Algebraic curves with many automorphisms

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Notation and Terminology

- $\mathbb{K}$ algebraically closed field of characteristic p.
- $\mathcal{X}$ projective, non-singular, geometrically irreducible, algebraic curve defined over $\mathbb{K}$ of genus $g = g(\mathcal{X})$.
- $\mathrm{Aut}(\mathcal{X}) = \{\phi : \mathcal{X} \to \mathcal{X} \mid \phi \text{ birational}\}$

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- If $\mathcal{X}$ is defined over a finite field, automorphisms of $\mathcal{X}$ can be lifted to permutation automorphisms of the corresponding AG codes

Basics

- If $g \geq 2$, then $\mathrm{Aut}(\mathcal{X})$ is a finite group
- For any $\mathbb{K}$ and any finite group G, there exists an algebraic curve $\mathcal{X}$ defined over $\mathbb{K}$ such that $\mathrm{Aut}(\mathcal{X}) \cong G$

Basics

- If $g \geq 2$, then $\mathrm{Aut}(\mathcal{X})$ is a finite group
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Question

Which constraints on $\mathrm{Aut}(\mathcal{X})$ are imposed by the invariants of $\mathcal{X}$?

Quotient Curves

Let $G < \mathrm{Aut}(\mathcal{X})$, G finite

- G acts faithfully on $\mathcal{X}$ and has a finite number of short orbits $\theta_1, ..., \theta_k$
- $\exists$ curve $\mathcal{Y}$ whose points are the G-orbits of $\mathcal{X}$
- $\mathcal{Y} := \mathcal{X}/G$ is called quotient curve of $\mathcal{X}$ by G

Riemann-Hurwitz Formula:

$$2g(\mathcal{X}) - 2 = |G|(2g(\mathcal{Y}) - 2) + \mathrm{Diff}(\mathcal{X}|\mathcal{Y})$$

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Riemann-Hurwitz Formula:

$$2g(\mathcal{X}) - 2 = |G|(2g(\mathcal{Y}) - 2) + \mathrm{Diff}(\mathcal{X}|\mathcal{Y})$$

if $p = 0$:

$$2g(\mathcal{X}) - 2 = |G|(2g(\mathcal{Y}) - 2) + \sum_{i=1}^{k}(|G| - |\theta_i|)$$

How many automorphisms?

- Hurwitz bound (1893): if $\mathbb{K} = \mathbb{C}$ and $g \geq 2$,

$$|\mathrm{Aut}(\mathcal{X})| \leq 84(g-1)$$

Example

Klein quartic (1879): $\mathcal{K} : X^3 + Y + XY^3 = 0$.

$$g = 3, \qquad \mathrm{Aut}(\mathcal{K}) = PSL(2,7), \qquad |\mathrm{Aut}(\mathcal{K})| = 168 = 84(3-1).$$

Fricke-Macbeath curve

- Macbeath (1961); infinite g's
- next example is for $g = 7$; $G \cong PSL(2,8)$, $|G| = 504$
- introduced as a Riemann surface by Robert Fricke in 1899
- explicit equations realizing it as an algebraic curve were presented by Macbeath in 1965
- Hendriks (2013): equations over the rationals
- Brock (2004, unpublished), Hidalgo (2015): plane model

$$1 + 7XY + 21X^2Y^2 + 35X^3Y^3 + 28X^4Y^4 + 2X^7 + 2Y^7 = 0$$

- no other example whose explicit definition in terms of polynomials is known

How many automorphisms when $p > 0$?

Hurwitz bound : if ~~$\mathbb{K} = \mathbb{C}$~~ p does not divide $|\mathrm{Aut}(\mathcal{X})|$ and $g \geq 2$,

$$|\mathrm{Aut}(\mathcal{X})| \leq 84(g-1)$$

Example

Hermitian curve: $\mathcal{H}(n) : X^{n+1} = Y^n + Y, \quad n = p^h.$

$$g = \frac{1}{2}n(n-1), \quad |\mathrm{Aut}(\mathcal{H}(n))| = |\mathrm{PGU}(3,n)| = (n^3+1)(n^2-1)(n^3)$$

One-point stabilizer

- Let G_P be the stabilizer of a point $P \in \mathcal{X}$ under the action of G

$$G_P = S_P \rtimes H$$

where H is a cyclic group, and S_P is the only (possibly trivial) Sylow p-subgroup of G_P.

