# Minimum weight codewords of Schubert codes

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# Summary of the Talk (Take Home Message!)

- Schubert codes constitute a nice class of linear codes which were introduced around the turn of the century. These correspond to the ($\mathbb{F}_q$-rational points of) Schubert varieties in Grassmannians, and include, as a special case, Grassmann codes whose study goes back to 1987.
- A conjecture about the minimum distance of Schubert codes remained open for almost a decade and was proved in the affirmative in 2008.
- In 2018, a new conjecture about the minimum weight codewords of Schubert codes was proposed, and moreover, it was "almost proved" in the affirmative.
- Nonetheless, a complete proof has not been found in the last five years or so. Recently, in a joint work with Mrinmoy Datta and Avijit Panja, we have revisited the problem and are able to make a small progress.
- But the problem is still open, in general.

# Grassmann Varieties : A Quick Introduction

$V$ : vector space of dimension $m$ over a field $\mathbb{F}$

For $1 \leq \ell \leq m$, we have the Grassmann variety:

$$G_{\ell,m} = G_\ell(V) := \{\ell\text{-dimensional subspaces of } V\}.$$

Plücker embedding: $G_{\ell,m} \hookrightarrow \mathbb{P}^{k-1}$, where $k := \binom{m}{\ell}$.

Explicitly, $\mathbb{P}^{k-1} = \mathbb{P}(\wedge^\ell V)$ and

$$W = \langle w_1, \ldots, w_\ell \rangle \longleftrightarrow [w_1 \wedge \cdots \wedge w_\ell] \in \mathbb{P}(\wedge^\ell V).$$

For example, $G_{1,m} = \mathbb{P}^{m-1}$. In terms of coordinates,

$$W = \langle w_1, \ldots, w_\ell \rangle \in G_\ell(V) \longleftrightarrow p(W) = \left(p_\alpha(A_W)\right)_{\alpha \in I(\ell,m)},$$

where $A_W = (a_{ij})$ is a $\ell \times m$ matrix whose rows are (the coordinates of) a basis of $W$ and $p_\alpha(A_W)$ is the $\alpha^{\text{th}}$ minor of $A_W$, viz., $\det\left(a_{i\alpha_j}\right)_{1 \leq i,j \leq \ell}$.

# Introduction to Grassmann Varieties Contd.

Notation: $I(\ell, m) := \{\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell : 1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m\}$.

Facts:

- $G_{\ell,m}$ is a projective algebraic variety given by the common zeros of certain quadratic homogeneous polynomials in $k$ variables. As a projective algebraic variety, $G_{\ell,m}$ is nondegenerate, irreducible, nonsingular, and rational.
- There is a natural transitive action of $\mathrm{GL}_m$ on $G_{\ell,m}$ and if $P_\ell$ denotes the stabilizer of a fixed $W_0 \in G_{\ell,m}$, then $P_\ell$ is a maximal parabolic subgroup of $\mathrm{GL}_m$ and $G_{\ell,m} \simeq \mathrm{GL}_m/P_\ell$.
- If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then $G_{\ell,m}$ is a (real or complex) manifold, and its cohomology spaces and Betti numbers are explicitly known. In fact, $b_\nu = \dim H^{2\nu}(G_{\ell,m}; \mathbb{C})$ is precisely the number of partitions of $\nu$ into at most $\ell$ parts, each part $\leq m - \ell$.

# Grassmannians Over Finite Fields

Suppose $\mathbb{F} = \mathbb{F}_q$ is the finite field with $q$ elements. Then $G_{\ell,m} = G_{\ell,m}(\mathbb{F}_q)$ is a finite set and its cardinality is given by the Gaussian binomial coefficient:

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q)\cdots(q^m - q^{\ell-1})}{(q^\ell - 1)(q^\ell - q)\cdots(q^\ell - q^{\ell-1})}.$$

This is a polynomial in $q$ of degree $\delta := \ell(m - \ell)$ and in fact,

$$|G_{\ell,m}(\mathbb{F}_q)| = \begin{bmatrix} m \\ \ell \end{bmatrix}_q = \sum_{\nu=0}^{\delta} b_\nu q^\nu = q^\delta + q^{\delta-1} + 2q^{\delta-2} + \cdots + 1,$$

where the coefficients $b_\nu$ are nonnegative integers that have combinatorial and topological interpretation mentioned earlier. Note that

$$\lim_{q\to 1} \begin{bmatrix} m \\ \ell \end{bmatrix}_q = \binom{m}{\ell}.$$

# Schubert Varieties in Grassmannians

Fix a basis $\{e_1, \ldots, e_m\}$ of $V$ and any $\alpha \in I(\ell, m)$, that is,

$$\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell, \; 1 \le \alpha_1 < \cdots < \alpha_\ell \le m.$$

