Grassmann geometries of codes

Luca Giuzzi
(Joint work with Ilaria Cardinali)

University of Brescia

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Grassmann geometries

Projective k-grassmannian

- Point-line geometry $\mathfrak{G}_{n,k} := (\mathcal{G}_{n,k}, \mathcal{L}_k)$:
  - Points $\mathcal{G}_{n,k}$: k-dimensional subspaces of V_n
  - Two elements C_1, C_2 are collinear if and only if $C_1 \cap C_2 \in \mathcal{G}_{n,k-1}$.
- $\Gamma_{n,k} := (\mathcal{G}_{n,k}, \mathcal{E}_k)$: collinearity graph of $\mathfrak{G}^{\delta}_{n,k}$.

Observation

Lines $\mathcal{L}_k$:

- if $k < n-1$: sets $\ell_{X,Y} := \{Z : X < Z < Y\}$ with $X \in \mathcal{G}_{n,k-1}$, $Y \in \mathcal{G}_{n,k+1}$;
- if $k = n-1$: sets $\ell_Z := \{Z : X < Z\}$ with $Z \in \mathcal{G}_{n,n-2}$.

Transparency

Question

The image $G_{n,k}$ of the Plücker embedding ε of a Grassmann geometry $\mathfrak{G}_{n,k}$ is an algebraic variety.

- How much information about the geometry can be read from just the variety?

In other words: can we recover the point-line geometry from just the image of the embedding?

Transparency

A full projective embedding $\varepsilon : \mathfrak{G} \to \mathrm{PG}(\bigwedge^k V)$ is *transparent* if the image of any line of $\mathfrak{G}$ is a line of $\mathrm{PG}(\bigwedge^k V)$ and, conversely the preimage of any line contained in $\varepsilon(\mathfrak{G})$ is a line of $\mathfrak{G}$.

Transparency

Remark

- For projective Grassmannians: Chow's theorem.

Theorem (I. Cardinali, LG, A. Pasini)

- *The Plücker embedding of*
  1. *a projective grassmannian $\mathfrak{G}_{n,k}$ is transparent.*
  2. …
- …

I.Cardinali, LG, A.Pasini, "*On transparent embeddings of point-line geometries*", J. Combin. Th. Series A **155** (2018), 190–224.

Codes on grassmannians

General problem

- A q-ary linear $[n,k,d]$-code is a k-dimensional subspace of $\mathbb{F}_q^n$.
- The linear $[n,k]$-codes are represented as points on the Grassmann geometry $\mathfrak{G}_{n,k}$.
- We study the *graph* of the codes.

Remark

- Related also to code density problems.
  (e.g. given $X \in \Gamma_{n,k}$ what is the spectrum of the distances from X in $\Gamma_{n,k}$ of the codes with given minimum distance.)

Codes on grassmannians

- M. Kwiatkowski, M. Pankov, "*On the distance between linear codes*", Finite Fields Appl. **39** (2016), 251–263
- M. Kwiatkowski, M. Pankov, A. Pasini, "*The graphs of projective codes*", Finite Fields Appl. **54** (2018), 15–29.
- I. Cardinali, LG, M. Kwiatowski, "*On the Grassmann graph of linear codes*", Finite Fields Appl. **75** (2021), 101895.
- M. Pankov, "*The graphs of non-degenerate linear codes*", J. Combin Th. A **195** (2023) 105720.
- I. Cardinali, LG, "*Grassmannians of codes*", in preparation.

Dual minimum distance

Construction

- C: $[n,k]$-linear code
- $d^{\perp}(C) := \min\{w_H(\mathrm{c}') : \mathrm{c}' \in C^{\perp} \setminus \{0\}\}$
  (dual minimum distance)
- $\mathscr{C}^{\delta}_{n,k} := \{C \in \mathscr{G}_{n,k} : d^{\perp}(C) \geq \delta + 1\}$
  [n,k]-codes with prescribed dual minimum distance $\delta + 1$.
  Any $\delta + 1$ columns of the generator matrix of C are linearly independent.
- $\Gamma_{n,k} := (\mathscr{G}_{n,k}, \mathscr{E}_k)$ collinearity graph of $\mathfrak{G}_{n,k}$
  $$(X, Y) \in \mathscr{E}_k \Leftrightarrow \dim(X \cap Y) = k - 1.$$
- $\Lambda^{\delta}_{n,k}$: subgraph of $\Gamma_{n,k}$ whose vertices are elements of $\mathscr{C}^{\delta}_{n,k}$.