Classification results ($g \geq 2$)

Theorem (Stichtenoth, 1973)

$|\mathrm{Aut}(\mathcal{X})| \leq 16g^4$, *unless* $\mathcal{X}$ *is a Hermitian curve.*

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Theorem (Henn, 1978)

if $|\mathrm{Aut}(\mathcal{X})| > 8g^3$, *then*

(A) $p > 2$, $\quad \mathcal{X} : Y^2 = X^{p^h} - X$

(B) $p = 2$, $\quad \mathcal{X} : Y^2 + Y = X^{2^k+1}, \quad k > 1$

(C) *Hermitian curve* $\mathcal{H}(n)$

(D) *Suzuki curve* $\mathcal{S}(n)$*:* $p = 2$,

$$\mathcal{X} : X^{n_0}(X^n + X) = Y^n + Y, \qquad n_0 = 2^r, r \geq 1, n = 2n_0^2$$

Other bounds

- (Stichtenoth, 1973) G cyclic, $p \nmid |G|$, G fixes a point

$$|G| \leq 4g + 2$$

- (Nakajima, 1987) G abelian.

$$|G| \leq \begin{cases} 4g+4 & \text{for} \quad p \neq 2 \\ 4g+2 & \text{for} \quad p = 2 \end{cases}$$

The p-rank

Jacobian variety $J(\mathcal{X})$, equivalently, the zero Picard group $Pic_0(\mathcal{X})$.

For any prime ℓ,

$$G_\ell := \{Q | Q \in Pic_0(\mathcal{X}), [\ell]Q = 0\}.$$

- if $\ell \neq p$, then $|G_\ell| = \ell^{2g}$
- if $\ell = p$, then $|G_\ell| \leq \ell^{g}$

If $|G_p| = p^\gamma$ then $\gamma = \gamma(\mathcal{X})$ is the p-rank of $\mathcal{X}$ (also called the Hasse-Witt invariant).

The p-rank

In general, $\gamma \leq g$. If $\gamma = g$ then $\mathcal{X}$ is said to be ordinary.

Deuring-Shafarevic Formula: If $|G| = p^h$ then

$$\gamma(\mathcal{X}) - 1 = |G|(\gamma(\mathcal{Y}) - 1) + \sum_{i=1}^{k}(|G| - |\theta_i|)$$

Remark:

- In all the examples from Henn's classification $\gamma = 0$
- For all maximal curves over finite fields have $\gamma = 0$

Nakajima's bounds

Theorem (Nakajima, 1987)

- if $\gamma = g$, then $|\mathrm{Aut}(\mathcal{X})| \leq 84(g^2 - g)$
- S a p-subgroup of $\mathrm{Aut}(\mathcal{X})$.
  - if $\gamma > 1$

$$p > 2, \qquad |S| \leq \frac{p}{p-2}(\gamma - 1) \leq \frac{p}{p-2}(g-1)$$

$$p = 2, \qquad |S| \leq 4(g-1)$$

  - if $\gamma = 0$ and $p \leq g$

$$|S| \leq \frac{4p}{(p-1)^2} g^2$$

(sharp!)

Type of problems

- Bounding the size of the group
- Classifying curves close to the bound
- Classifying groups for specific curves

The spirit

combining ramification theory with finite group theory

Example

- Henn's proof: if $|G| > 8g^3$ then G acts on one of its orbits as a 2-transitive permutation group
- Kantor-O'Nan-Seitz Theorem: classification of 2-transivite permutation groups whose 2-point stabilizer is cyclic

Zero p-rank curves

Deuring-Shafarevic Formula: If $|G| = p^h$ and $\gamma(\mathcal{X}) = 0$ then

$$-1 = -|G| + \sum_{i=1}^{k}(|G| - |\theta_i|)$$

Corollary

If $\gamma(\mathcal{X}) = 0$, then

- *every Sylow p-subgroup of $\mathrm{Aut}(\mathcal{X})$ fixes exactly one point*
- *any two distinct Sylow p-subgroups of $\mathrm{Aut}(\mathcal{X})$ have trivial intersection*

Trivial Intersection of Sylow p-subgroups

Theorem (Burnside-Gow)

Let G be a finite solvable group. If Sylow p-subgroups have trivial intersection, then one of the following occurs for any Sylow p-subgroup S_P:

- *S_p is normal,*
- *S_p is cyclic,*
- *$p = 2$ and S_2 is a generalized quaternion group.*

Non-solvable case

Finite groups whose Sylow 2-subgroups have trivial intersection were classified by Suzuki, Shult and Hering

d-groups

RH genus formula for a d-group U, $d \neq p$, $\mid U \mid \geq d^2$:

$$2g - 2 = |U|(2\bar{g} - 2) + \sum_{i=1}^{k} (|U| - |\theta_i|)$$

Remark

- If $d \not| \, 2g - 2$, U is cyclic
- If $d^2 \not| \, 2g - 2$, U has a cyclic subgroup of index d

d-groups

Theorem (12.5.1 in Hall's book)

For a prime d, the groups of order d^n which contain a cyclic subgroup of index d are:

