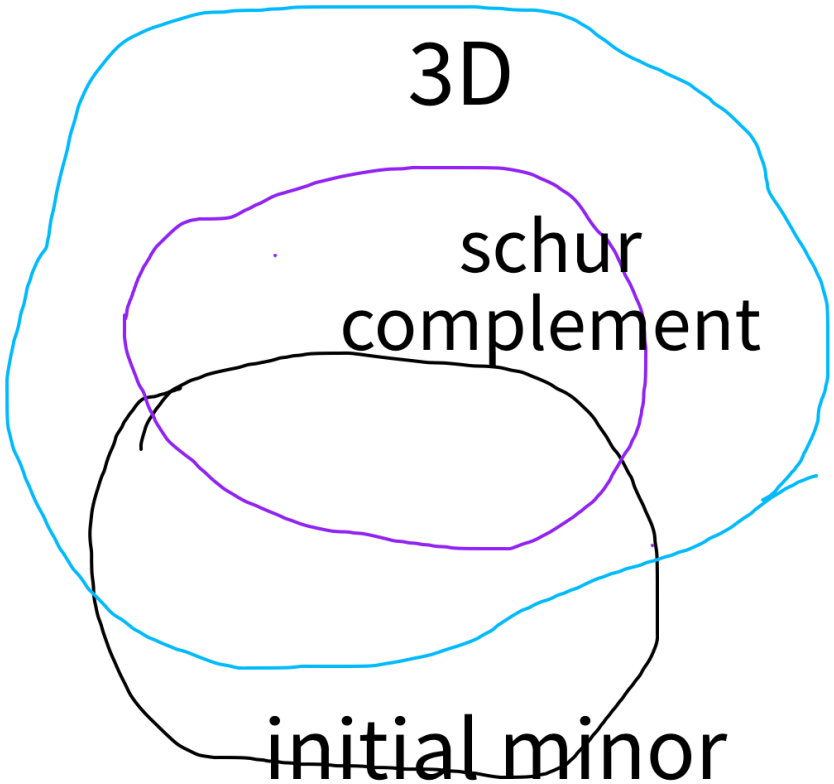

Conjecture (Initial minors)

For any matrix $\mathcal{M}$, there is a set of minors such if every minor in that set is non-zero then all minors are non-zero.