Dual minimum distance

Remarks

- $\mathscr{C}^1_{n,k}$: *non-degenerate linear codes:*
  no zero column in the generator matrix
- $\mathscr{C}^2_{n,k}$: *projective codes:*
  no proportional columns in the generator matrix
- $\mathscr{C}^3_{n,k}$: *caps:*
  no three columns on a line in the generator matrix

Codes on grassmannians

Basic questions

- Is the subgraph $\Lambda^{\delta}_{n,k}$ of $[n,k]$-codes with dual minimum distance at least $\delta+1$ connected?
- Are there nice axiomatic descriptions for $\Lambda^{\delta}_{n,k}$?
- What is the relationship between the automorphisms of $\Lambda^{\delta}_{n,k}$ and those of $\Gamma_{n,k}$?
- Are there manageable equations describing the embedding $\varepsilon(\Lambda^{\delta}_{n,k})$?
- What is the (average) valency of vertices of $\Lambda^{\delta}_{n,k}$?
- …

Codes on grassmannians: some results

Theorem (I. Cardinali, LG, M. Kwiatkowski, 2021)

Suppose $1 \leq \delta \leq k \leq n$ *and that* $\mathbb{F}$ *is a field with* $|\mathbb{F}| \geq \binom{n}{\delta}$*. Then*

- $\Lambda^{\delta}_{n,k}$ *is connected;*
- $\Lambda^{\delta}_{n,k}$ *is isometrically embedded into* $\Gamma_{n,k}$*;*
- $\Lambda^{\delta}_{n,k}$ *and* $\Gamma_{n,k}$ *have the same diameter.*

Remark (I. Cardinali, LG)

The graph $\Lambda^{2}_{q+1,3}$ over $\mathbb{F}_q$ with q odd is connected. Its diameter is larger than 3.

Equivalent codes

Theorem (I. Cardinali, LG)

The equivalence[a] class of a $[n,k]$-code is connected in $\Lambda^{\delta}_{n,k}$.

[a] up to monomial transformations

Proof.

- The monomial group $\mathscr{M}$ acts as $\mathbb{F}_q^* \wr S_n$ on the columns of a generator matrix for a code $C \in \mathscr{C}^{\delta}_{n,k}$;
- $\mathscr{M}$ acts on $\mathscr{C}^{\delta}_{n,k}$ (it preserves the dual distance);
- There is a set of generators for $\mathscr{M}$ each of which sends elements of $\Lambda^{\delta}_{n,k}$ to adjacent elements of $\Lambda^{\delta}_{n,k}$.

□

Codes on grassmannians: Chow-style theorems

Theorem (M. Pankov, 2023)

Suppose $1 < k < n-1$, $q < \infty$.
Then

- *If* $q \geq 3$ *or* $k \geq 3$, *then every isomorphism of* $\Lambda^1_{n,k}$ *to a subgraph of* $\Gamma_{n,k}$ *can be uniquely extended to an automorphism of* $\Gamma_{n,k}$.
- *If* $q = k = 2$, *then there are subgraphs of* $\Gamma_{n,2}$ *isomorphic to* $\Lambda^1_{n,2}$ *and such that isomorphisms between these subgraphs cannot be extended to automorphisms of* $\Gamma_{n,2}$.

Codes on grassmannians

Remark

- Let $C, C' \in \mathscr{C}^{\delta}_{n,k}$ with (C, C') adjacent in $\Lambda^{\delta}_{n,k}$ (equivalent to being adjacent in $\Gamma_{n,k}$);
- Take ℓ: unique line of $\mathfrak{G}_{n,k}$ containing C, C';
- There *might* be $X \in \ell$ with $X \notin \mathscr{C}^{\delta}_{n,k}$.

$$\Downarrow$$

- The collinearity of $\Lambda^{\delta}_{n,k}$ does not determine lines *of codes*.