- Abelian
  - $n \geq 1$, cyclic
  - $n \geq 2$, $a^{d^{n-1}} = 1$, $b^d = 1$, $ab = ba$
- Non-Abelian
  - d odd, $n \geq 3$, $a^{d^{n-1}} = 1$, $b^d = 1$, $ba = a^{1+d^{n-2}}b$
  - $d = 2$, $n \geq 3$, $a^{2^{n-1}} = 1$, $b^2 = 1$, $ba = a^{-1}b$ (Dihedral group)
  - $d = 2$, $n \geq 3$, $a^{2^{n-1}} = 1$, $b^2 = a^{2^{n-2}}$, $ba = a^{-1}b$ (Generalized quaternion group)
  - $d = 2$, $n \geq 4$, $a^{2^{n-1}} = 1$, $b^2 = 1$, $ba = a^{1+2^{n-2}}b$
  - $d = 2$, $n \geq 4$, $a^{2^{n-1}} = 1$, $b^2 = 1$, $ba = a^{-1+2^{n-2}}b$

Even genus curves

Remark for $d = 2$

If g is even and $p > 2$, then **any** 2-**subgroup** U **of** G **has a cyclic subgroup of index** 2.

$$2g - 2 = |U|(2\bar{g} - 2) + \sum_{i=1}^{m}(|U| - |\theta_i|)$$

Even genus curves

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If g is even and $p > 2$, then **any** 2**-subgroup** U **of** G **has a cyclic subgroup of index** 2.

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Information on 2**-subgroups is very useful:**

- groups of even order with that property have a particular structure: $G = N \rtimes S_2$, where S_2 is a Sylow 2-subgroup (apart from few exceptions).
- every group of odd order is solvable

Even genus curves

Remark

If $\gamma = g$, $p > 2$, and S is a p-group such that $\mathcal{X}/S$ is rational, then g is even.

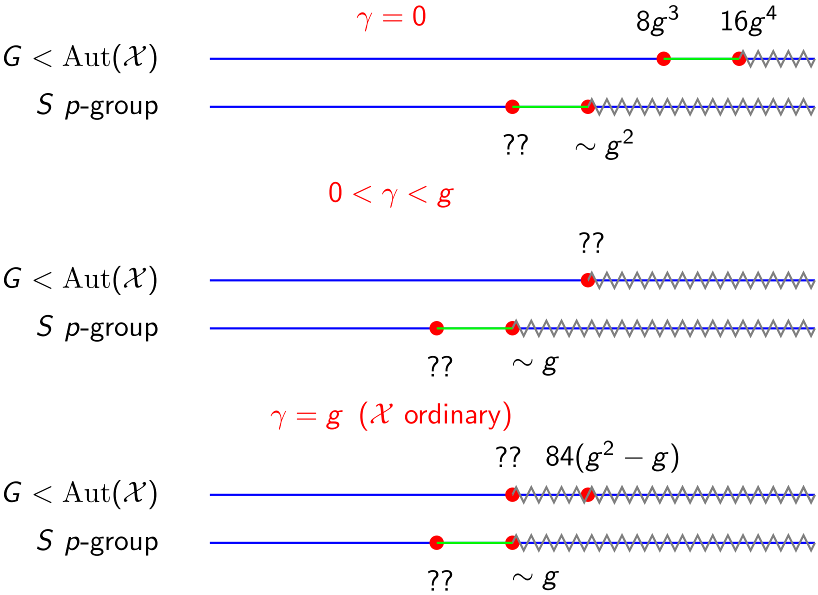
$$g - 1 = \gamma - 1 = -|S| + \sum_{i=1}^{k}(|S| - |\theta_i|)$$

Remark

If $p \equiv 1 \pmod 4$ and S is a p-group such that $\mathcal{X}/S$ is rational, then g is even.

$$2g - 2 = -2|S| + \sum_{P \in \mathcal{X}} \left(\sum_{i \geq 0}(|S_P^{(i)}| - 1) \right)$$

Recap



$\gamma = 0$
$8g^3$
$16g^4$
$G < \mathrm{Aut}(\mathcal{X})$
S p-group
??
$\sim g^2$
$0 < \gamma < g$
??
$G < \mathrm{Aut}(\mathcal{X})$
S p-group
??
$\sim g$
$\gamma = g$ ($\mathcal{X}$ ordinary)
??
$84(g^2 - g)$
$G < \mathrm{Aut}(\mathcal{X})$
S p-group
??
$\sim g$

Problem (1a): $\gamma = 0$, large p-group

$\gamma = 0$, S p-group

Lehr-Matignon, Matignon-Rocher, Rocher (2005-2009)

- If $|S| \geq \frac{4g^2}{(p-1)^2}$, then
  - $S' \cong \mathbb{Z}/p\mathbb{Z}$
  - $\mathcal{X} \cong \mathcal{X}_F : Y^p - Y = XF(X) + cX$, F additive
- If $|S| \geq \frac{4g^2}{(p^2-1)^2}$, then
  - $S' \cong (\mathbb{Z}/p\mathbb{Z})^n$, $1 \leq n \leq 3$
  - Characterization given