The corresponding Schubert variety is defined by

$$\Omega_\alpha := \{W \in G_{\ell,m} : \dim(W \cap A_i) \ge i \; \forall \, i = 1, \ldots, \ell\},$$

where $A_i = \langle e_1, \ldots, e_{\alpha_i} \rangle$ for $1 \le i \le \ell$. Alternatively,

$$\Omega_\alpha := \{[v_1 \wedge \ldots \wedge v_\ell] : v_1, \ldots, v_\ell \in V \text{ linearly independent and } v_i \in A_i \; \forall \, i\}.$$

The Plücker embedding of $G_{\ell,m}$ induces a nondegenerate embedding

$$\Omega_\alpha(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k_\alpha - 1} \quad \text{where} \quad k_\alpha = |\{\beta \in I(\ell, m) : \beta \le \alpha\}|,$$

with $\le$ being the componentwise partial order (Bruhat-Chevalley order):

$$\beta = (\beta_1, \ldots, \beta_\ell) \le \alpha = (\alpha_1, \ldots, \alpha_\ell) \iff \beta_i \le \alpha_i \; \forall \, i = 1, \ldots, \ell.$$

## Basics of (Linear) Codes; A Quick Recap

- $q$-ary (linear) code of length $n$: $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$
- $[n,k]_q$-code: $k$-dimensional subspace $C$ of $\mathbb{F}_q^n$.
- Hamming weight of $c = (c_1, \dots, c_n) \in \mathbb{F}_q^n$:

$$\mathrm{wt}(c) := |\{i : c_i \neq 0\}|.$$

- Minimum distance of an $[n,k]_q$-code $C$:

$$d(C) := \min\{\mathrm{wt}(c) : c \in C,\ c \neq 0\}.$$

- $c \in C$ is a minimum weight codeword of $C$ if $\mathrm{wt}(c) = d(C)$.
- The dual of an $[n,k]_q$-code $C$ is $C^\perp := \{x \in \mathbb{F}_q^n : x \cdot c = 0 \text{ for all } c \in C\}$.
- An $[n,k]_q$-code $C$ is nondegenerate if $C \not\subseteq$ coordinate hyperplane of $\mathbb{F}_q^n$.

# Grassmann Codes and Schubert Codes

Fix $\alpha \in I(\ell, m)$ and representatives $P_1, \ldots, P_{n_\alpha}$ in $\bigwedge^\ell V$ of points of the Schubert variety $\Omega_\alpha(\mathbb{F}_q) \subseteq G_{\ell,m}(\mathbb{F}_q) \subseteq \mathbb{P}(\bigwedge^\ell V)$. We have the evaluation map

$$\bigwedge^{m-\ell} V \longrightarrow \mathbb{F}_q^{n_\alpha} \quad \text{given by} \quad f \longmapsto c_f = (f \wedge P_1, \ldots, f \wedge P_{n_\alpha}) \, .$$

This is clearly linear and the image is denoted by $C_\alpha(\ell, m)$ and called the Schubert code. When $\alpha = (m - \ell + 1, \ldots, m)$, it is called the Grassmann code and denoted by $C(\ell, m)$. In this case, the evaluation map is injective and thus the length $n$ and the dimension $k$ of the Grassmann code $C(\ell, m)$ are given by

$$n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q \quad \text{and} \quad k = \binom{m}{\ell}.$$

Question: What is the minimum distance of the Schubert code $C_\alpha(\ell, m)$ and what are its minimum weight codewords? Also what are the length and dimension of $C_\alpha(\ell, m)$?

# Geometric Counterpart of the Question

Fix $\alpha \in I(\ell, m)$ and consider $\Omega_\alpha(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k_\alpha - 1}$. Finding the minimum distance of $C_\alpha(\ell, m)$ corresponds to finding

$$e^\alpha(\ell, m) := \max_H |\Omega_\alpha(\mathbb{F}_q) \cap H|$$

where the maximum is over all hyperplanes $H$ in $\mathbb{P}^{k_\alpha - 1}$, or equivalently, all hyperplanes $H$ in $\mathbb{P}(\bigwedge^\ell V)$ such that $\Omega_\alpha \not\subseteq H$.