Grassmannians of codes

Definition

- Point-line geometry $\mathfrak{C}^{\delta}_{n,k} = (\mathscr{C}^{\delta}_{n,k}, \mathscr{L}^{\delta}_{n,k})$:
  - Points $\mathscr{C}^{\delta}_{n,k}$: $[n,k]$-codes of dual minimum distance $> \delta$.
  - Two elements C_1, C_2 are collinear if and only if $C_1 \cap C_2 \in \mathscr{C}^{\delta}_{n,k-1}$.
- $\Theta^{\delta}_{n,k}$: collinearity graph of $\mathfrak{C}^{\delta}_{n,k}$.

Observation

Lines $\mathscr{L}^{\delta}_{n,k}$:

- if $k < n-1$: sets $\ell_{X,Y} := \{Z : X < Z < Y\}$ with $X \in \mathscr{C}^{\delta}_{n,k-1}$, $Y \in \mathscr{C}^{\delta}_{n,k+1}$.
- if $k = n-1$: sets $\ell_Z := \{Z : X < Z\}$ with $X \in \mathscr{C}^{\delta}_{n,n-2}$.

Grassmannians of codes: collinearity graph

Remarks

- $\mathfrak{C}^{\delta}_{n,k}$ *is constructed in analogy to* $\mathfrak{G}_{n,k}$ *by replacing* $\mathscr{G}_{n,k-1}$ *and* $\mathscr{G}_{n,k+1}$ *with* $\mathscr{C}^{\delta}_{n,k-1}$ *and* $\mathscr{C}^{\delta}_{n,k+1}$.
- $\Theta^{\delta}_{n,k}$ *is analogous to* $\Gamma_{n,k}$.
- *The graph* $\Lambda^{\delta}_{n,k}$ *is a subgraph of* $\Gamma^{\delta}_{n,k}$.
- $\Theta^{\delta}_{n,k}$ *and* $\Lambda^{\delta}_{n,k}$ *have the same vertices.*
- *Adjacency in* $\Theta^{\delta}_{n,k}$ *implies adjacency in* $\Lambda^{\delta}_{n,k}$ *but the converse is false.*
- *We study the geometry of the codes.*
- *In general* $\Theta^{\delta}_{n,k}$ *is not connected.*

Definition

$$\nu_{\delta}(k;q) := \min\{n : \Theta^{\delta}_{n,k} \text{ not connected}\}.$$

Index of a code

Remark

The codes $C \in \mathscr{C}^{\delta}_{n,k}$ which do not contain as a subspace any code in $\mathscr{C}^{\delta}_{n,k-1}$ are isolated points of $\Theta^{\delta}_{n,k}$.

- $\mathscr{I}^{\delta}_{n,k} := \{C \in \mathscr{C}^{\delta}_{n,k} : C \text{ isolated}\}$;
- $\nu^{+}_{\delta}(k;q) := \min\{n : \mathscr{I}^{\delta}_{n,k} \neq \emptyset\}$;
- $\overline{\Theta^{\delta}_{n,k}}$: subgraph of $\Theta^{\delta}_{n,k}$ induced by $\mathscr{C}^{\delta}_{n,k} \setminus \mathscr{I}^{\delta}_{n,k}$.

Remark

- If there is $n' := \nu_{\delta}(k;q)$ such that $\Theta^{\delta}_{n',k}$ is not connected, then $\Theta^{\delta}_{n,k}$ is not connected for any $n > n'$.

