Locally recoverable codes from towers of function fields

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Conference **O**n al**G**ebraic varieties over fi**N**ite fields and **A**lgebraic geometry **C**odes
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LRC Codes

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Dimension $k = \log_q(|\mathscr{C}|)$
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Clearly, given any $n-d+1$ coordinates of a codeword x, it is posible to recover all the remaining $d-1$ coordinates.
¿It is possible to recover any coordinate x_i with fewer than $n-d+1$ other coordinates?

Gopalan gave a first definition of an LRC code, which now state as

Definition (LRC Codes)

A code $\mathscr{C} \subset \mathbb{F}_q^n$ is locally recoverable (LRC) with locality r if the value of every coordinate of the codeword can be found by accessing at most r other coordinates of this codeword. That is, for every $i \in [n]$ exists $A_i \subset [n] \setminus \{i\}$ with $|A_i| \leq r$ and a function $\phi_i : \mathbb{F}_q^{|A_i|} \to \mathbb{F}_q$ s.t. $\forall x \in \mathscr{C}$

$$x_i = \phi_i(x_{j_1}, \ldots, x_{j_{|A_i|}})$$

where $A_i = \{j_1, \ldots, j_{|A_i|}\}$

For $\mathscr{C} \subset \mathbb{F}_q^n$ of length n, dimension k and locality r, we will say $\mathscr{C}$ is a (n, k, r) LRC code.

As an example

$$\mathscr{C} = \{000000, 001111, 111100, 110011\} \subset \mathbb{F}_2^6$$

is a repetition code in which every codeword $x \in \mathscr{C}$ verifies $x_{2j} = x_{2j-1}$ for $j = 1, 2, 3$, so $\mathscr{C}$ is a $(6, 2, 1)$ LRC code with locality 1.

It has enormous *redundancy*, since only two out of the six coordinates (1st and 3rd coordinate) carry information.

LRC RS Codes

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Barg and Tamo gave the following construction of LRC codes based on the Reed Solomon codes:

- Consider parameters (n, k, r) with $n \leq q$, $r+1 \mid n$ and $r \mid k$.
- Let $A = \{P_1, \ldots, P_n\} \subset \mathbb{F}_q$ distinct points,
- $\mathscr{A} = \{A_1, \ldots, A_{\frac{n}{r+1}}\}$ a partition of A.
- Suppose exists $g(x) \in \mathbb{F}_q[x]$ of degree $r+1$, constant on each A_i.

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The space

$$V = \left\langle g(x)^j x^i : 0 \leq i \leq r-1, 0 \leq j \leq \frac{k}{r} - 1 \right\rangle \subset \mathbb{F}_q[x]$$

is of dimension $r \cdot \frac{k}{r} = k$ with

$$a = (a_{ij}) \in \mathbb{F}_q^k \mapsto f_a = \sum_{i=0}^{r-1} \sum_{j=0}^{k/r-1} a_{ij} g(x)^j x^i.$$

The evaluation map $ev : V \to \mathbb{F}_q^n$ with

$$f_a \mapsto (f_a(P_1), \ldots, f_a(P_n))$$

is injective, so this gives rise to a $(n, k, d \geq n - k - \frac{k}{r} + 2)$ code $\mathscr{C}$.

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It is LRC with locality r: each f_a restricted to A_i can be viewed as a polynomial $\delta_i(x)$ of degree less at most $r - 1$, and thus can be determined by polynomial interpolation in any r elements of A_i.

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For $R = \frac{k}{n}$ and $\delta = \frac{d}{n}$ it holds $R < \frac{r}{r+1}(1-\delta) < \frac{r}{r+1}$

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Proposition (Barg,Tamo,Vladut bound)

There exist $\mathscr{C} \subset \mathbb{F}_{q^2}$ s.t. its relative parameters verify

$$R \geq \frac{r}{r+1}\left(1 - \delta - \frac{3}{q+1}\right),$$

AG Codes

We now give a brief definition of an AG Codes, based on the RS codes

Definition

Let X be a smooth projective irreducible curve defined over $\mathbb{F}_q$. Set $\mathbb{F}_q(X)$ its function field, and chose a divisor $D = \sum a_i Q_i$ of situable degree $l \geq 1$.