Problem (1a): $\gamma = 0$, large p-group

$\gamma = 0$, S p-group

Theorem (M.G.-Korchmáros, 2009)

If $g \geq 2$, $|S| > \frac{2p}{p-1} g$ and $\mathrm{Aut}(\mathcal{X})$ fixes no point, then one of the following occurs for $\mathcal{X}$:

- *(B),(C),(D) in Henn's classification result*
- *(E) Ree curve $\mathcal{R}(n)$: $p = 3$, $\mathcal{X}$ non-singular model of*

$$Y^n - Y = X^{n_0}(X^n - X), \quad Z^n - Z = X^{n_0}(Y^n - Y),$$

for $n_0 = 3^r, r \geq 0, n = 3n_0^2$.

Problem (1b): $\gamma = 0$, large $\mathrm{Aut}(\mathcal{X})$, $p = 2$

Theorem (M.G.-Korchmáros, 2009)

Let $p = 2$, $g \geq 2$ and $\gamma = 0$. Assume that $G = \mathrm{Aut}(\mathcal{X})$ fixes no point of $\mathcal{X}$.

- *If G is solvable, then*

$$|G| \leq 72(g-1)$$

- *If $|G| \geq 24g(g-1)$, then G' is isomorphic to one of the groups:*

$$\mathrm{PSL}(2,v),\ \mathrm{PSU}(3,v),\ \mathrm{SU}(3,v),\ \mathrm{Sz}(v)$$

with $v = 2^r \geq 4$, and the possible genera of $\mathcal{X}$ are computed from the order of G'.

Problem (1b): $\gamma = 0$, large $\mathrm{Aut}(\mathcal{X})$, $p > 2$

Theorem (M.G.-Korchmáros, 2019)

Let $p > 2$, $g \geq 2$ and $\gamma = 0$. Assume that $G = \mathrm{Aut}(\mathcal{X})$ fixes no point of $\mathcal{X}$. If G is solvable, then

$$|G| \leq \frac{p}{p-2} 84 g^2$$

Theorem (M.G.-Korchmáros, 2019)

If $p > 2$, $g \geq 2$ even, $|G| > 900g^2$ then

- *$p(G)$ is isomorphic to one of the groups:*

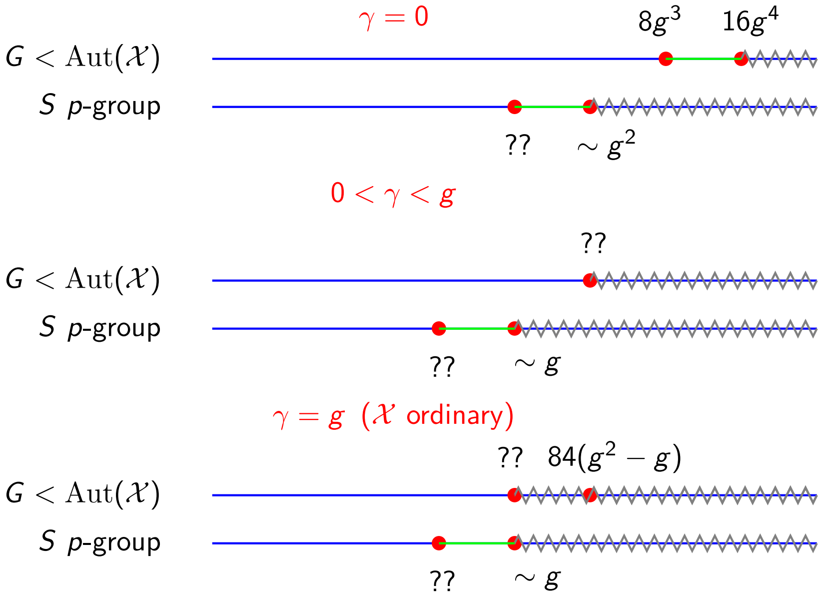
$$\mathrm{PSL}(2,q) \text{ for } q \geq 5, \qquad \mathrm{PSU}(3,q) \text{ for } q \equiv 1 \pmod 4,$$

$$\mathrm{SL}(2,q) \text{ for } q \geq 5, \qquad \mathrm{SU}(3,q) \text{ for } q \equiv 5 \pmod{12},$$

for a power q of p

(here $p(G)$ is the subgroup generated by p-groups)

Problem (2): Nakajima extremal curves



Problem (2a): Nakajima extremal curves $p > 2$, $\gamma = g$

$\gamma = g$, S p-group

- Nakajima bound for $p > 2$:

$$|S| \leq \frac{p}{p-2}(g-1)$$

Theorem (M.G.-Korchmáros (2017))