And finding the minimum weight codewords of $C_\alpha(\ell, m)$ corresponds to finding the hyperplanes $H$ in $\mathbb{P}^{k_\alpha - 1}$ for which the maximum is attained; we call these the maximal hyperplane sections of the Schubert variety $\Omega_\alpha$. Enumerating the minimum weight codewords corresponds to determining

$$M^\alpha(\ell, m) := |\{H : |\Omega_\alpha(\mathbb{F}_q) \cap H| = e^\alpha(\ell, m)\}|.$$

Answers to these questions are known when $\alpha = (m - \ell + 1, \dots, m - 1, m)$, that is, when $\Omega_\alpha = G_{\ell,m}$ or when $C_\alpha(\ell, m)$ is the Grassmann code $C(\ell, m)$.

# Minimum Distance of Grassmann Codes

**Theorem (Nogin, 1996)**

*With notation as above,*

$$d\left(C(\ell,m)\right) = q^{\delta} \quad \text{and} \quad |\{\text{Min wt codewords of } C(\ell,m)\}| = (q-1)\begin{bmatrix} m \\ \ell \end{bmatrix}_q.$$

*Or equivalently,*

$$e(\ell,m) = \begin{bmatrix} m \\ \ell \end{bmatrix}_q - q^{\delta} \quad \text{and} \quad M(\ell,m) = \begin{bmatrix} m \\ \ell \end{bmatrix}_q.$$

*In fact, the hyperplanes $H$ that attain $e(\ell,m)$ are precisely those that correspond to decomposable elements of $\bigwedge^{m-\ell} V = \left(\bigwedge^{\ell} V\right)^*$, that is, those $f \in \bigwedge^{m-\ell} V$ such that $f = f_1 \wedge \ldots \wedge f_{m-\ell}$ for some linearly independent $f_1, \ldots, f_{m-\ell} \in V$.*

# The Case of Schubert Codes

For Schubert codes, determining the minimum distance presented some difficulties. But the following result was not difficult to prove.

**Proposition (G - Lachaud (2000))**

*For any* $\alpha \in I(\ell, m)$,

$$d\left(C_\alpha(\ell, m)\right) \le q^{\delta_\alpha} \quad \textit{where} \quad \delta_\alpha := \sum_{i=1}^{\ell} (\alpha_i - i).$$

When $\alpha = (m - \ell + 1, \ldots, m - 1, m)$, the inequality is an equality, thanks to Nogin. The following conjecture was made in the same paper:

**Minimum Distance Conjecture (MDC)**

For any $\alpha \in I(\ell, m)$,

$$\boxed{d\left(C_\alpha(\ell, m)\right) = q^{\delta_\alpha}.}$$

# Length of Schubert Codes

- If $\ell = 2$ and $\alpha = (m-h-1, m)$, then

$$n_\alpha = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)} - \sum_{j=1}^{h}\sum_{i=1}^{j} q^{2m-j-2-i}$$

and

$$k_\alpha = \frac{m(m-1)}{2} - \frac{h(h+1)}{2}.$$

[Hao Chen (2000)]

- In general,

$$n_\alpha = \sum \prod_{i=0}^{\ell-1} \begin{bmatrix} \alpha_{i+1}-\alpha_i \\ k_{i+1}-k_i \end{bmatrix}_q q^{(\alpha_i-k_i)(k_{i+1}-k_i)}$$

where the sum is over $(k_1, \ldots, k_{\ell-1}) \in \mathbb{Z}^\ell$ satisfying $i \le k_i \le \alpha_i$ and $k_i \le k_{i+1}$ for $1 \le i \le \ell-1$; by convention, $\alpha_0 = 0 = k_0$ and $k_\ell = \ell$.