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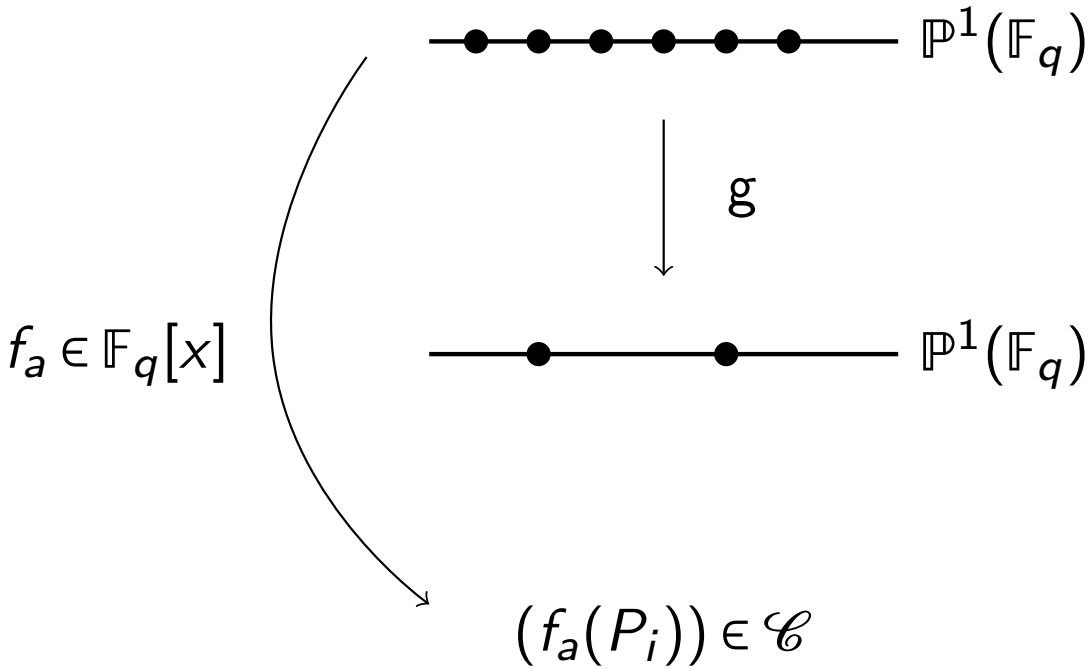
Suppose $\mathscr{B} \subset X(\mathbb{F}_q)$ is a subset of n $\mathbb{F}_q$-rational points disjoint of $supp(D)$. For $\mathscr{L}(D)$ the Rieman-Roch space associated to D, the code $\mathscr{C} = \mathscr{C}(D, \mathscr{B})$ given by the evaluation map

$$\begin{aligned} ev : \mathscr{L}(D) &\to \mathbb{F}_q^n \\ f &\mapsto (f(P_1), \ldots, f(P_n)) \end{aligned}$$

is a (n, k, d) code where $k = \ell(D) = \dim \mathscr{L}(D)$ and $d \geq n - l$.


$\mathbb{P}^1(\mathbb{F}_q)$
g
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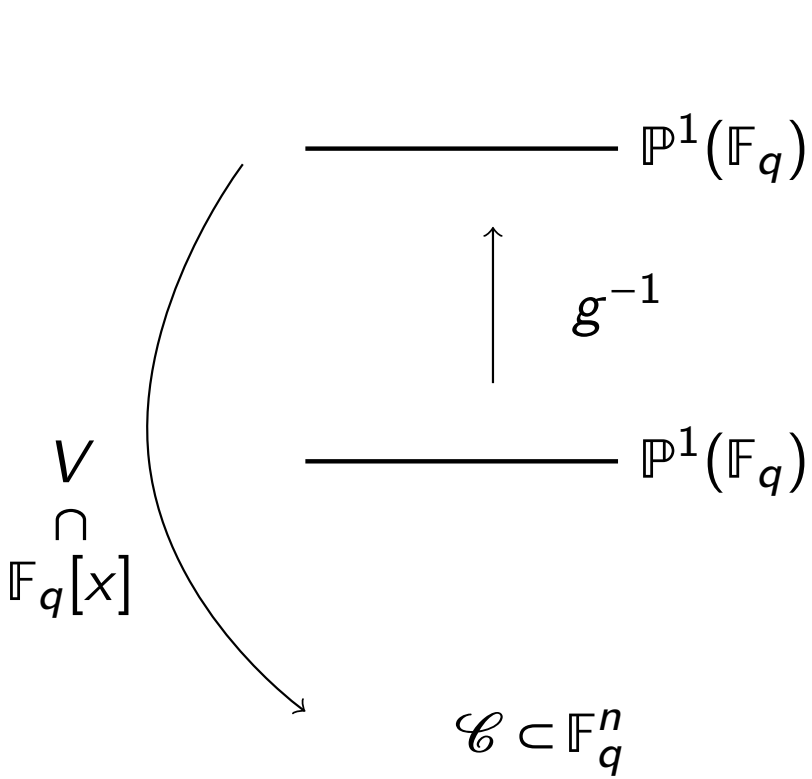
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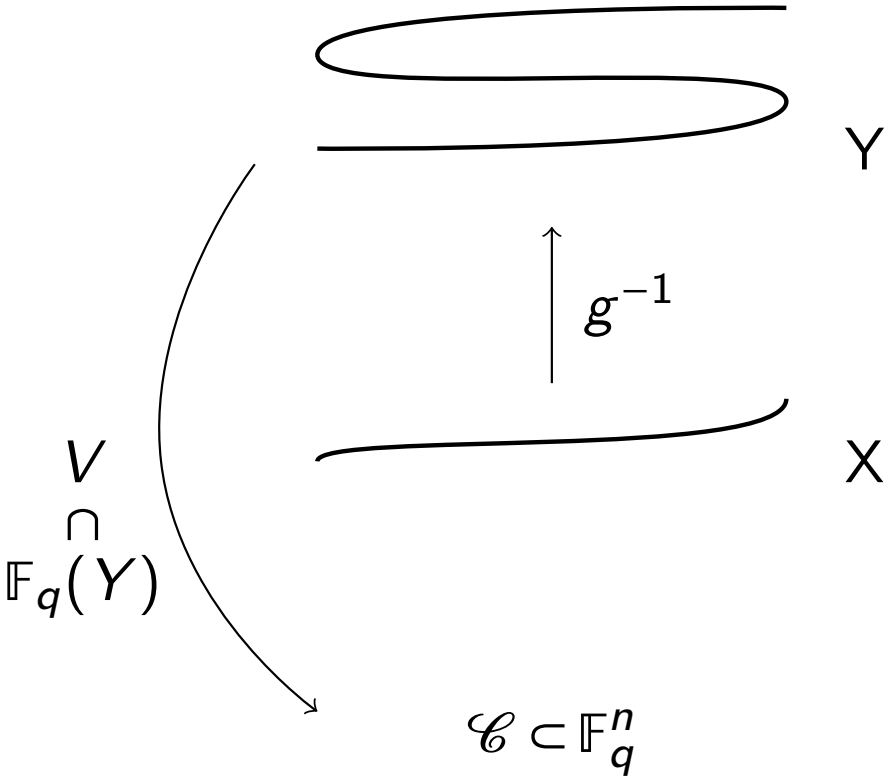
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More precisely, we have a construction of LRC AG codes