If equality holds, then either

(i) $|S| = p$, *and* $g = p - 1$, *or*

(ii) $\mathcal{X}$ *is unramified Galois extension of a curve given in* (i)

If (ii) *holds, either*

- $\mathrm{Aut}(\mathcal{X})$ *is the semidirect product of* S *by a subgroup of a dihedral group of order* $2(p-1)$, *or*
- $p = 3$ *and, for some subgroup* M *of* S *of index* 3, M *is a normal subgroup of* $\mathrm{Aut}(\mathcal{X})$ *and* $\mathrm{Aut}(\mathcal{X})/M$ *is isomorphic to a subgroup of* $GL(2,3)$.

Examples

- $X(Y^p - Y) - X^2 - 1 = 0$, $g = (p-1)$, $|S| = p$
- $(X^p - X)(Y^p - Y) = 1$, $g = (p-1)^2$, $|S| = p^2$ (Subrau)
- $(X^p - X)(Y^p - Y) = 1, \qquad Z^q - Z + X^p Y - XY^p = 0$

  $g = p^2(p-2) + 1$, $|S| = p^3$
- curve whose function field is generated by all cyclic unramified p-extensions of degree p^N of the function field of the Subrau curve

$$g = p^{N(p-1)^2+1}(p-2) + 1, \qquad |S| = p^{N(p-1)^2+2}$$

Problem (2b): Nakajima bound $p > 2$, $1 < \gamma < g$

$1 < \gamma < g$, S p-group

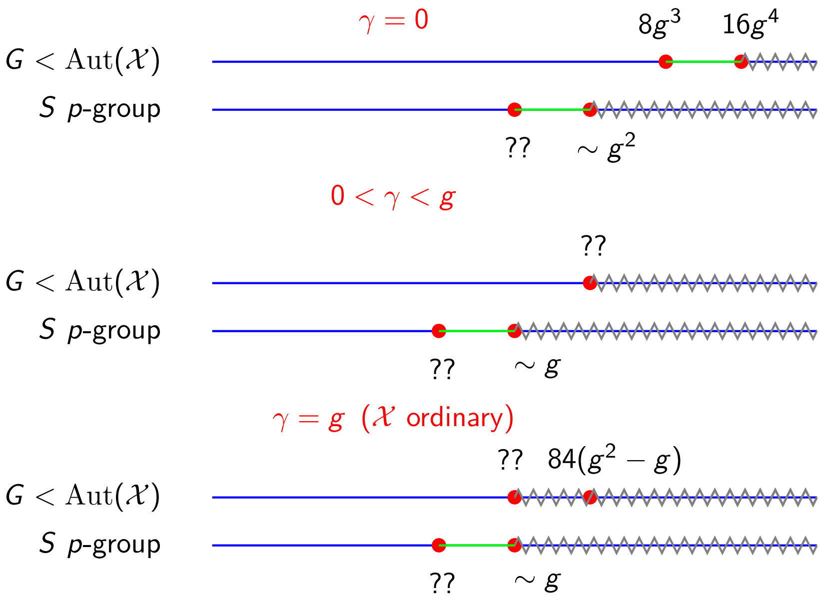
- Nakajima bound for $p > 2$, $\gamma > 1$:

$$|S| \leq \frac{p}{p-2}(\gamma - 1)$$

Theorem (M.G.-Korchmáros (2017))

$$|S| \leq \frac{p^2}{p^2 - p - 1}(g - 1)$$

Problem (3): Nakajima bound for ordinary curves



Examples with $\gamma = g$, $|G| \sim g^{3/2}$

- Subrau (1975):

$$(X^p - X)(Y^p - Y) = 1$$

$g = \gamma = (p-1)^2$, $|G| \geq 2p^2(p-1)$

- Nakajima (1987):

$$X(Y^{q^2} - Y) = X^{q+1} + 1$$

$g = \gamma = q^2 - 1$, $|G| \geq q^2(q+1)$

- Other examples: Guralnick-Mueller-Zieve, Guralnick-Rosenberg-Zieve, Stichtenoth, Korchmáros-Montanucci-Speziali

Examples with $\gamma = g$, $|G| \sim g^{3/2}$

Stichtenoth (2016), Korchmáros-Montanucci-Speziali (2018)

For $q = p^h$ odd, $(m, p) = 1$ the curve $\mathcal{S}$ with equations

$$Y^q + Y = X^m + 1/X^m \qquad Z^q + Z = X^m$$

is such that

- $g(\mathcal{S}) = (q-1)(qm-1), \qquad \gamma(\mathcal{S}) = (q-1)^2$
- $Aut(\mathcal{S})$ is solvable and contains a subgroup $Q \rtimes U$ of index 2 where Q elementary abelian of order q^2 and U cyclic of order $m(q-1)$,
- if $m = 1$, $\mathcal{S}$ is ordinary and $|Aut(\mathcal{S})| > 2g^{3/2}$.