[Vincenti (2001)]

# Length of Schubert Codes (Contd.)

- $n_\alpha = \sum_{\beta \leq \alpha} q^{\delta_\beta}$, [Ehresmann (1934); G - Tsfasman (2005)]
- Suppose $\alpha$ has $u+1$ consecutive blocks:

  $\alpha = (\alpha_1, \ldots, \alpha_{p_1}, \ \ldots, \ \alpha_{p_u+1}, \ldots, \alpha_{p_{u+1}})$. Then

$$n_\alpha = \sum_{s_1=p_1}^{\alpha_{p_1}} \cdots \sum_{s_u=p_u}^{\alpha_{p_u}} \prod_{i=0}^{u} \lambda(\alpha_{p_i}, \alpha_{p_{i+1}}; s_i, s_{i+1})$$

  where, $s_0 = p_0 = 0$; $s_{u+1} = p_{u+1} = \ell$, and

  $\lambda(a, b; s, t) := \sum_{r=s}^{t} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} a-s \\ r-s \end{bmatrix}_q \begin{bmatrix} b-r \\ t-r \end{bmatrix}_q$. [G - Tsfasman (2005)]

- $n_\alpha = \det \left( q^{(j-i)(j-i-1)/2} \begin{bmatrix} \alpha_j - j + 1 \\ i - j + 1 \end{bmatrix}_q \right)_{1 \leq i,j \leq \ell}$. [G - Krattenthaler]

# Dimension of Schubert Codes [G-Tsfasman (2005)]

- Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in I(\ell, m)$. The dimension of $C_\alpha(\ell, m)$ is the $\ell \times \ell$ determinant:

$$k_\alpha = \det_{1 \le i,j \le \ell} \left( \binom{\alpha_j - j + 1}{i - j + 1} \right) = \begin{vmatrix} \binom{\alpha_1}{1} & 1 & 0 & \ldots & 0 \\ \binom{\alpha_1}{2} & \binom{\alpha_2 - 1}{1} & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ \binom{\alpha_1}{\ell} & \binom{\alpha_2 - 1}{\ell - 1} & \binom{\alpha_3 - 2}{\ell - 2} & \ldots & \binom{\alpha_\ell - \ell + 1}{1} \end{vmatrix}.$$

- If $\alpha_1, \ldots, \alpha_\ell$ are in arithmetic progression, i.e., $\alpha_i = c(i-1) + d \ \forall i$ for some $c, d \in \mathbb{Z}$, then

$$k_\alpha = \frac{\alpha_1}{\ell!} \prod_{i=1}^{\ell-1} (\alpha_{\ell+1} - i) = \frac{\alpha_1}{\alpha_{\ell+1}} \binom{\alpha_{\ell+1}}{\ell},$$

  where $\alpha_{\ell+1} = c\ell + d = \ell\alpha_2 + (1-\ell)\alpha_1$.

- $k_\alpha = \sum_{s_1=p_1}^{\alpha_{p_1}} \sum_{s_2=p_2}^{\alpha_{p_2}} \cdots \sum_{s_u=p_u}^{\alpha_{p_u}} \prod_{i=0}^{u} \binom{\alpha_{p_{i+1}} - \alpha_{p_i}}{s_{i+1} - s_i}.$

# What do we know about the MDC?

Recall that the MDC states that

$$d\left(C_\alpha(\ell, m)\right) = q^{\delta_\alpha}, \quad \text{where} \quad \delta_\alpha := (\alpha_1 - 1) + \cdots + (\alpha_\ell - \ell).$$

The MDC is:

- True if $\alpha = (m - \ell + 1, \ldots, m-1, m)$. [Nogin (1996)]
- True if $\ell = 2$. [Hao Chen (2000)]; independently [Guerra-Vincenti (2002)].
  In general, one has a lower bound for $d\left(C_\alpha(\ell, m)\right)$ [G-V (2002)]:

$$\frac{q^{\alpha_1}(q^{\alpha_2} - q^{\alpha_1}) \cdots (q^{\alpha_\ell} - q^{\alpha_{\ell-1}})}{q^{1+2+\cdots+\ell}} \geq q^{\delta_\alpha - \ell}.$$

- True for $C_{(2,4)}(2,4)$. [Vincenti (2001)]
- True for all Schubert divisors in $G_{\ell,m}$. [G - Tsfasman (2005)]
- True, in general! [Xu Xiang (2008)], [G - Singh (2018)]

# Minimum Weight Codewords of Schubert Codes

The first natural question is the following.

Question: Does every decomposable elements of $\bigwedge^{m-\ell} V$ correspond to a minimum weight codeword of $C_\alpha(\ell, m)$?