Theorem (Barg, Tamo, Vladut)

For $S = \{P_1, \ldots, P_s\} \subset X(\mathbb{F}_q)$ a set of rational points,
$\mathscr{B} = g^{-1}(S) = \{P_{ij}\} \subset Y$ a set of rs rational points,
$\mathscr{L}(D) = \langle f_1, \ldots, f_m \rangle$ for a suitable divisor, and $y \in \mathbb{F}_q(Y)$ such that $\mathbb{F}_q(Y) = \mathbb{F}_q(X)(y)$, then for

$$V = \left\langle f_l x^i : 1 \leq l \leq m, 0 \leq i \leq r-1 \right\rangle$$

one obtain a LRC code $\mathscr{C}$ of locality $r = deg(g)$ and parameters

$$n = (r+1)s, \quad k = r\ell(D) \geq r(deg(D) - g_X + 1),$$

$$d \geq n - deg(D)(r+1) - (r-1)deg(x)$$

provided d is a positive integer.

For the case of more than one recovery set, there also is another important construction

Theorem (Haymaker, Malmskog, Matthews)

For $S = \{P_1, \ldots, P_s\} \subset X$ a set of rational points, $Y_1, \ldots, Y_t$ curves such that $\mathbb{F}_q(Y_j) = \mathbb{F}_q(X)(x_j)$, $Y = Y_1 \times_X \ldots \times_X Y_t$ the fiber product, $\mathscr{L}(D) = \langle f_1, \ldots, f_m \rangle$ for a suitable divisor of X, then for

$$V = \left\langle f_l x_1^{e_1} \cdots x_t^{e_t} : 1 \leq l \leq m, 0 \leq e_i \leq d_i - 2 \right\rangle$$

one obtain a LRC code $\mathscr{C}$ of locality $d_i - 1$ and parameters

$$n = rs, \quad k = \ell(D)(d_1 - 1) \cdots (d_t - 1),$$

$$d \geq n - deg(D)r - \sum (d_i - 1) deg(x_i)[\mathbb{F}_q(Y) : \mathbb{F}_q(Y_j)]$$

provided d is a positive integer.

Construction on Function Fields

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Theorem

$F/\mathbb{F}_q$ function field, and let E/F function field extension of degree m. $S \subset \mathbb{P}(F)$ set of s places that split completely in E/F s.t.

$$\{Q \in \mathbb{P}(E) : Q \cap F \in S\} \cap \{Q \in \mathbb{P}(E) : v_Q(x) < 0\} = \emptyset$$

and let $\mathscr{B} = \{Q \in \mathbb{P}(E) : Q = P \cap F$ for some P in $S\}$, so $|\mathscr{B}| = sm$.

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$$\begin{array}{rccc} ev: & V & \longrightarrow & \mathbb{F}_q^{(r+1)s} \\ & f & \to & (f(P_{11}), \ldots, f(P_{ms})) \end{array}$$

is well defined and gives a LRC code $\mathscr{C} = C(S, D)$ with locality r. The code coordinates are naturally partitioned into s subsets $A_j = \{P_{ij}\}_{i=1,\ldots,r+1}$.