Examples with $\gamma = g$, $|G| > g^{3/2}$, $p > 2$

- For $p > 3$, no hope from Nakajima extremal curves:

$$|G| \leq 2\frac{p^2 - p}{p - 2}(g - 1)$$

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Dickson (1911)

- $q = p^h > 2$

$$D_1(x,y,z) = \begin{vmatrix} x & x^q & x^{q^3} \\ y & y^q & y^{q^3} \\ z & z^q & z^{q^3} \end{vmatrix}, \quad D_2(x,y,z) = \begin{vmatrix} x & x^q & x^{q^2} \\ y & y^q & y^{q^2} \\ z & z^q & z^{q^2} \end{vmatrix};$$

- $F(x,y,z) = \frac{D_1(x,y,z)}{D_2(x,y,z)}$ is $GL(3,q)$-invariant

A Dickson curve

- $F(X, Y, Z)$ is an absolutely irreducible polynomial
- The curve $\mathcal{D} : F(X, Y, Z) = 0$ has genus
$$g = \frac{1}{2}q(q-1)(q^3-2q-2)+1$$
- $\mathrm{PGL}(3, q) \leq \mathrm{Aut}(\mathcal{D})$.
- if $q = p > 2$, then $\gamma(\mathcal{D}) = g$
- $|\mathrm{Aut}(\mathcal{D})| \sim g^{8/5}$.

Borges (2009/2022) M.G.-Korchmáros-Timpanella (2019)

Several properties: (unique) double Frobenius non-classical curve over $\mathbb{F}_q$ and $\mathbb{F}_{q^3}$, nice combinatorial properties of $\mathcal{D}(\mathbb{F}_{q^3})$, $\mathrm{Aut}(\mathcal{D}) = \mathrm{PGL}(3, q)$

Examples with $\gamma = g$, $|G| > g^{3/2}$, $p = 2$

Zieve (2016)

The curve defined by

$$(X^q + X)(Y^q + Y) = 1$$

and

$$Z^q + Z = X^{q+1}/(X^q + X)$$

- has genus $g = \gamma = q^3 - q^2 - q + 1$
- its automorphism group contains a subgroup of size $O(q^5)$ containing $PGL(2, q)$

Improving Nakajima's bound

Guralkick-Zieve (unpublished result)

If p is odd then $O(g^{8/5})$ is the best bound possible for the automorphism group of an ordinary curve. If $p = 2$ then the bound is $O(g^{5/3})$.

Gunby-Smith-Yuan (2015)

There exists a constant $c = c(p)$ such that any ordinary curve $\mathcal{X}$ over a field of characteristic p with genus $g > c$ satisfies the inequality

$$|\mathrm{Aut}(\mathcal{X})| \leq 6(g^2 + 12\sqrt{21}g^{3/2})$$

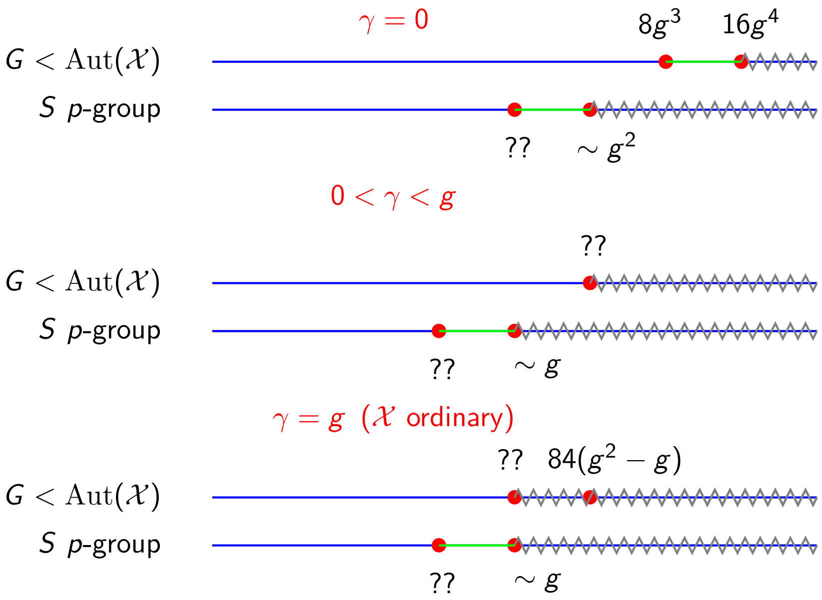
An alternative proof of Nakajima's bound

Lia-Timpanella (2020)