The answer is No, in general. For example, consider $\alpha = (\alpha_1, \alpha_2) \in I(2, m)$ with $\alpha_1 \geq 2$. As before, let $A_1 = \langle e_1, \ldots, e_{\alpha_1} \rangle$ and $A_2 = \langle e_1, \ldots, e_{\alpha_2} \rangle$, where $\{e_1, \ldots, e_m\}$ is a fixed basis of $V$. Let

$$f = e_3 \wedge \cdots \wedge e_m \in \bigwedge^{m-2} V \text{ and } c_f \text{ the corresponding codeword in } C_\alpha(2, m).$$

Then it can be shown that $\mathrm{wt}(c_f) = q^{\alpha_1+\alpha_2-3} + q^{\alpha_1+\alpha_2-4} - q^{2\alpha_1-3}$, and so

$$\mathrm{wt}(c_f) = q^{\delta(\alpha)} \iff \alpha_2 = \alpha_1 + 1, \text{ i.e., } C_\alpha(2, m) = C(2, \alpha_2).$$

On the other hand, $h = e_1 \wedge e_3 \wedge e_5 \wedge \cdots \wedge e_m \in \bigwedge^{m-2} V$ is decomposable and it can be seen that $\mathrm{wt}(c_h) = q^{\alpha_1+\alpha_2-3}$.

# Schubert Decomposability

It turns out that we need a notion more subtle than decomposability. Let us

- write $\alpha$ uniquely as

$$\alpha = (\alpha_1, \ldots, \alpha_{p_1},\ \alpha_{p_1+1}, \ldots, \alpha_{p_2},\ \ldots,\ \alpha_{p_{u-1}+1}, \ldots, \alpha_{p_u},\ \alpha_{p_u+1}, \ldots, \alpha_\ell)$$

  so that $1 \le p_1 < \cdots < p_u < \ell$ and $\alpha_{p_i+1}, \ldots, \alpha_{p_{i+1}}$ are consecutive for $0 \le i \le u$, and moreover, $\alpha_{p_j+1} - \alpha_{p_j} \ge 2$ for $1 \le j \le u$. By convention, $p_0 = 0$ and $p_{u+1} = \ell$.

- $\alpha$ is called completely nonconsecutive if $\alpha_i - \alpha_{i-1} \ge 2$ for all $2 \le i \le \ell$

### Definition

A decomposable element $f = f_1 \wedge \ldots \wedge f_{m-\ell} \in \bigwedge^{m-\ell} V$ is said to be Schubert decomposable if $\dim(\mathrm{V_f} \cap \mathrm{A_{p_i}}) = \alpha_{\mathrm{p_i}} - \mathrm{p_i}$ for all $i = 1, \ldots, u$, where $V_f$ denotes the *annihilator* of $f$, i.e., $V_f := \{v \in V : v \wedge f = 0\} = \langle f_1, \ldots, f_{m-\ell} \rangle$.

# Schubert Decomposability and Min Weight Codewords

Note that in the Grassmann case, i.e., when $\alpha = (m-\ell+1, \dots, m-1, m)$, or more generally, when $\alpha_1, \dots, \alpha_\ell$ are consecutive, we have $u = 0$ and in this case, the notions of decomposability and Schubert decomposability coincide.

**Conjecture**

Minimum weight codewords of the Schubert code $C_\alpha(\ell, m)$ are precisely the codewords corresponding to Schubert decomposable elements of $\bigwedge^{m-\ell} V$.

**Theorem (G – Singh, 2018)**

If $f \in \bigwedge^{m-\ell} V$ is Schubert decomposable, then $c_f$ is a minimum weight codeword of the Schubert code $C_\alpha(\ell, m)$.

**Theorem (G – Singh, 2018)**

Assume that $f \in \bigwedge^{m-\ell} V$ is decomposable. If $c_f$ is a minimum weight codeword of $C_\alpha(\ell, m)$, then $f$ is Schubert decomposable.

# What remains to be proved?

In view of the above results, to prove the above conjecture, it suffices to answer the following.

Question: If $c_f$ is a minimum weight codeword of $C_\alpha(\ell, m)$, then is there is a decomposable element $h \in \bigwedge^{m-\ell} V$ such that $c_f = c_h$?

It is rather easy to see that answer is Yes when $\alpha \in I(\ell, m)$ is consecutive. Curiously, the conjecture is also true when $\alpha$ is at the other extreme.