Denoting by $g(F)$ the genus of F and $h = [E : \mathbb{F}_q(x)]$, we have that the parameters of the code satisfy

$$\begin{aligned} n &= (r+1)s \\ k &= r\ell \geq r(l+1-g(F)) \\ d &\geq n - l(r+1) - (r-1)h \end{aligned}$$

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Local recovery of an erased symbol $c_{ij} = f(P_{ij})$ can be performed by polynomial interpolation through the positions of the points in the recovery set $A_j \setminus \{P_{ij}\}$.

Towers of Function Fields

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We consider a sequence $\mathscr{F} = (F_0, F_1, F_2, \ldots)$ of function fields such that

- $F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots$
- F_{i+1}/F_i is finite separable for any $i \geq 0$,
- $\mathbb{F}_q \subset F_i$ is algebraically closed for any $i \geq 0$

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Although is not needed, we may suppose that exist some polynomial $f(x, y)$ such that $F_0 = \mathbb{F}_q(x_0)$ and $F_{i+1} = F_i(x_{i1})$ where $f(x_i, x_{i+1}) = 0$ and

$$[F_{i+1} : F_i] = \deg f.$$

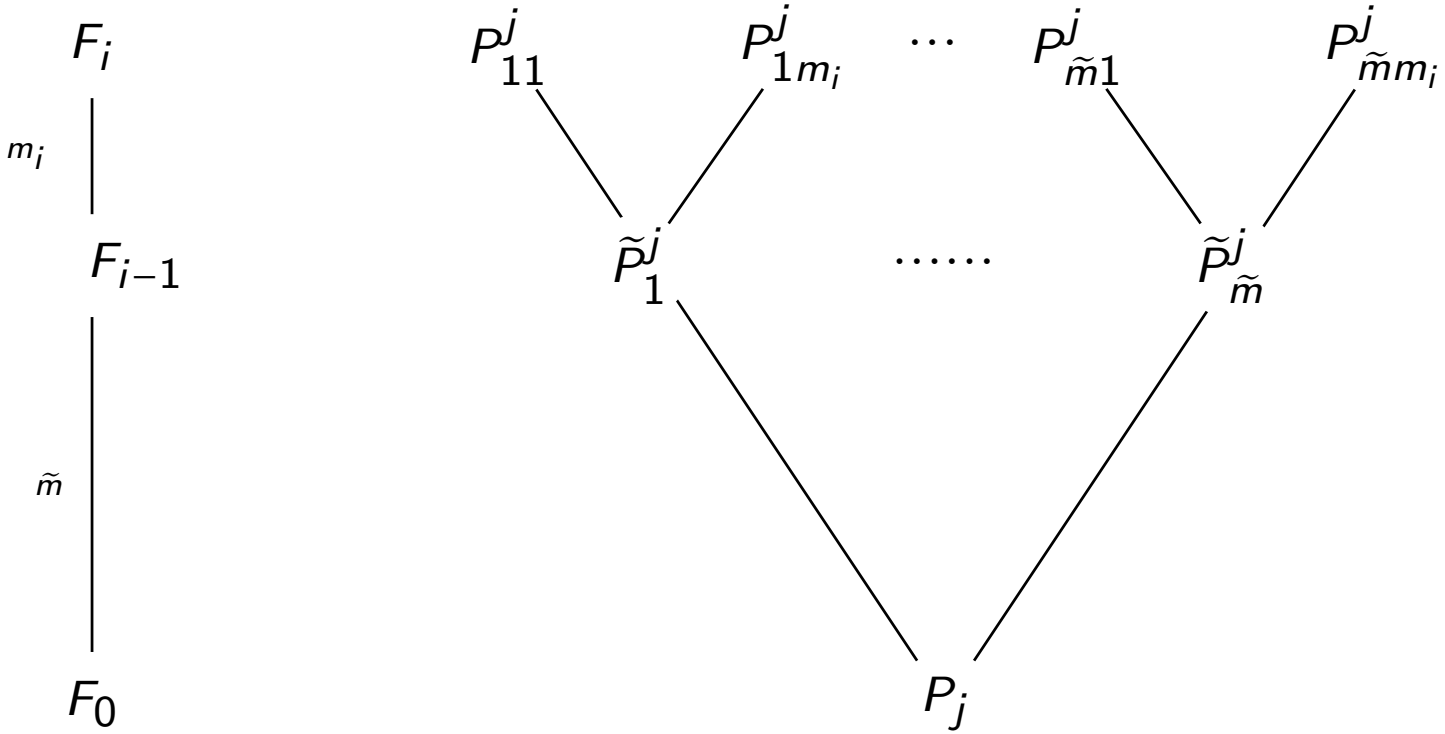

Figure: Diagram of an splitting place P_j of F_0 in F_i

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Consider the function field $E/\mathbb{F}_{q^2}$ defined by

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It can be shown that

1. $[E : F] = [E : \mathbb{F}_{q^2}(y)] = q$,
2. x has exactly one pole in E (namely, Q_∞) totally ramified in E/F,
3. If P_α is the only zero of $x - \alpha$ for $\alpha \in S$, then P_α splits completely.
   Otherwise, it is totally ramified if $\alpha \neq 0$.