Let $\mathcal{X}$ be a curve of genus $g \geq 2$ with $\gamma > 0$ and let G be a subgroup of $\mathrm{Aut}(\mathcal{X})$. If for every $P \in \mathcal{X}$, $G_P^{(2)} = \{1\}$ then

$$|G| \leq 48(g-1)^2. \tag{1}$$

Problem (4a): large groups, $0 < \gamma < g$



Problem (4a): large solvable groups for $0 < \gamma < g$

$$g \geq 2, \qquad p > 2, \qquad 0 < \gamma < g$$

Theorem (M.G.–Korchmáros, 2019)

If G is solvable, then

$$|G| \leq 84 \frac{p}{p-2} g^2$$

S-KMS for $m > 1$: $g^{3/2}$

Problem (4b): large groups, $0 < \gamma < g$, g even

$$g \geq 2, \qquad p > 2, \qquad 0 < \gamma < g$$

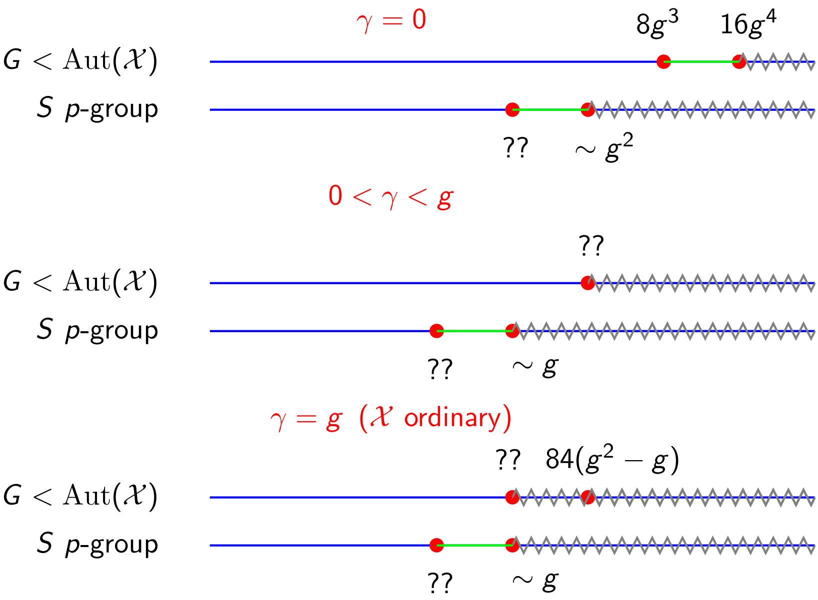
Theorem (M.G.-Korchmáros, 2019)

If g is even then

$$|G| \leq 900g^2$$

Dickson: $g^{8/5}$ for $q = p^3$, $p \equiv 3 \pmod 4$

Problem (5): solvable/even genus case for ordinary curves



Problem (5): solvable/even genus case for ordinary curves

$\gamma = g \geq 2$

Theorem (Korchmáros-Montanucci, 2020)

If $p > 2$, G solvable then

$$|G| \leq 34(g+1)^{3/2} < 68\sqrt{2}g^{3/2}$$

Sharp: S-KMS curve

Theorem (Montanucci-Speziali, 2020)

If $p > 2$, g even, G is not solvable then

$$|G| \leq 822g^{7/4}$$

Theorem (Montanucci-Speziali, 2019)

If $p = 2$, g even, G is solvable then

$$|G| \leq 35(g+1)^{3/2}$$

Problem (6): Large special groups

- *d*-groups, $d \neq p$: Korchmáros-Montanucci (2020)
- nilpotent groups: Anbar-Gunes (2022)
- groups of prime order: Homma (1980), Arakelian-Speziali (2021)

Problem (6a): curves with large d-groups, $d \neq p$

Korchmáros-Montanucci (2020)

$g \geq 2$, G a d-subgroup of $\mathrm{Aut}(\mathcal{X})$ with $d \neq p$ and $d > 2$. Then

$$|G| \leq \begin{cases} 9(g-1), & \text{if } d = 3, \\ \frac{2d}{d-3}(g-1), & \text{if } d > 3. \end{cases}$$

sharp for $d \geq 5$: **Fermat curve** $\mathcal{F}_d : x^d + y^d + 1 = 0$

$$g(\mathcal{F}) = (d-1)(d-2)/2 \qquad C_d \times C_d \cong G < \mathrm{Aut}(\mathcal{F}_d)$$

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$$|G| \leq \begin{cases} 9(g-1), & \textit{if } d = 3, \\ \frac{2d}{d-3}(g-1), & \textit{if } d > 3. \end{cases}$$

sharp for $d \geq 5$: **Fermat curve** $\mathcal{F}_d : x^d + y^d + 1 = 0$

$$g(\mathcal{F}) = (d-1)(d-2)/2 \qquad C_d \times C_d \cong G < \mathrm{Aut}(\mathcal{F}_d)$$