Questions

$$\nu_\delta(k;q) \leq \nu_\delta^+(k;q).$$

Questions

- When is $\Theta_{n,k}^\delta$ connected?
  Determine $\nu_\delta(k;q) := \min\{n : \Theta_{n,k}^\delta$ not connected$\}$
- For $n \geq \nu_\delta(k;q)$, when is $\overline{\Theta_{n,k}^\delta}$ the only connected component of $\Theta_{n,k}^\delta$ with more than one element?
  In other words: when does $\Theta_{n,k}^\delta$ have a connected component as large as possible?
- When do we have $\nu_\delta(k;q) = \nu_\delta^+(k;q)$?
- Chracterize the elements of $\mathscr{I}_{n,k}^\delta$.

$\delta = 1$: non–degenerate codes

Theorem (I. Cardinali, LG)

1. *Either the graph $\Theta^1_{n,k}$ is connected or it consists of a large connected component coinciding with $\overline{\Theta^1_{n,k}}$ and the isolated vertices of $\mathscr{I}^1_{n,k}$.*
2. *The elements of $\mathscr{I}^1_{n,k}$ are exactly the codes whose generator matrix contains as columns representatives for all the points of $\mathrm{PG}(k-1,q)$.*
3. $$\nu_1(k;q) = \nu_1^+(k;q) = \frac{q^k - 1}{q - 1}.$$
4. $\mathrm{diam}(\overline{\Theta^1_{n,k}}) \leq k+1.$

$\delta = 1$: transparency

Theorem (I. Cardinali, LG)

The Plücker embedding of $\mathfrak{C}^1_{n,k}$ is transparent.

Remark

- If two codes $C_1, C_2 \in \mathscr{C}^1_{n,k}$ are collinear in $\mathfrak{C}^1_{n,k}$, then for all elements C of the line of $\mathfrak{G}_{n,k}$ through C_1 and C_2 we have $C \in \mathscr{C}^1_{n,k}$.
- Conversely, if $C_1, C_2 \in \mathscr{C}^1_{n,k}$ are not collinear, then there is X in the line of $\mathfrak{G}_{n,k}$ through C_1 and C_2 with $X \notin \mathscr{C}^1_{n,k}$.

$\delta = 2$: projective codes

Theorem (I. Cardinali, LG)

- *The codes in* $\mathscr{I}^2_{n,k}$ *correspond to the* 1*-saturating sets of* $\mathrm{PG}(k-1,q)$*, i.e. the secants to the projective system of the columns of the generator matrices of codes in* $\mathscr{I}^2_{n,k}$ *cover all points of* $\mathrm{PG}(k-1,q)$*.*
- $\Theta^2_{n,k}$ *consist of the union of* $\overline{\Theta^2_{n,k}}$ *and the isolated vertices in* $\mathscr{I}^2_{n,k}$*.*
- $\nu_2(k;q) = \nu_2^+(k;q)$.

Theorem (I. Cardinali, LG)

The Plücker embedding of $\mathfrak{C}^2_{n,k}$ *is transparent.*

On $\mathscr{I}^2_{k,n}$ and $v_2(k;q)$

Bounds

- $v_2^+(k;q) = \min\{|\Omega| : \Omega$ saturating set of $\mathrm{PG}(k-1,q)\}$.
- Trivial bound:

$$\binom{n}{2}(q-1)+n \geq \frac{q^k-1}{q-1}.$$

- Many "*highly nontrivial*" bounds from the theory of saturating sets.

Saturating sets

Definition

$\Omega \subseteq \mathrm{PG}(k-1,q)$ is *δ-saturating* if

$$\forall x \in \mathrm{PG}(k-1,q) \colon \exists p_0, p_1, \ldots, p_\delta \in \Omega \colon x \in \langle p_0, \ldots, p_t \rangle.$$

Remarks

- *There is an extensive literature on saturating sets: Bartocci, Bartoli, Brualdi, Pless, Wilson, Davydov, Denaux, Faina, Gács, Giulietti, Janwa, Kovács, Marcugini, Östergård, Pambianco, Szőnyi, Ughi, etc.*
- *Dual code with covering radius $\delta + 1$.*
- *Bounds on the minimal size of δ-saturating sets are upper bounds for $\nu_\delta(k;q)$.*

$\delta \geq 2$: the set $\mathscr{I}_{n,k}^{\delta}$

Remark

In $\mathscr{C}_{n,k}^{\delta}$ any set of δ points is independent.