We can consider $D = lP_\infty$ and

$$S = \{P_\alpha \in \mathbb{P}(F_0) : \alpha \in \mathbb{F}_{q^2} \text{ and } \alpha^q + \alpha \neq 0\},$$

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and thus a

$$(n = q(q^2 - q), k = l(q-1), r = q-1) \text{ LRC code.}$$

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Theorem (Chara, G., Martinez-Moro)

Let $\mathscr{F} = \{F_j\}_{j=0}^{\infty}$ be a sequence of function fields such that $F_0 = \mathbb{F}_q(x_0)$ and $F_j = F_{j-1}(x_j)$ for $j > 0$. Let $m_j = [F_j : F_{j-1}]$ and $E = F_i$, for some $i \geq 2$.

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$$\{Q \in \mathbb{P}(E) : Q \cap F_0 \in S\} \cap \left(\bigcup_{j=1}^{i} \{Q \in \mathbb{P}(E) : v_Q(x_j) < 0\} \right) = \emptyset$$

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and let $\mathscr{B} = \{Q \in \mathbb{P}(E) : Q \cap F_0 \in S\}$. Then, if $s = |S| > 0$ we have that $|\mathscr{B}| = sm$ where $m = m_i \dots m_1$. For a divisor D of F_0 of degree l such that $supp(D) \cap S = \emptyset$, let $\{f_1, \dots, f_\ell\}$ a basis for $\mathscr{L}(D)$.

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Consider the space V generated by

$$\{f_w x_1^{e_1} \cdots x_i^{e_i} : 1 \leq w \leq \ell; 0 \leq e_i \leq m_i - 2 \text{ and } 0 \leq e_j \leq m_j - 1 \text{ for } j = 1, \ldots, i-1\}.$$

Then $ev : V \to \mathbb{F}_q^{ms}$

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$$A_t^j = \{P_{tu}^j : 1 \le u \le m_i\} = \{Q \in \mathbb{P}(F_i) : Q \cap F_{i-1} = P_{t1}^j \cap F_{i-1} = \widetilde{P_t}^j\}$$

where $1 \le j \le s$; $1 \le t \le \tilde{m}$ and $\tilde{m} = m_{i-1} \ldots m_1 = m/m_i$.

Denoting by $h_j = [F_i : \mathbb{F}_q(x_j)]$, we have that the parameters of the code satisfy

$$
\begin{aligned}
&n = ms = m_i \dots m_1 s \\
&k = \ell(m_i - 1)m_{i-1} \cdots m_1 \geq (l+1)(m_i - 1)m_{i-1} \cdots m_1 \\
&d \geq n - lm - (m_1 - 1)h_1 - \cdots - (m_{i-1} - 1)h_{i-1} - (m_i - 2)h_i
\end{aligned}
$$

provided that the right-hand side of the inequality for d is a positive integer.

Local recovery of an erased symbol $f(P^j_{tu})$ can be performed by polynomial interpolation through the positions of the points in the recovery set $A^j_t \setminus \{P^j_{tu}\}$.

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The Garcia Stichtenoth Tower

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We recall the Garcia-Stichtenoth Tower, defined recursively for $i \geq 0$ as

$$F_0 = \mathbb{F}_{q^2}(x_0), \quad F_{i+1} = F_i(x_{i+1}),$$

where

$$x_{i+1}^q + x_{i+1} = \frac{x_i^q}{x_i^{q-1} + 1}.$$

Garcia-Stichtenoth showed that $N(F_j) \geq q^j(q^2 - q) + 1$ and moreover

$$S = \{P_\alpha \in \mathbb{P}(F_0) : \alpha \in \mathbb{F}_{q^2} \text{ and } \alpha^q + \alpha \neq 0\}$$

is a set of places that splits completely on $\mathscr{F}$.

We can consider, then, for each $j \geq 1$ the following subspace of F_j

$$V = \left\langle x_0^{e_0} x_1^{e_1} \cdots x_j^{e_j} : 0 \leq e_0 \leq l, 0 \leq e_i \leq q-1; 1 \leq i \leq j \right\rangle.$$

for suitable bound l.

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This gives a k dimensional linear subspace, with $k = (l+1)q^{j-1}(q-1)$, and considering $\mathscr{B} = \{Q \in \mathbb{P}(F_j) : Q|P \text{ for some } P \in S\}$ we can define an (n, k, r) code $\mathscr{C} = C_j(S, D)$ with

$$n = q^j(q^2 - q),\ k = (l+1)q^{j-1}(q-1),\ r = q-1$$

$$d \geq n - lq^j - \sum_{t=1}^{j}(q-1)q^j = (q^2 - 2q + 2 - l - (q-1)(j-1))q^j$$

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Note that the lower bound is negative for fixed q and j large enough!