Korchmáros-Montanucci, 2020

Construction of extremal curves for every $(g, |G|) = (3^h + 1, 3^{h+2})$

Problem (6b): nilpontent groups

Zomorrodian (1985): the case Char(K) $= 0$

G nilpotent subgroup of $\mathrm{Aut}(\mathcal{X})$

$$|G| \leq 16(g-1)$$

bound is sharp if and only if $g - 1 = 2^k$

Theorem (Anbar-Gunes, 2022)

Zomorrodian results holds for $p > 0$, unless G is a p-group which fixes a point

Bound is sharp for infinite g's

Problem (6c): curves with large groups of prime order

$$g \geq 2$$

Homma (1980)

If a prime d is the order of an automorphism, then $d \leq g+1$, or $d = 2g \pm 1$

Homma (1980)

If $d = 2g+1$, then

- $\mathcal{X}$ is a tame $(2g+1)$-curve if and only if $\mathcal{X}$ is

$$\mathcal{X}_{m,n} : Y^{2g+1} = X^{m-n}(X-1)^n \; (1 \leq n < m \leq g+1).$$

- $\mathcal{X}$ is a wild $(2g+1)$-curve if and only if $\mathcal{X}$ is

$$\mathcal{R} : Y^2 = X^{2g+1} - X.$$

Problem (6c): curves with large groups of prime order

Arakelian-Speziali (2021)

Let $\mathcal{X}$ be a $(g+1)$-curve with $g > 2$.

- If $\mathcal{X}$ is tame, then $\mathcal{X} \cong$ to one of the following plane curves:

$$\mathcal{X}_{r,s,t,a} : Y^{g+1} = X^r(X-1)^s(X-a)^t,$$

  $a \in \mathbb{K} \setminus \{0,1\}$, $r, s, t < g+1$, with $r+s+t \not\equiv 0$ mod $(g+1)$.

- If $\mathcal{X}$ is wild, then $\mathcal{X} \cong$ to either a (hyperelliptic) curve

$$\mathcal{Y}_{a,b,c} : Y^{g+1} - Y = \frac{aX^2+bX+c}{X(X-1)}$$

  with $(a,b,c) \neq (0,0,0)$ and $\gcd(aX^2+bX+c, X(X-1)) = 1$, or

$$\mathcal{Z}_{d,e,\ell} : Y^{g+1} - Y = X^3 + dX^2 + eX + \ell,$$

  for $d, e, \ell \in \mathbb{K}$

Problem (6d): simple automorphism groups of curves of even genus

$$g \geq 2$$

Theorem (M.G.-Korchmáros, 2019)

g even, $p > 2$, G a nonabelian simple group then one of the following cases occur where $v = d^k$, d prime, k odd:

(i) $G \cong PSL(2, v)$ *with* $v \geq 5$;

(ii) $G \cong PSL(3, v)$ *with* $v \equiv 3 \pmod 4$;

(iii) $G \cong PSU(3, v)$ *with* $v \equiv 1 \pmod 4$;

(iv) $G \cong \mathrm{Alt}_7$;

(v) $G \cong M_{11}$.

Examples for each case

Problem (6e): groups fixing a point

$G < \mathrm{Aut}(\mathcal{X})$, G fixes a point, $g \geq 2$

- Singh (1974):

$$|G| \leq \frac{4pg^2}{p-1}\left(\frac{2g}{p-1}+1\right)$$

- $\mathrm{Aut}(H(n))_P \sim 4\sqrt{2}g^{5/2}$

M.G.- Korchmáros (2019)

$p > 2$, $|G| > 30(g-1)$ then either $\gamma = p^k - p^a - p^b + 1$, or $\gamma = 0$

Lia-Timpanella (2020)

G fixes a point $P \in \mathcal{X}$ and $G_P^{(2)} = \{1\}$. If $|G| > 12(g-1)$ then either

(i) $\mathcal{X} : L_1(y) = ax + 1/x$, where $L_1(T)$ p-linearized, or

(ii) $p \neq 3$ and $\mathcal{X} : L_2(y) = x^3 + bx$, where $L_2(T)$ p-linearized

In Case (i) $\rightarrow$ $\mathcal{X}$ is an ordinary hyperelliptic curve

In Case (ii) $\rightarrow$ $\mathcal{X}$ has zero p-rank

In progress: $0 < \gamma < g$, G non fixing any point

M.G.-Korchmáros-Lia-Timpanella

If $\mathcal{X}/S_P$ is rational for some P, then either

(i) $|G| \leq 10(g-1)(2\gamma+3)$, or

(ii) there exists $h \in G$ such that $S_p \cap S_{h(P)} = \{id\}$ where $S_{h(P)} = hS_P h^{-1}$.

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