Theorem (I. Cardinali, LG)

- *If $\delta \geq 2$ then $C \in \mathscr{I}_{n,k}^{\delta}$ if and only if the set of all columns of C, regarded as points of $\mathrm{PG}(k-1,q)$, are a $(\delta-1)$-saturating set.*

Remark

- Links with ℓ-secant varieties.

$\delta \geq 2$: main lemma

Lemma (I. Cardinali, LG)

If the graph $\Lambda^{\delta}_{n,k-1}$ is connected, then $\Theta^{\delta}_{n,k}$ is the union of $\overline{\Theta^{\delta}_{n,k}}$ and the isolated vertices in $\mathscr{I}^{\delta}_{n,k}$. So $\nu_{\delta}(k;q) = \nu^{+}_{\delta}(k;q)$.

Strategy

We can prove that $\Theta^{\delta}_{n,k}$ is connected by showing that

- $\Lambda^{\delta}_{n,k-1}$ is connected;
- $\mathscr{I}^{\delta}_{n,k} = \emptyset$.

$\delta \geq 2$: connectedness

Theorem (I. Cardinali, LG)

- *If $\delta = k-1$, then $\mathscr{C}_{n,k}^{k-1} = \mathscr{I}_{n,k}^{k-1}$.* MDS codes
- *If $\delta < k-1$ and $q > \binom{n}{\delta}$ then $\mathscr{I}_{n,k}^{\delta} = \emptyset$ and $\Theta_{n,k-1}^{\delta}$ is connected.*

Remark

- The theorem does not help for computing $\nu_{\delta}(k;q)$ as the value of q depends on n.

Open problem

- Lower the bound $q > \binom{n}{\delta}$.

$\delta \geq 2$: equivalent codes

Theorem (I.Cardinali, LG)

Take $C_1, C_2 \in \mathscr{C}^{\delta}_{n,k} \setminus \mathscr{I}^{\delta}_{n,k}$ to be two equivalent codes. Then C_1 and C_2 belong to the same connected component of $\Theta^{\delta}_{n,k}$.

Proof.

- $\theta \in \mathscr{M}$: monomial morphism from C_1 to $C_2 = \theta(C_1)$.
- $D_1 < C_1$: code with $D_1 \in \mathscr{C}^{\delta}_{n,k-1}$.
- D_1 is in the same connected component as $\theta(D_1)$ in $\Lambda^{\delta}_{n,k-1}$.
- Lift the path from D_1 to $\theta(D_1)$ to a path in $\Theta^{\delta}_{n,k}$ from C_1 to a code C_1' containing $\theta(D_1)$.
- There is a path from C_1' to C_2 in $\Theta^{\delta}_{n,k}$.

□

In general the chosen generators of $\mathscr{M}$ do not send codes to adjacent codes in $\Theta^{\delta}_{n,k'}$.

General observations

- The *interesting* families to describe are those $C \in \mathscr{I}^{\delta}_{n,k}$, for they are minimal with respect to the dimension of their embedding.
- In order to have the geometry of codes of prescribed minimum distance, we consider the action of a duality $\mathfrak{G}_{n,k} \to \mathfrak{G}_{n,n-k}$.
- In the case of codes with prescribed minimum distance the isolated vertices are codes which are *dimension-maximal*.

Future work and open questions

- Lower the bound $q > \binom{n}{\delta}$ for $\Theta^{\delta}_{n,k}$ to be connected.
- Do there exist cases where $\nu_\delta(k;q) < \nu^+_\delta(k;q)$?
- What can we say of $\liminf_{q\to\infty} \frac{\nu^+_\delta(k;q)}{\nu_\delta(k;q)}$?
- For any given q, k and $\delta \geq 2$ determine effective bounds on $\nu_\delta(k;q)$.
- Consider Grassmann geometries of selected families of codes.
- Experiment with different notions of collinearity inherited from the incidence structure of $\mathfrak{G}_{n,k}$.