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Proposition

For $q \geq 5$, the code $C_2(S, D)$ over the Garcia-Stichtenoth Tower, with $D = qP_\infty$, is a locally recoverable code over $\mathbb{F}_{q^2}$, with locality $r = q - 1$, whose parameters are

$$n = q^2(q^2 - q),$$
$$k = (q+1)q(q-1) = q^3 - q,$$
$$d = q^2(q^2 - 2q + 2 - q - (q-1)) = q^2(q^2 - 4q + 3).$$

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We must show that exist some $f \in V$ such that f has at least $q^2(3q-3)$ zeroes, so the minimum distance of the evaluation code $C_2(S,D)$ is at most

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Consider the set $S_0=\{\alpha \in \mathbb{F}_{q^2} : \alpha^q+\alpha \neq 0\}$. This set is naturally partitioned into $q-1$ disjoints subsets $S_1,\ldots,S_{q-1}$ of size q where

$$S_i=\{\alpha \in \mathbb{F}_{q^2} : \frac{\alpha^q}{\alpha^{q-1}+1}=\frac{\alpha^{q+1}}{\alpha^q+\alpha}=\beta_i\}, \quad \beta_i \in \mathbb{F}_q^*$$

For a rational place $P \in \mathbb{P}(F_j)$ we denote

$$N(P) = N_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x_j(P)) = x_j(P)^{q+1},$$

$$Tr(P) = Tr_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x_j(P)) = x_j(P)^q + x_j(P),$$

where $x_j \in F_j$ is the transcendental element defining the Tower.

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so in particular we see that for each $P \in \mathscr{B} \cap F_{j-1}$ there are at most two places $Q_1, Q_2 \in \mathscr{B}$ such that