**Theorem (G – Singh, 2018)**

Assume that $\alpha$ is completely non-consecutive. If $c$ is a minimum weight codeword of $C_\alpha(\ell, m)$, then $c = c_h$ for some decomposable $h \in \bigwedge^{m-\ell} V$.

**Corollary**

If $\ell \leq 2$, then the above Conjecture holds in the affirmative.

# Enumeration of Minimum Weight Codewords

The question about the number of maximal hyperplane sections of Schubert varieties is answered by the following, modulo the conjecture.

**Theorem (G – Singh, 2018)**

The number of codewords of $C_\alpha(\ell, m)$ corresponding to Schubert decomposable elements of $\bigwedge^{m-\ell} V$ is equal to

$$N_\alpha(\ell, m) := (q-1)q^{\mathsf{P}} \prod_{j=0}^{u} \begin{bmatrix} \alpha_{p_{j+1}} - \alpha_{p_j} \\ p_{j+1} - p_j \end{bmatrix}_q,$$

where

$$\mathsf{P} = \sum_{j=1}^{u} p_j \left(\alpha_{p_{j+1}} - \alpha_{p_j} - p_{j+1} + p_j\right).$$

# The case of Schubert divisors and its Variants

**Theorem (Datta–Panja–G, 2022)**

*Suppose $\ell > 1$ and $\alpha = (\alpha_1, \dots, \alpha_\ell) \in I(\ell, m)$ is such that $\alpha_2, \dots, \alpha_\ell$ are consecutive and $\alpha_1 = \alpha_2 - 2$. If $c_f$ is a minimum weight codeword in $C_\alpha(\ell, m)$ for some $f \in \bigwedge^{m-\ell} V$, then $f$ is decomposable.*

Note that the Schubert divisor in $G_\ell(V)$ corresponds to the variety $\Omega_\alpha$ with $\alpha = (m - \ell, m - \ell + 2, \dots, m)$ , which is a special case of the above theorem. The following result corresponds to another variant of Schubert divisors.

**Theorem (Datta–Panja–G, 2022)**

*Let $\ell > 1$ and $\alpha = (\alpha_1, \dots, \alpha_\ell) \in I(\ell, m)$ be such that $\alpha_1, \dots, \alpha_{\ell-1}$ are consecutive and $\alpha_\ell - \alpha_{\ell-1} > 1$. If $c_f$ be a minimum weight codeword in $C_\alpha(\ell, m)$ for some $f \in \bigwedge^{m-\ell} V$, then $c_f = c_h$ for some decomposable element $h$ in $\bigwedge^{m-\ell} V$.*

**Thank you!**

# Linear Sections of Schubert Varieties

More generally, for a fixed $\alpha \in I(\ell, m)$ and $1 \leq r \leq k_\alpha$, we consider

$$e_r^\alpha(\ell, m) := \max_L |\Omega_\alpha(\mathbb{F}_q) \cap L|$$

where the maximum is over all linear subvarieties $L$ in $\mathbb{P}^{k_\alpha - 1}$ of codimension $r$. As before, we also consider

$$M_r^\alpha(\ell, m) := \left|\{L : L \subseteq \mathbb{P}^{k_\alpha - 1} \text{ linear of codim } r \text{ with } |\Omega_\alpha(\mathbb{F}_q) \cap L| = e_r^\alpha(\ell, m)\}\right|.$$

Again, in the special case when $\alpha = (m - \ell + 1, \ldots, m - 1, m)$, that is, when $\Omega_\alpha = G_{\ell,m}$, we may denote $e_r^\alpha(\ell, m)$ and $M_r^\alpha(\ell, m)$ simply by $e_r(\ell, m)$ and $M_r(\ell, m)$, respectively.

The question about the determination of $e_r^\alpha(\ell, m)$ and $M_r^\alpha(\ell, m)$ are open, in general. They are also closely related to questions in Coding Theory.

# Linear Sections and Higher Weights

Let $C$ be a $[n, k]_q$-code. For $1 \leq r \leq k$, the $r^{\text{th}}$ higher weight or the $r^{\text{th}}$ generalized Hamming weight of $C$ is defined by

$$d_r(C) := \min\{\mathrm{wt}(D) : D \text{ a subspace of } C \text{ with } \dim D = r\},$$

where for $D \subseteq C$, by $\mathrm{wt}(D)$ we mean the support weight of $C$, i.e.,

$$\mathrm{wt}(D) := |\{i \in \{1, \ldots, n\} : \exists\, c = (c_1, \ldots, c_n) \in D \text{ with } c_i \neq 0\}|$$

Note that $d_1(C) = d(C)$ and also that; $C$ is nondegenerate $\Longleftrightarrow d_k(C) = n$.

