Plane algebraic curves with many symmetries

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joint work with H. Borghes and P. Speziali.

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What about the plane curves $\mathcal{C}$ hitting the minimum?

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Corollary *Let $|G| > d^2$. Then $d(G) \leq d$. If* $\deg(C) = d(G)$ *then C is either in Λ, or C is a nonlinear component of a curve in Λ.*
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G-fixed pencil of plane algebraic curves

$F_1, F_2 \in \overline{\mathbb{F}_q}[X_1, X_2, X_3]$, homogenous polynomials of degree d
$C_\lambda :=$ (degree d) plane curve of equation $F_1 + \lambda F_2 = 0$
C_∞ :=the plane curve of equation $F_2 = 0$
pencil $\Lambda := \{C_\lambda | \lambda \in \overline{\mathbb{F}_q}\} \cup \{C_\infty\} = < C_0, C_\infty >$.
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$PGL(3,q)$ has exactly seven maximal (non sporadic) proper subgroups G.

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(i) $PSL(3, q)$ for $q \equiv 1 \pmod 3$, having order $\frac{1}{3}(q^2 + q + 1)q^3(q + 1)(q - 1)^2$

(ii) the stabilizer of a point of $PG(2, q)$, having order $q^3(q + 1)(q - 1)^2$

(iii) the stabilizer of a line of $PG(2, q)$, having order $q^3(q + 1)(q - 1)^2$

(iv) the stabilizer of an Hermitian curve of $PG(2, q)$ for $q = n^2$, having order $n^3(n^3 + 1)(n - 1)^2$

(v) the stabilizer of a triangle of $PG(2, q)$, having order $6(q - 1)^2$

(vi) the stabilizer of an imaginary triangle (i.e., a triangle in $PG(2, q^3) \setminus PG(2, q)$), having order $3(q^2 + q + 1)$

(vii) for q odd, the stabilizer of an irreducible conic, having order $q(q + 1)(q - 1)$

(viii) sporadic subgroups (of order ≤ 2520)

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Some details.

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For $q \equiv 1 \pmod 3$, $PSL(3,q)$ is a maximal subgroup of $PGL(3,q)$ of index 3, but the same results hold.

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All $AGL(2,q)$-invariant irreducible curves of degree q^3-q^2 belong, up to projectivity, to the above pencil

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the DGZ curve is the unique $\overline{AGL}(2,q)$-invariant irreducible plane curve of degree q^3-q^2
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Case $G = \overline{AGL}(2,q)$, $|G| = q^2(q-1)(q^3-q)$

$\overline{AGL}(2,q)$ is viewed as the subgroup of $PGL(3,q)$ fixing a point.
an $\overline{AGL}(2,q)$-invariant pencil is:

$$\frac{(X^{q^3}-X)(Y^q-Y)-(Y^{q^3}-Y)(X^q-X)}{(X^{q^2}-X)(Y^q-Y)-(Y^{q^2}-Y)(X^q-X)}-$$

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All $\overline{AGL}(2,q)$-invariant irreducible curves of degree q^3-q belong, up to projectivity, to the pencil with $\lambda \neq 1$.

Case $G = PGU(3, n)$, $q = n^2$, $|G| = (n^3 + 1)n^3(n^2 - 1) = (q^{3/2} + 1)q^{3/2}(q - 1)$

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For $\lambda = 1$, the curve splits into n^3+1 lines.

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$$\lambda(X^{q-1} + Y^{q-1} + Z^{q-1})^2 + (XY)^{q-1} + (YZ)^{q-1} + (ZX)^{q-1}) = 0.$$

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the $q + 1$ points of $\mathcal{C}^2$ in $PG(2, q)$ are simple points of $\mathcal{C}$,
the $\frac{1}{2}q(q - 1)$ internal points to $\mathcal{C}^2$ are double points of $\mathcal{C}$.