$$x_{j-1}(Q_1) = x_{j-1}(Q_2) = x_{j-1}(P) \in S_i.$$

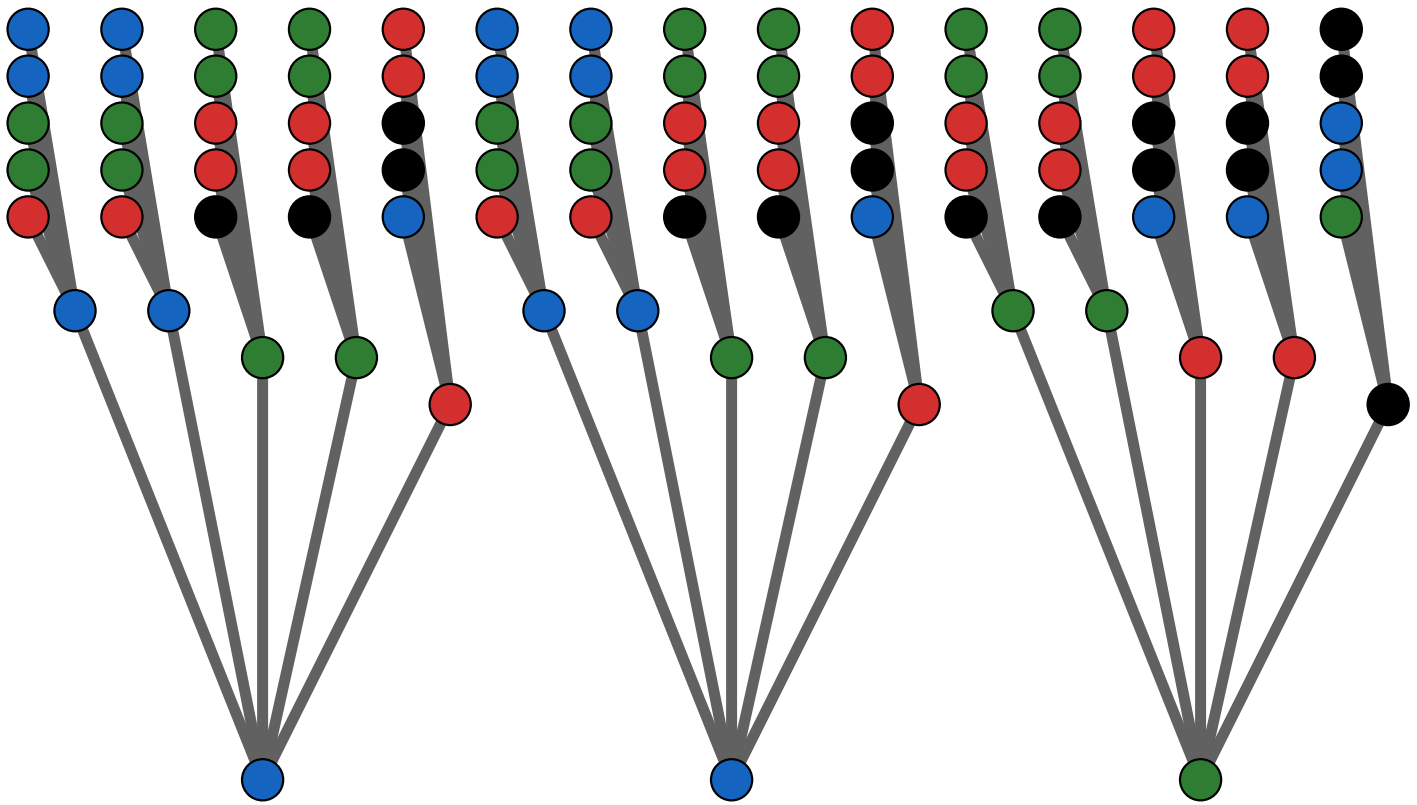

Diagram of three splitting places P_j of F_0 in F_2/F_0, for $q = 5$. Each colour, in each function field, represent a set S_i, for $1 \leq i \leq 4 = q - 1$

Moreover, if σ is the only nontrivial automorphism of $\mathbb{F}_{q^2}/\mathbb{F}_q$ then $x_j(Q_1) = \sigma(x_j(Q_2))$ so that, since q is odd, there is exactly one place $Q|P$ such that $x_{j-1}(Q) = x_{j-1}(P) \in S_i$ and $\sigma(x_j(Q)) = x_j(Q)$.

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In other words, we can define $q-1$ disjoint subsets

$$B_i = \{\alpha \in \mathbb{F}_{q^2} : Tr(\alpha) = \beta_i\}, \beta_i \in \mathbb{F}_q^*$$

such that for any indexes $1 \leq i, k \leq q-1$ we have

$$|S_i \cap B_k| = |\{Q \in \mathbb{P}(F_j) : x_j(Q) \in S_i \text{ and } x_j(Q) \in B_k\}| \leq 2$$

with

$$|\{Q \in \mathbb{P}(F_j) : x_j(Q) \in S_i \text{ and } x_j(Q) \in B_k\}| = 1 \iff x_j(Q) \in \mathbb{F}_q$$

Now, in the notation of the theorem, we can consider $i = 2$, $l = q$, and $D = qP_\infty$ so that

$$V = \left\langle x_0^{e_0} x_1^{e_1} x_2^{e_2} : 0 \leq e_0 \leq l, 0 \leq e_1 \leq q-1, 0 \leq e_2 \leq q-2 \right\rangle.$$

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Chose some $\beta_1 \in \mathbb{F}_q^*$ and set

$$h_0 = \prod_{\alpha \in S_1} (x_0 - \alpha),$$

so that h_0 is a function with exactly $q^3 = |S_1|[F_2 : F_0]$ zeroes $Q \in \mathscr{B}$. These places verify $x_0(Q) \in S_1$ and thus $x_1(Q) \in B_1$ by the recursive definition of the Tower.

Now, since q is odd, chose β_i such that $|S_i \cap B_1| = 1$, and set

$$h_1 = \prod_{\alpha \in S_i \setminus B_1} (x_1 - \alpha),$$

so we have $(q-1)q^2 = (|S_i|-1)[F_2 : F_1]|\{Q \in \mathbb{P}(F_1) : x_1(Q) = \alpha\}$ new places $Q \in \mathscr{B}$ that are zeroes of h_1.

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Since $|B_1| = q$, we can write B_1 as a disjoint union of $(q+1)/2$ subsets

$$B_1 = \bigcup_{k=1}^{(q+1)/2} B_1 \cap S_{i_k}$$

and therefore if $Q \in \mathscr{B}$ is such that $x_0(Q) \in S_1$ or $x_1(Q) \in S_i$, then $x_2(Q)$ is among $q(q+1)/2$ different values of $\mathbb{F}_{q^2}$.

For $q \geq 5$ we have

$$q(q+1)/2 < (q^2 - q) - (q-2)$$

so we can chose $q-2$ values $\gamma_1, \ldots, \gamma_{q-2}$ in S_0 and form

$$h_2 = \prod_{i=1}^{q-2} (x_2 - \gamma_i)$$

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Therefore, $f = h_0 h_1 h_2 \in V$ has exactly

$$q^3 + q^2(q-1) + q^2(q-2) = q^2(3q-3)$$

different zeroes, attaining the bound provided by the theorem. $\square$

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Remark

The ideas used in the previous proposition can be used to show that exist a LRC code $C_2(S,D)$ *with* $D = \frac{q(q-1)}{2}P_\infty$ *with locality* $r = q-1$ *and parameters*

$$n = q^2(q^2-q),$$

$$k = \left(\frac{q(q-1)}{2}+1\right)q(q-1)$$

$$d = \frac{1}{2}q^2(q^2-3q+6) = q^2(q^2-4q+3).$$

Extending the construction for $j \geq 3$

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The construction attain the bound of the distance for $j = 2$, that is, for two steps in the tower. For $j \geq 3$ it can be shown that

$$d > (q^2 - 2q + 2 - l - (q-1)(j-1))q^j,$$

in the case x_0 and x_3 have a common zero, which is the case of the García-Stichtenoth Tower. Thus, the bound can be slightly improved.