The $r^{\text{th}}$ higher weights of the Schubert code $C_\alpha(\ell, m)$ correspond to maximal sections of $\Omega_\alpha(\mathbb{F}_q)$ by linear subvarieties of $\mathbb{P}^{k_\alpha - 1}$ of codimension $r$, since

$$d_r\left(C_\alpha(\ell, m)\right) = n_\alpha - e_r^\alpha(\ell, m) \quad \text{for } r = 1, \ldots, k_\alpha,$$

and $M_r^\alpha(\ell, m)$ corresponds to the number of minimal subcodes of dimension $r$.

# Higher Weights of Grassmann Codes

The first result in this direction is:

**Theorem (Nogin (1996), Ghorpade-Lachaud(2000))**

*More generally, for* $1 \leq r \leq \mu$ *we have*

$$d_r\left(C(\ell, m)\right) = q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-r+1},$$

*where* $\mu := \max\{\ell, m-\ell\} + 1$.

The alternative proofs in Ghorpade-Lachaud(2000) used a characterization of the so-called close families of $\ell$-subsets of the finite set $\{1, \ldots, m\}$. This can be viewed as an analogue for uniform hypergraphs of the following elementary result in graph theory: *A simple graph in which any two edges are incident is either a star or a triangle.*

Note: $\mu = \max\{\ell, m-\ell\} + 1$ is usually much smaller than $k = \binom{m}{\ell}$ and so the above theorem doesn't give all the higher weights.

# Initial & Terminal Higher Weights of Grassmann Codes

Recall: $\mu := \max\{\ell, m-\ell\}+1$ and we had:

**Theorem (Nogin (1996), Ghorpade-Lachaud(2000))**

*For* $1 \leq r \leq \mu$*, we have*

$$d_r\left(C(\ell, m)\right) = q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-r+1}.$$

One has the following counterpart from the other end.

**Theorem (Hansen-Johnsen-Ranestad (2007))**

*On the other hand, for* $0 \leq r \leq \mu$*,*

$$d_{k-r}\left(C(\ell, m)\right) = n - (1 + q + \cdots + q^{r-1}).$$

These results cover several initial and terminal elements of the weight hierarchy of $C(\ell, m)$. Yet, a considerable gap remains.

# Narrowing the gap

Examples:

- $(\ell, m) = (2, 5)$. Here $k = 10$, $\mu = 4$ and we know:

$$d_1, \ldots, d_4 \quad \text{as well as} \quad d_6, \ldots, d_{10}.$$

  But $d_5$ seems to be unknown.
- $(\ell, m) = (2, 6)$. Here $k = 15$, $\mu = 5$ and $d_6, \ldots, d_9$ are not known.
- For $C(2, m)$ with $m \geq 2$, the values of $d_r$ for $m \leq r < \binom{m-1}{2}$ do not seem to be known.

**Theorem (Hansen-Johnsen-Ranestad (2007))**

$$d_5(C(2,5)) = q^6 + q^5 + 2q^4 + q^3 = d_4 + q^4.$$

**Conjecture (Hansen-Johnsen-Ranestad (2007))**

$d_r - d_{r-1}$ is always a power of $q$.

# One step forward

Theorem (Ghorpade-Patil-Pillai (2009))

*Assume that* $\ell = 2$ *and* $m \geq 4$ *so that*

$$\mu = \max\{2, m-2\} + 1 = m - 1 \text{ and } k = \binom{m}{2}.$$

*Then*

$$d_{\mu+1}(C(2,m)) = d_\mu + q^{\delta-2}$$

*and*

$$d_{k-\mu-1}(C(2,m) = n - \left(1 + q + \cdots + q^\mu + q^2\right).$$

Corollary. Complete weight hierarchy of $C(2,5)$.

Remark. The proof of the above theorem uses a characterization of decomposable subspaces of $\wedge^\ell V$ where $V$ is an $m$-dimensional vector space, and this auxiliary result can be viewed as an algebraic analogue of the structure theorem for close families of $\ell$-subsets of an $m$-set.

**Thank you!**

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