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Proposition

For odd $q \geq 5$, $2 \leq i \leq q-1$ and $1 \leq l \leq (q-1)(q-i)$ the parameters $R = \frac{k}{n}$ and $\delta = \frac{d}{n}$ of $C_i(S, D)$ verify

$$R + \frac{q-1}{q}\delta > \frac{q-1}{q}\left(\frac{q-i}{q}\right)$$

However, for the general case of an arbitrary tower $\mathscr{F}$ we don't know how to improve the bound, if possible.

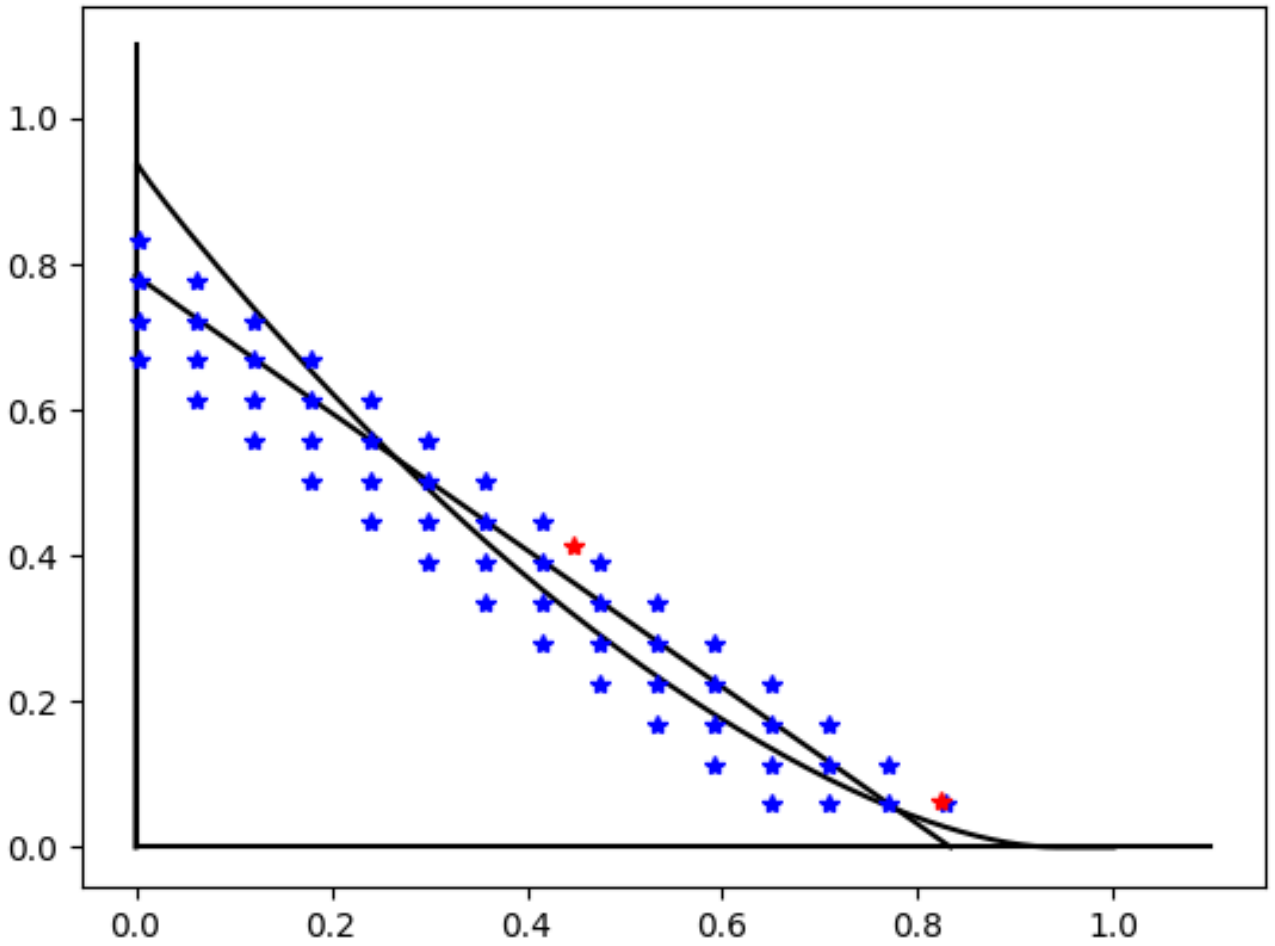

Figure: BTV and GV bounds (shown in black) are improved. For $2 \leq i \leq 5$, lower bounds for $C_i(S, D)$ are shown in blue. Parameters of $C_2(S, D)$ are shown in red.

Tanks for your attention :)